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LINEAR OPERATORS

PART II:

SPECTRAL THEORY

Self Adjoint Operators in Hilbert Space

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Preface

Because of the large amount of material presented, we have been prevented from including all the topics originally announced for Part II of *Linear Operators*. The present volume includes all of the material of our earlier announcement associated with the classical spectral theorem for self adjoint operators in Hilbert space. While there are some isolated discussions of nonselfadjoint operators, such as that giving the completeness of the generalized eigenfunctions of Hilbert-Schmidt operators in Section XI.6, the general theory of spectral operators and the discussion of nonselfadjoint differential boundary value problems have been postponed for inclusion in the forthcoming Part III of this treatise.

Since Part II deals largely with operators in Hilbert space we have reproduced, for convenient reference, Definition IV.2.6 and Section IV.4, as an appended section on Hilbert space immediately following Chapter XIV. This appended section gives basic definitions and the geometric properties of Hilbert space which are used repeatedly in this volume. Thus, for the reader who is familiar with the contents of Chapters I, II, III, and VII, there will be only occasional need for reference to Part I. An interdependence chart showing the dependence of the various sections in Part II upon each other is to be found in the front of the book; the corresponding graph for Part I is reproduced in the back of the book.

Chapter IX is an introduction to the much larger subject of *B*-algebras; it serves us as a basis for the spectral theory of bounded self adjoint operators which is presented in Chapter X. Chapters X and XII constitute a relatively complete discussion of the abstract spectral theory for bounded and unbounded self adjoint operators in Hilbert space. Chapter XI contains a variety of applications of the spectral theorem for bounded operators and of some related topics: the theory of Hilbert-Schmidt operators, and the Riesz-Calderon-Zygmund inequalities. Chapter XIII gives a detailed and

extensive presentation of applications of the spectral theorem for unbounded self adjoint operators to the theory of self adjoint ordinary differential boundary value problems. Chapter XIV gives a brief introduction to the theory of linear partial differential equations; it is presented here to show more of the rich variety of applications of the spectral theorem.

The reader familiar with the similarly-titled work of Stone will find little overlap between the present volume and Stone's well-known work. Our presentation of the general theory is based on the researches of Gelfand, and our choice of applications is different from that of Stone.

We are indebted to many students and colleagues for calling our attention to misprints and errors that have occurred in Part I. Such corrections appear in a list of errata. We wish to thank Doctors E. Koppelman, R. Langlands, G. Leibowitz, N. Metas, and E. Thorp for pointing out many such misprints. In particular R. Langlands has shown that Exercise III.9.20 is false and as a result has made a decided improvement (see list of errata) on a result of Alexandroff (III.5.13). We are indebted to Miss R. M. Castroll for her competent editorial assistance in preparing the manuscript for the printer, as well as in reading all of the proof.

We should also like to thank the staffs of the Air Force Office of Scientific Research and the Office of the Naval Research for their continued encouragement in the preparation of this treatise.

Orwell, Vermont
New York City
July 1963

Nelson Dunford
Jacob Schwartz

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CHAPTER IX

B-Algebras

1. Preliminary Notions

Many of the classes of functions which are of frequent occurrence in mathematical analysis are algebras. For example, the classes consisting of the bounded functions on a set, the bounded continuous functions on a topological space, functions of bounded variation, almost periodic functions, functions with n continuous derivatives, and analytic functions are familiar examples of algebras. In all these examples the algebraic operations are the *natural* ones as expressed by the equations

$$(\alpha f)(s) = \alpha f(s), \qquad (f+g)(s) = f(s)+g(s), \qquad (fg)(s) = f(s)g(s).$$

Some familiar function spaces, such as $L_1(-\infty, \infty)$, are not closed under the natural algebraic operations (the product of two integrable functions need not be integrable) but nevertheless are algebras in a very useful sense. In $L_1(-\infty, \infty)$, for example, if one uses the natural notions of addition and scalar multiplication and defines the product of two functions $f, g \in L_1(-\infty, \infty)$ as their convolution

$$(f * g)(s) = \int_{-\infty}^{\infty} f(s-t)g(t)dt,$$

then $L_1(-\infty, \infty)$ is an algebra. Thus, in one sense or another, many of the linear spaces encountered in mathematics are also algebras. In this chapter will be found an axiomatic treatment of several types of algebras that occur in analysis. Numerous applications of the general theory to special algebras will be found in the following two chapters.

1 Definition. A *B-algebra*, or *Banach algebra*, is a complex B-space $\mathfrak{X}$ which is an algebra with unit e over the field of complex numbers and which has the properties

$$|e| = 1, \qquad |xy| \leq |x||y|, \qquad x, y \in \mathfrak{X}.$$

A B-algebra $\mathfrak{X}$ is *commutative* in case $xy = yx$ for all x and y in $\mathfrak{X}$. An *involution* in a B-algebra $\mathfrak{X}$ is a mapping $x \to x^*$ of $\mathfrak{X}$ into itself with the properties

$$(x+y)^* = x^* + y^*, \qquad (xy)^* = y^*x^*$$
$$(\alpha x)^* = \bar{\alpha}x^*, \qquad (x^*)^* = x.$$

All of the examples mentioned above, with the exception of $L_1(-\infty, \infty)$ and the class of analytic functions, are commutative B-algebras with involutions. In each of these cases the involution may be defined by the equation $x^*(s) = \overline{x(s)}$. The algebra $L_1(-\infty, \infty)$ with convolution as multiplication is a commutative algebra with an involution defined by $f^*(s) = \overline{f(-s)}$. It fails to be a B-algebra because it lacks a unit e. We shall show how a unit may be adjoined to such an algebra so that the extended algebra is a B-algebra. Let $\mathfrak{X}$ be an algebra satisfying all the requirements of a B-algebra except that $\mathfrak{X}$ has no unit. Let $\mathfrak{X}_1 = \Phi \times \mathfrak{X}$ where Φ is the complex number system. The elements of $\mathfrak{X}_1$ are then all ordered pairs $[\alpha, x]$ with $\alpha \in \Phi$, $x \in \mathfrak{X}$. The operations in $\mathfrak{X}_1$ are defined by the equations

$$[\alpha, x] + [\beta, y] = [\alpha+\beta, x+y],$$
$$[\alpha, x][\beta, y] = [\alpha\beta, \alpha y + \beta x + xy],$$
$$\lambda[\alpha, x] = [\lambda\alpha, \lambda x], \qquad |[\alpha, x]| = |\alpha| + |x|.$$

It is readily seen that $\mathfrak{X}_1$ is a B-algebra whose unit $e = [1, 0]$ has norm $|e| = 1$ and that the map $x \to [0, x]$ is an isometric isomorphism of $\mathfrak{X}$ into $\mathfrak{X}_1$.

There is another type of algebra which fails to satisfy the requirements for a B-algebra but whose study nevertheless may be subsumed under that of B-algebras. Suppose that $\mathfrak{X}$ is a complex B-space which is also an algebra with unit e over the field of complex numbers. We do not assume that $|e| = 1$ or that $|xy| \leq |x||y|$ but do assume that $|e| \neq 0$ and that the product xy is a continuous function in x for each fixed y and is also continuous in y for each fixed x. Now let $\tau : x \to T_x$ be the map of $\mathfrak{X}$ into the algebra $B(\mathfrak{X})$ of all continuous linear operators in $\mathfrak{X}$ which is defined by the equation $T_x y = xy$. The map τ is clearly an isomorphism of $\mathfrak{X}$ onto a subalgebra $\tau(\mathfrak{X})$ of $B(\mathfrak{X})$. It will be shown that τ is also a homeomorphism. This will show that the

given algebra $\mathfrak{X}$ is algebraically and topologically equivalent to the B-algebra $\tau(\mathfrak{X})$. Since $|x| = |xe| = |T_x e| \leq |e||T_x|$ it follows that the inverse map τ^{-1} is continuous. To see that τ is also continuous it will first be shown that $\tau(\mathfrak{X})$ is closed in $B(\mathfrak{X})$. To do this the following criterion is useful: an element $T \in B(\mathfrak{X})$ is in $\tau(\mathfrak{X})$ if and only if $(Ty)z = T(yz)$. For, if T has this property, let $x = Te$ so that $Ty = T(ey) = (Te)y = xy$ and thus $T = T_x \in \tau(\mathfrak{X})$. Now let $T_n \in \tau(\mathfrak{X})$ and $T_n \to T$; then, since

$$(Ty)z = \lim_n (T_n y)z = \lim_n T_n(yz) = T(yz),$$

it follows that $T \in \tau(\mathfrak{X})$ and thus that $\tau(\mathfrak{X})$ is closed. The mapping τ is therefore a one-to-one linear map of one B-space onto another B-space which has a continuous inverse. The closed graph theorem (II.2.4) shows that τ is a homeomorphism. Thus the algebra $\mathfrak{X}$ is topologically and algebraically equivalent to the B-algebra $\tau(\mathfrak{X})$.

The above discussion shows in particular that every B-algebra $\mathfrak{X}$ is isometrically isomorphic to a subalgebra $\tau(\mathfrak{X})$ with unit I of the algebra $B(\mathfrak{X})$ of all continuous linear operators on the B-space $\mathfrak{X}$. In this connection it is desirable to note that an element $T_x \in \tau(\mathfrak{X})$ has an inverse as an element of $B(\mathfrak{X})$ if and only if x has an inverse in $\mathfrak{X}$ and that when this inverse T_x^{-1} exists, then $T_x^{-1} = T_{x^{-1}}$. Clearly if x^{-1} exists then $T_{x^{-1}}T_x = T_x T_{x^{-1}} = I$. If T_x^{-1} exists in $B(\mathfrak{X})$, then

$$T_x[(T_x^{-1}y)z] = yz, \qquad (T_x^{-1}y)z = T_x^{-1}(yz),$$

and if $a = T_x^{-1}e$, then $az = T_x^{-1}z$ for every $z \in \mathfrak{X}$. Also

$$xa = T_x a = e = T_x^{-1}(T_x e) = T_x^{-1}(ex) = (T_x^{-1}e)x = ax.$$

Thus x^{-1} exists and $T_x^{-1}z = x^{-1}z$.

2 Definition. An element x in a B-algebra $\mathfrak{X}$ is said to be *regular* in case x^{-1} exists in $\mathfrak{X}$. It is *singular* if it is not regular. The *spectrum* $\sigma(x)$ of x consists of those complex numbers λ such that $\lambda e - x$ is singular. The *spectral radius* $|\sigma(x)|$ of x is $\sup_{\lambda \in \sigma(x)} |\lambda|$. The *resolvent set* $\rho(x)$ of x is the set of those complex numbers λ such that $\lambda e - x$ is regular. The *resolvent* of x is the function defined for $\lambda \in \rho(x)$ as $(\lambda e - x)^{-1}$. The element x is a *right (left) topological divisor of zero* if there exists a sequence $\{x_n\}$ in $\mathfrak{X}$ with $|x_n| = 1$, $n = 1, 2, \ldots$, and

$x_n x \to 0$ ($x x_n \to 0$). A *two-sided topological divisor of zero* is an element which is both a right and a left topological divisor of zero.

The preceding discussion shows that $\sigma(x) = \sigma(T_x)$ and $\rho(x) = \rho(T_x)$. It also enables one to state many of the lemmas that follow as corollaries of the corresponding results in operator theory. However, most of these lemmas are quite elementary and we shall, for the sake of completeness, give their proofs here.

3 LEMMA. *The multiplicative group G of regular elements in a B-algebra $\mathfrak{X}$ is open in $\mathfrak{X}$ and the map $x \to x^{-1}$ is a homeomorphism in G.*

PROOF. We first show that G contains the sphere $\{x \mid |e-x| < 1\}$. If $|e-x| < 1$ the series $y = \sum_{n=0}^{\infty} (e-x)^n$ converges (here the symbol a^0 is defined as e). Now

$$yx = xy = y-(e-x)y = \sum_{n=0}^{\infty}(e-x)^n - \sum_{n=1}^{\infty}(e-x)^n = e.$$

Thus $y = x^{-1}$ and

$$|x^{-1}-e| = |\sum_{n=1}^{\infty}(e-x)^n| \leq \frac{|e-x|}{1-|e-x|}.$$

Thus G contains a neighborhood of e and x^{-1} is a continuous function of x at $x = e$. Now let $x \in G$ and let $|y-x| < |x^{-1}|^{-1}$. Then

$$|x^{-1}y-e| = |x^{-1}(y-x)| \leq |x^{-1}||y-x| < 1,$$

and so, by what has just been proved, $x^{-1}y \in G$ and therefore $y \in G$. If $y_n \to y$ then $y_n y^{-1} \to e$ and hence $y y_n^{-1} = (y_n y^{-1})^{-1} \to e$. Thus $y_n^{-1} \to y^{-1}$. Q.E.D.

4 LEMMA. *Every boundary point of the group of regular elements in a B-algebra is a two-sided topological divisor of zero.*

PROOF. Let $x \notin G$, $x_n \in G$, $x_n \to x$. If $\{|x_n^{-1}|\}$ is bounded then $x_n^{-1}x - e = x_n^{-1}(x-x_n) \to 0$ and thus, by Lemma 3, $x_n^{-1}x \in G$ for large n. This implies that $x \in G$. Hence $\{|x_n^{-1}|\}$ is not bounded and we may assume that $|x_n^{-1}| \to \infty$. Let $y_n = x_n^{-1}/|x_n^{-1}|$ so that $|y_n| = 1$, and

$$y_n x = y_n(x-x_n) + \frac{e}{|x_n^{-1}|} \to 0,$$

$$x y_n = (x-x_n)y_n + \frac{e}{|x_n^{-1}|} \to 0,$$

which shows that x is a two-sided divisor of zero. Q.E.D.

5 Lemma. *The spectrum $\sigma(x)$ of an element x in a B-algebra is a non-void bounded closed set. The resolvent $x(\lambda) = (\lambda e - x)^{-1}$ of x is an analytic function vanishing at ∞ and satisfying the resolvent equation*

$$x(\lambda) - x(\mu) = (\mu - \lambda)x(\lambda)x(\mu), \qquad \lambda, \mu \in \rho(x).$$

Proof. That $\rho(x)$ is open and hence $\sigma(x)$ closed follows from Lemma 3. Also this lemma shows that for large λ, $(\lambda e - x) = \lambda(e - x/\lambda)$ is in G (since $e - x/\lambda$ is near e) which proves that $\sigma(x)$ is bounded. As $|\lambda| \to \infty$, we have $e - x/\lambda \to e$ and so, by Lemma 3, $x(\lambda) = \lambda^{-1}(e - x/\lambda)^{-1} \to 0$, as $|\lambda| \to \infty$. For $\mu, \lambda \in \rho(x)$ the elements $x(\mu)$, $x(\lambda)$ commute and

$$(\lambda e - x)x(\lambda)x(\mu) = x(\mu),$$
$$(\mu e - x)x(\lambda)x(\mu) = x(\lambda),$$
$$x(\mu) - x(\lambda) = (\lambda - \mu)x(\lambda)x(\mu),$$

and thus

$$\frac{x(\mu) - x(\lambda)}{\mu - \lambda} = -x(\lambda)x(\mu).$$

Now Lemma 3 shows that $x(\lambda)$ is continuous in λ and the above equation therefore proves that $x'(\lambda) = -x(\lambda)^2$ and shows that $x(\lambda)$ is analytic on $\rho(x)$.

Finally, if $\sigma(x)$ is void, let x^* be a continuous linear functional on $\mathfrak{X}$ so that $x^*x(\lambda)$ is an entire function which vanishes at ∞ and hence vanishes identically. Since x^* is an arbitrary point in $\mathfrak{X}^*$ this means (II.3.15) that $0 = x(\lambda) = x(\lambda)(e\lambda - x) = e$ which contradicts the assumption that $|e| = 1$. Q.E.D.

6 Theorem. *If a B-algebra has no non-zero two-sided topological divisors of zero, it is isometrically isomorphic to the field of complex numbers.*

Proof. Let $x \in \mathfrak{X}$. Then, by Lemma 5, the spectrum $\sigma(x)$ is non-void and bounded so that there is a point λ in its boundary. Thus, by Lemma 4, $\lambda e - x$ is a two-sided topological divisor of zero and therefore $x = \lambda e$. Since $|e| = 1$, $|x| = |\lambda|$. Q.E.D.

➜ 7 Corollary. *If a B-algebra is a division ring, then it is isometrically isomorphic to the field of complex numbers.*

8 LEMMA. *The spectral radius of an element x in a B-space $\mathfrak{X}$ has the properties*

$$|\sigma(x)| = \lim_n |x^n|^{1/n} \leqq |x|.$$

PROOF. For $|\lambda| > |x|$ the series $\sum_0^\infty x^n/\lambda^{n+1}$ converges and, since

$$(\lambda e - x)\sum_0^\infty \frac{x^n}{\lambda^{n+1}} = \sum_0^\infty \left(\frac{x^n}{\lambda^n} - \frac{x^{n+1}}{\lambda^{n+1}}\right) = e,$$

it represents the resolvent $x(\lambda)$ for $|\lambda| > |x|$. Thus $|\sigma(x)| \leqq |x|$. By Lemma 5, $x(\lambda)$ is analytic on $\rho(x)$ and hence, for $x^* \in \mathfrak{X}^*$, the scalar valued analytic function $x^*x(\lambda)$ has its singularities all in the disc $|\lambda| \leqq |\sigma(x)|$. Thus the series $x^*x(\lambda) = \sum_{n=0}^\infty x^*x^n/\lambda^{n+1}$ converges for $|\lambda| > |\sigma(x)|$ and for such λ therefore

$$\sup_n \left|\frac{x^* x^n}{\lambda^{n+1}}\right| < \infty.$$

Since x^* is an arbitrary linear functional on $\mathfrak{X}$ the principle of uniform boundedness (II.3.20) shows that

$$\left|\frac{x^n}{\lambda^{n+1}}\right| \leqq M_\lambda < \infty, \qquad n = 1, 2, \ldots,$$

and hence that

$$\limsup_n |x^n|^{1/n} \leqq |\lambda|.$$

Since λ is an arbitrary number with $|\lambda| > |\sigma(x)|$ it follows that

$$\limsup_n |x^n|^{1/n} \leqq |\sigma(x)|.$$

To complete the proof we observe that, since $\lambda e - x$ is a factor of $\lambda^n e - x^n$, $\lambda^n e - x^n$ is singular if $\lambda e - x$ is singular. Thus if $\lambda \in \sigma(x)$ then $\lambda^n \in \sigma(x^n)$ and hence $|\lambda^n| \leqq |x^n|$ which shows that $|\lambda| \leqq \liminf_n |x^n|^{1/n}$ and therefore that

$$|\sigma(x)| \leqq \liminf_{n\to\infty} |x^n|^{1/n}. \qquad \text{Q.E.D.}$$

A given element x in a *B*-algebra $\mathfrak{X}$ may also be an element of a *B*-subalgebra (i.e., a closed subalgebra) $\mathfrak{X}_0$ of $\mathfrak{X}$. As such it has a spectrum $\sigma_0(x)$ which may include properly or be properly included in $\sigma(x)$. However, the preceding lemma shows that the spectral radii

$|\sigma_0(x)|$ and $|\sigma(x)|$ are equal. An element x in a B-subalgebra of the form $\mathfrak{X}_0 = e_0\mathfrak{X}e_0$ where e_0 is an idempotent with $0 \neq e_0 \neq e$ clearly has $\sigma_0(x) \subseteq \sigma(x)$. The following lemma shows that the opposite inclusion holds in case $\mathfrak{X}_0$ has the same unit as $\mathfrak{X}$.

9 Lemma. *Let x be an element of a B-subalgebra $\mathfrak{X}_0$ of $\mathfrak{X}$ whose unit is the same as that of $\mathfrak{X}$. Then $\sigma(x) \subseteq \sigma_0(x)$ while the boundary of $\sigma_0(x)$ is contained in the boundary of $\sigma(x)$.*

Proof. Since the unit e in $\mathfrak{X}$ is also in $\mathfrak{X}_0$, it follows that a regular element in $\mathfrak{X}_0$ is regular in $\mathfrak{X}$. Thus $\rho_0(x) \subseteq \rho(x)$ or $\sigma(x) \subseteq \sigma_0(x)$. If $\lambda \in \text{bdy}\,\sigma_0(x)$, the boundary of $\sigma_0(x)$, then $\lambda e - x$ is on the boundary of the group of regular elements of $\mathfrak{X}_0$. Thus, by Lemma 4, $\lambda e - x$ is a two-sided topological divisor of zero in $\mathfrak{X}_0$ and hence in $\mathfrak{X}$. Therefore $\lambda \in \sigma(x)$, which, since $\rho_0(x) \subseteq \rho(x)$, proves that

$$\overline{\rho_0(x)} \cap \sigma_0(x) = \text{bdy}\,\sigma_0(x) \subseteq \overline{\rho(x)} \cap \sigma(x) = \text{bdy}\,\sigma(x). \quad \text{Q.E.D.}$$

10 Corollary. *If in addition to the assumptions of Lemma 9, $\sigma_0(x)$ is nowhere dense, then $\sigma(x) = \sigma_0(x)$. Also if $\rho(x)$ is connected then $\sigma_0(x) = \sigma(x)$.*

Proof. If $\sigma_0(x)$ is nowhere dense then, since it is closed, $\sigma_0(x) = \text{bdy}\,\sigma_0(x) \subseteq \text{bdy}\,\sigma(x) \subseteq \sigma(x) \subseteq \sigma_0(x)$. If $\rho(x)$ is connected and if there is a point $\lambda \in \sigma_0(x)\rho(x)$ then λ may be connected with ∞ by a continuous path contained entirely in $\rho(x)$. In this case there is a boundary point of $\sigma_0(x)$ which is in $\rho(x)$ and this is a contradiction to the theorem. Thus $\sigma_0(x)\rho(x)$ is void and $\sigma_0(x) \subseteq \sigma(x) \subseteq \sigma_0(x)$. Q.E.D.

11 Corollary. *If $\sigma_0(x)$ is real, then $\sigma_0(x) = \sigma_1(x)$ where $\sigma_1(x)$ is the spectrum of x as an element of any B-subalgebra $\mathfrak{X}_1$ of $\mathfrak{X}$ which contains the unit of $\mathfrak{X}$.*

Proof. If $\sigma_0(x)$ is real, then so is its subset $\sigma(x)$. By Lemma 5 the spectrum $\sigma(x)$ is bounded. Thus $\rho(x)$ is connected and Corollary 10 shows that $\sigma_0(x) = \sigma(x) = \sigma_1(x)$. Q.E.D.

A *right* (*left*) *ideal* in $\mathfrak{X}$ is a non-void proper linear manifold $\mathfrak{J}$ in $\mathfrak{X}$ for which $\mathfrak{J}\mathfrak{X} = \mathfrak{J}$ ($\mathfrak{X}\mathfrak{J} = \mathfrak{J}$). A *two-sided ideal* is one which is both a right and left ideal. The sets $\mathfrak{X}$ and $\{0\}$ are called *trivial ideals*. Since an ideal $\mathfrak{J}$ is a proper subset of $\mathfrak{X}$ it cannot contain any regular element and hence $\mathfrak{J}$ is contained in the complement G' of the group of

regular elements. Since G is open (Lemma 3) the closure $\bar{\mathfrak{J}} \subseteq G'$ and hence $\bar{\mathfrak{J}} \neq \mathfrak{X}$. The continuity of the algebraic operations shows that $\bar{\mathfrak{J}}$ satisfies the other requirements for an ideal. Thus the closure of a right, left, or two-sided ideal is also a right, left, or two-sided ideal. This shows that a maximal ideal is closed. Let $\mathfrak{J}$ be a right ideal and order by inclusion the family of all right ideals which contain $\mathfrak{J}$. An application of Zorn's lemma shows that this family contains a maximal element. Thus any right (and similarly for left and two-sided) ideal is contained in a maximal right ideal. In particular if x is a singular element the ideal $x\mathfrak{X}$ is contained in a maximal ideal. Thus an element is contained in a maximal right (left, two-sided) ideal if and only if it has no inverse. The above facts about ideals are summarized in the following lemma.

12 LEMMA. *The following statements concerning ideals apply to right, left, or two-sided ideals.*

(a) *An ideal contains no regular element.*

(b) *The closure of an ideal is an ideal.*

(c) *A maximal ideal is closed.*

(d) *Every ideal is contained in a maximal ideal.*

(e) *An element x is contained in a maximal right (left) ideal if and only if it has no right (left) inverse.*

We recall that if $\mathfrak{J}$ is a two-sided ideal in the algebra $\mathfrak{X}$, the cosets $x+\mathfrak{J}$, $x \in \mathfrak{X}$ form an algebra under the following definitions

$$(x+\mathfrak{J})+(y+\mathfrak{J}) = (x+y)+\mathfrak{J}$$
$$\alpha(x+\mathfrak{J}) = (\alpha x)+\mathfrak{J}, \qquad (x+\mathfrak{J})(y+\mathfrak{J}) = (xy)+\mathfrak{J}.$$

This algebra is the *quotient algebra* of $\mathfrak{X}$ by $\mathfrak{J}$ and is denoted by $\mathfrak{X}/\mathfrak{J}$. The *norm* in the quotient $\mathfrak{X}/\mathfrak{J}$ of a B-algebra $\mathfrak{X}$ by a two-sided ideal $\mathfrak{J}$ is given by

$$|x+\mathfrak{J}| = \inf_{y \in \mathfrak{J}} |x+y|.$$

13 LEMMA. *If $\mathfrak{J}$ is a closed two-sided ideal in the B-algebra $\mathfrak{X}$ then $\mathfrak{X}/\mathfrak{J}$ is a B-algebra.*

PROOF. For brevity let us denote the class $x+\mathfrak{J}$ by $\bar{x}$. It is clear that $\mathfrak{X}/\mathfrak{J}$ is an algebra with unit $\bar{e}$ so that only the required properties of the norm will be explicitly proved. If $|\bar{x}| = 0$ then there is a sequence

$x_n \in \mathfrak{J}$ with $|x+x_n| \to 0$ and thus, since $\mathfrak{J}$ is closed, $x \in \mathfrak{J}$. Hence $\bar{x} = \bar{0}$ is the zero element of $\mathfrak{X}/\mathfrak{J}$. Now let z, u, v vary independently over $\mathfrak{J}$. Then

$$\begin{aligned} |\bar{x}\bar{y}| = |\overline{xy}| &= \inf_z |xy+z| \\ &\leq \inf_{u,v} |(x+u)(y+v)| \\ &\leq |\bar{x}||\bar{y}|. \end{aligned}$$

Thus $|\bar{e}| = |\bar{e}^2| \leq |\bar{e}|^2$ and, since $|\bar{e}| \neq 0$, we have $|\bar{e}| \geqq 1$. On the other hand $|\bar{e}| = \inf |e+z| \leq |e| = 1$. Hence $|\bar{e}| = 1$. The triangle inequality is verified by a similar computation as follows:

$$\begin{aligned} |\bar{x}+\bar{y}| = |\overline{x+y}| &= \inf_z |x+y+z| \\ &= \inf_{u,v} |x+u+y+v| \leq |\bar{x}|+|\bar{y}|. \end{aligned}$$

It is clear that $|\alpha\bar{x}| = |\alpha||\bar{x}|$. Only completeness of $\mathfrak{X}/\mathfrak{J}$ remains to be proved. Let $\{\bar{x}_n\}$ be a Cauchy sequence in $\mathfrak{X}/\mathfrak{J}$ and $x_n \in \bar{x}_n$, $n = 1, 2, \ldots$. Choose a subsequence $\{\bar{x}_n'\}$ such that $\sum_{n=1}^{\infty} |\bar{x}_n' - \bar{x}_{n+1}'| < \infty$. Fix $z_1 \in \mathfrak{J}$ and inductively choose $z_{i+1} \in \mathfrak{J}$ such that

$$|x_{i+1}' + z_{i+1} - (x_i' + z_i)| \leqq 2|\bar{x}_{i+1}' - \bar{x}_i'|.$$

The sequence $y_n = x_n' + z_n$ is then a Cauchy sequence, for

$$\begin{aligned} |y_{n+p} - y_n| &= \Big|\sum_{k=0}^{p-1} y_{n+k+1} - y_{n+k}\Big| \\ &\leq \sum_{k=0}^{p-1} |y_{n+k+1} - y_{n+k}| \\ &\leq 2\sum_{k=0}^{p-1} |\bar{x}_{n+k+1} - \bar{x}_{n+k}|. \end{aligned}$$

Let $x = \lim y_n$, then

$$|\bar{x}_n' - \bar{x}| = |\bar{y}_n - \bar{x}| \leqq |y_n - x| \to 0.$$

Thus the original Cauchy sequence $\{\bar{x}_n\}$ has a convergent subsequence and so it must itself converge. Q.E.D.

2. Commutative B-Algebras

In case $\mathfrak{X}$ is a commutative B-algebra every ideal $\mathfrak{I}$ is two-sided and the quotient algebra $\mathfrak{X}/\mathfrak{I}$ is again a commutative algebra. It will be a B-algebra if $\mathfrak{I}$ is closed (1.13). It is readily seen that every ideal $\mathfrak{J}$ in $\mathfrak{X}$ which contains $\mathfrak{I}$ properly determines an ideal $\tilde{\mathfrak{J}}$ in $\mathfrak{X}/\mathfrak{I}$ defined as the set of all $\tilde{x} = x+\mathfrak{I}$ with x in $\mathfrak{J}$. Conversely every ideal in $\mathfrak{X}/\mathfrak{I}$ is of this form.

1 THEOREM. *If $\mathfrak{I}$ is a closed ideal in the commutative B-algebra $\mathfrak{X}$ then the quotient algebra $\mathfrak{X}/\mathfrak{I}$ is isometrically isomorphic to the field of complex numbers if and only if $\mathfrak{I}$ is maximal.*

PROOF. If $\mathfrak{I}$ is not maximal it is properly contained in an ideal and so $\mathfrak{X}/\mathfrak{I}$ has non-trivial ideals and is thus not a field. If $\mathfrak{I}$ is maximal then $\mathfrak{X}/\mathfrak{I}$ contains no non-trivial ideals and hence is a field. The desired conclusion follows from Lemma 1.13 and Theorem 1.6. Q.E.D.

Let $\mathscr{M}$ be the set of all maximal ideals in the commutative B-algebra $\mathfrak{X}$. It is seen from Theorem 1 that for each $\mathfrak{M} \in \mathscr{M}$ and each x in $\mathfrak{X}$ there is a complex number $x(\mathfrak{M})$ such that $x+\mathfrak{M} = x(\mathfrak{M})e+\mathfrak{M}$. This mapping $x \to x(\mathfrak{M})$ of $\mathfrak{X}$ into the field Φ of complex numbers is clearly a homomorphism. Since $|x(\mathfrak{M})| \leq |x|$ this homomorphism is continuous.

2 LEMMA. *Let μ be a non-zero homomorphism of the commutative B-algebra $\mathfrak{X}$ onto the field of complex numbers and let*

$$\mathfrak{M}_\mu = \{x|\mu(x) = 0\}$$

be its kernel. Then $\mathfrak{M}_\mu$ is a maximal ideal in $\mathfrak{X}$ such that $x(\mathfrak{M}_\mu) = \mu(x)$. This correspondence $\mu \to \mathfrak{M}_\mu$ is a one-to-one correspondence between the set of all non-zero homomorphisms and the set of maximal ideals.

PROOF. Since $\mu(x)e-x$ is in $\mathfrak{M}_\mu$ the element $\mu(x)e-x$ is singular, i.e., $\mu(x)$ is in $\sigma(x)$, and Lemma 1.8 shows that $|\mu(x)| \leq |x|$. Thus μ is continuous, $\mathfrak{M}_\mu$ is closed and $\mathfrak{X}/\mathfrak{M}_\mu$ is a B-algebra (1.13). Since μ is linear it is constant on all residue classes $x+\mathfrak{M}_\mu$ and therefore defines an isomorphism of $\mathfrak{X}/\mathfrak{M}_\mu$ into the field Φ of complex numbers. Since $\mu(\alpha e) = \alpha$, this isomorphism must map $\mathfrak{X}/\mathfrak{M}_\mu$ onto all of Φ. This shows that $\mathfrak{X}/\mathfrak{M}_\mu$ is a field and therefore that $\mathfrak{M}_\mu$ is a maximal ideal. The

equation $\mu(x) = x(\mathfrak{M}_\mu)$ follows from the definition of $x(\mathfrak{M}_\mu)$. This correspondence $\mu \to \mathfrak{M}_\mu$ is one-to-one, for if x is in $\mathfrak{M}_1$ and x is not in $\mathfrak{M}_2$ then $x(\mathfrak{M}_1) = 0 \neq x(\mathfrak{M}_2)$. Q.E.D.

3 COROLLARY. *Every homomorphism of a commutative B-algebra into the complex number system is continuous.*

4 LEMMA. *Let $\mathscr{M}$ be the set of maximal ideals in the commutative B-algebra $\mathfrak{X}$. Then $x(\mathscr{M}) = \sigma(x)$ and*

$$\sup_{\mathfrak{M} \in \mathscr{M}} |x(\mathfrak{M})| = \lim_n |x^n|^{1/n}.$$

PROOF. Since the element $x(\mathfrak{M})e - x$ belongs to the maximal ideal $\mathfrak{M}$, it is singular (1.12e) and thus $x(\mathfrak{M})$ is in $\sigma(x)$. Conversely if λ is in $\sigma(x)$ then the singular element $\lambda e - x$ belongs to a maximal ideal $\mathfrak{M}$ (1.12e) and thus $\lambda = x(\mathfrak{M})$. The final conclusion follows from Lemma 1.8. Q.E.D.

5 DEFINITION. A *topological nilpotent* in a B-algebra $\mathfrak{X}$ is an element x such that $|x^n|^{1/n} \to 0$. The *radical* of $\mathfrak{X}$ is the set $\mathfrak{R}$ of all topological nilpotents in $\mathfrak{X}$. The algebra $\mathfrak{X}$ is *semi-simple* if its radical $\mathfrak{R} = \{0\}$.

6 LEMMA. *The radical in a commutative B-algebra is the intersection of all its maximal ideals.*

PROOF. Noting that $x(\mathfrak{M}) = 0$ if and only if $x \in \mathfrak{M}$, this lemma is a corollary of Lemma 4. Q.E.D.

7 DEFINITION. The *structure space* of a commutative B-algebra $\mathfrak{X}$ is the set $\mathscr{M}$ of all maximal ideals in $\mathfrak{X}$ with the topology determined by all neighborhoods of the form

$$N(\mathfrak{M}_0; \varepsilon, A) = \{\mathfrak{M} | \mathfrak{M} \in \mathscr{M}, |x(\mathfrak{M}) - x(\mathfrak{M}_0)| < \varepsilon, x \in A\},$$

where A is an arbitrary finite set of elements of $\mathfrak{X}$ and $\varepsilon > 0$.

It is clear that the family of all such neighborhoods satisfy the requirements listed in Lemma I.4.7 and thus define a topology in $\mathscr{M}$.

8 LEMMA. *The structure space $\mathscr{M}$ of a commutative B-algebra is a compact Hausdorff space and for each x in $\mathfrak{X}$ the function $x(\mathfrak{M})$, $\mathfrak{M}$ in $\mathscr{M}$, is continuous.*

Proof. The continuity of $x(\cdot)$ at each point $\mathfrak{M}_0$ of $\mathscr{M}$ follows from Definition 7. Thus $x(\cdot)$ is continuous on $\mathscr{M}$ (I.4.16a). Now let Q_x be the closed disc $\{\lambda \,|\, |\lambda| \leqq |x|\}$ in the complex plane and let $Q = P_{x \in \mathfrak{X}} Q_x$ be the Cartesian product of all such discs. Since $|x(\mathfrak{M})| \leqq |x|$ it is seen that $x(\mathfrak{M})$ is in Q_x, $x \in \mathfrak{X}$, and thus each $\mathfrak{M}$ in $\mathscr{M}$ determines a point q in Q with $q(x) = x(\mathfrak{M})$. Two different ideals $\mathfrak{M}_1, \mathfrak{M}_2$ determine different points in Q, since if $x \in \mathfrak{M}_1$ and $x \notin \mathfrak{M}_2$, then $x(\mathfrak{M}_1) = 0 \neq x(\mathfrak{M}_2)$. If Q is endowed with its product topology, it is, by a theorem of Tychonoff (I.8.5), compact and $\mathscr{M}$ is topologically equivalent to a subset of Q. To prove the lemma it will be sufficient therefore to prove that $\mathscr{M}$, regarded as a subset of Q, is closed (I.5.7). Let $\lambda \in \bar{\mathscr{M}}$, $\varepsilon > 0$, $A = \{x, y, x+y\}$ where x, y are arbitrary elements of $\mathfrak{X}$. Then the neighborhood $N(\lambda; \varepsilon, A)$ intersects $\mathscr{M}$ and so there is an $\mathfrak{M} \in \mathscr{M}$ with

$$|\lambda(x) - x(\mathfrak{M})| < \varepsilon, \qquad |\lambda(y) - y(\mathfrak{M})| < \varepsilon,$$
$$|\lambda(x+y) - (x+y)(\mathfrak{M})| < \varepsilon.$$

Since $(x+y)(\mathfrak{M}) = x(\mathfrak{M}) + y(\mathfrak{M})$ and $\varepsilon > 0$ is arbitrary, we see that $\lambda(x+y) = \lambda(x) + \lambda(y)$. In a similar fashion it may be shown that $\lambda(e) = 1$, $\lambda(\alpha x) = \alpha\lambda(x)$, and $\lambda(xy) = \lambda(x)\lambda(y)$. Thus λ is a non-zero homomorphism and, by Lemma 2, there is a point $\mathfrak{M}_\lambda \in \mathscr{M}$ with $x(\mathfrak{M}_\lambda) = \lambda(x)$, $x \in \mathfrak{X}$. Thus $\lambda \in \mathscr{M}$ and $\mathscr{M}$ is closed. Q.E.D.

→ 9 **Theorem.** *Let $\mathscr{M}$ be the structure space of the commutative B-algebra $\mathfrak{X}$ and let $C(\mathscr{M})$ be the B-algebra of all complex continuous functions on $\mathscr{M}$. Then the mapping $x \to x(\cdot)$ is a continuous homomorphism of $\mathfrak{X}$ into $C(\mathscr{M})$ with* $\sup_{\mathfrak{M} \in \mathscr{M}} |x(\mathfrak{M})| \leqq |x|$. *It is an isomorphism if and only if $\mathfrak{X}$ is semi-simple.*

Proof. The fact that the map $x \to x(\cdot)$ is a homomorphism follows from the definition of $x(\mathfrak{M})$. It was proved in Lemma 8 that $x(\cdot) \in C(\mathscr{M})$. The inequality $|x(\mathfrak{M})| \leqq |x|$ follows from the definition of the norm in a quotient algebra (it also follows from Lemma 4). If $\mathfrak{X}$ is semi-simple it follows from Lemma 6 that $x(\mathfrak{M}) = 0$ for $\mathfrak{M} \in \mathscr{M}$ only if $x = 0$ and thus the map $x \to x(\cdot)$ is an isomorphism. Conversely, if $x \to x(\cdot)$ is an isomorphism, then Lemma 6 shows that $\mathfrak{X}$ is semi-simple. Q.E.D.

10 **Definition.** The *B*-algebra $\mathfrak{X}$ is said to be *generated by a set*

$Y \subseteqq \mathfrak{X}$ if $\mathfrak{X}$ is the smallest closed subalgebra of itself which contains Y and the unit e in $\mathfrak{X}$.

11 THEOREM. *The structure space of a commutative B-algebra generated by a set Y is homeomorphic to a closed subset of the Cartesian product $P\sigma(y)$ where y varies over Y.*

PROOF. By Lemma 4, $y(\mathscr{M}) = \sigma(y)$ and thus the correspondence $\mathfrak{M} \to y(\mathfrak{M})$ defines a map of $\mathscr{M}$ into $P\sigma(y)$. This is a continuous map since the functions $y(\cdot)$ are all continuous on $\mathscr{M}$. Suppose that $y(\mathfrak{M}_1) = y(\mathfrak{M}_2)$ for all y in Y. Then, since Y generates $\mathfrak{X}$, it follows that $x(\mathfrak{M}_1) = x(\mathfrak{M}_2)$ for all x in $\mathfrak{X}$. Thus $\mathfrak{M}_1 = \mathfrak{M}_2$ for, if $x \in \mathfrak{M}_1$ and $x \notin \mathfrak{M}_2$, then $x(\mathfrak{M}_1) = 0$, $x(\mathfrak{M}_2) \neq 0$. Thus the mapping $\mathfrak{M} \to y(\mathfrak{M})$ is a continuous one-to-one map of the compact (Lemma 8) space $\mathscr{M}$ into the Hausdorff space $P\sigma(y)$ and is hence (I.5.8) a homeomorphism. Q.E.D.

12 COROLLARY. *The structure space of a commutative B-algebra generated by a set Y has, as a base for its open sets, neighborhoods of the form*

$$N(\mathfrak{M}_0; \varepsilon, A) = \{\mathfrak{M} | \mathfrak{M} \in \mathscr{M}, |y(\mathfrak{M}) - y(\mathfrak{M}_0)| < \varepsilon, y \in A\}$$

where A is a finite set in Y.

13 COROLLARY. *The structure space of a B-algebra generated by one element is homeomorphic to the spectrum of the generating element.*

14 THEOREM. *A subset of the complex plane is homeomorphic to the structure space of a B-algebra with one generator if and only if it is compact and has a connected complement.*

PROOF. Let $\mathscr{M}$ be the structure space of the B-algebra $\mathfrak{X}$ generated by the element z. By Corollary 13 the compact set $\sigma(z)$ is homeomorphic to $\mathscr{M}$. Suppose that the complement of $\sigma(z)$ is not connected and let G be a bounded component of the complement of $\sigma(z)$. For each x in $\mathfrak{X}$ there is a sequence $\{P_n\}$ of polynomials with $|P_n(z) - x| \to 0$ and thus, if $\lambda = z(\mathfrak{M})$,

$$|P_n(\lambda) - x(\mathfrak{M})| = |(P_n(z) - x)(\mathfrak{M})| \leqq |P_n(z) - x| \to 0$$

uniformly for $\mathfrak{M}$ in $\mathscr{M}$. Since $z(\mathscr{M}) = \sigma(z)$ the sequence $\{P_n(\lambda)\}$ converges uniformly on $\sigma(z)$. It follows from the theory of functions of a

complex variable that $\{P_n(\lambda)\}$ also converges uniformly on G. For each λ in G and each x in $\mathfrak{X}$ define $x(\lambda) = \lim P_n(\lambda)$ where $\{P_n\}$ is a sequence of polynomials with $|P_n(z)-x| \to 0$. The number $x(\lambda)$ is clearly independent of the particular sequence $\{P_n\}$ used to define it. For a fixed $\lambda_0 \in G$ the map $x \to x(\lambda_0)$ is a homomorphism of $\mathfrak{X}$ into the field of complex numbers. Thus by Lemma 2 there is a maximal ideal $\mathfrak{M}_0$ with $x(\mathfrak{M}_0) = x(\lambda_0)$ for every x in $\mathfrak{X}$. In particular $\lambda_0 = z(\lambda_0) = z(\mathfrak{M}_0)$. This contradicts the facts that $z(\mathscr{M}) = \sigma(z)$ (Lemma 4) and that λ_0 is in the complement of $\sigma(z)$. Conversely, let σ be a compact set in the complex plane whose complement is connected. Let $C(\sigma)$ be the B-algebra of all continuous complex functions defined on σ with norm

$$|f| = \sup_{\lambda \in \sigma} |f(\lambda)|.$$

Let z be the element in $C(\sigma)$ with $z(\lambda) = \lambda$, $\lambda \in \sigma$, and let $\mathfrak{X}_0$ be the B-subalgebra of $C(\sigma)$ generated by z and the unit element of $C(\sigma)$. Let $\sigma_0(z)$ be the spectrum of z as an element of $\mathfrak{X}_0$ and $\sigma(z)$ its spectrum as an element of $C(\sigma)$. Clearly $\sigma(z) = \sigma$ and therefore the complement $\rho(z)$ of $\sigma(z)$ is connected. By Corollary 1.10, $\sigma_0(z) = \sigma(z) = \sigma$. Q.E.D.

We shall conclude our discussion of the structure space of a commutative B-algebra by applying the foregoing theory to obtain the existence of the Stone-Čech compactification of a completely regular topological space.

15 Definition. A topological space Λ is *completely regular* if sets consisting of single points are closed and if for any point $\lambda_0 \in \Lambda$ and any closed set $\Lambda_0 \subset \Lambda$ with $\lambda_0 \notin \Lambda_0$ there is a continuous function f defined on Λ with $0 \leqq f(\lambda) \leqq 1$, $\lambda \in \Lambda$; $f(\lambda) = 0$, $\lambda \in \Lambda_0$; and $f(\lambda_0) = 1$.

16 Theorem. (*Stone-Čech compactification theorem*) *Every completely regular topological space Λ is homeomorphic with a dense subset $\mathscr{M}_\Lambda$ of a compact Hausdorff space $\mathscr{M}$ such that every bounded continuous complex function on $\mathscr{M}_\Lambda$ has a unique continuous extension to $\mathscr{M}$.*

Proof. Let $C(\Lambda)$ be the B-algebra of all bounded continuous complex functions on Λ and let $\mathscr{M}$ be the structure space of $C(\Lambda)$. The space $\mathscr{M}$ is compact (Lemma 8). The map $x \to x(\lambda)$ of $C(\Lambda)$ into the complex number system is a homomorphism and thus (Lemma 2) there is a maximal ideal $\mathfrak{M}_\lambda$ with $x(\mathfrak{M}_\lambda) = x(\lambda)$. Since Λ is completely

regular, the map $\lambda \to \mathfrak{M}_\lambda$ of Λ into $\mathscr{M}$ is one-to-one. Let $\mathscr{M}_\Lambda$ be the set in $\mathscr{M}$ of all $\mathfrak{M}_\lambda$ with $\lambda \in \Lambda$. To see that $\mathscr{M}_\Lambda$ is dense in $\mathscr{M}$ suppose the contrary and let

$$\{\mathfrak{M}||x_i(\mathfrak{M})-x_i(\mathfrak{M}_0)| < \varepsilon, \qquad i = 1, \ldots, n\}$$

be a neighborhood in $\mathscr{M}$ which does not intersect $\mathscr{M}_\Lambda$. Then, if $y_i = x_i - x_i(\mathfrak{M}_0)e$, there is for each $\lambda \in \Lambda$ an integer $i \leqq n$ with $|y_i(\lambda)| \geqq \varepsilon$. If

$$y(\lambda) = \sum_{i=1}^{n} y_i(\lambda)\overline{y_i(\lambda)}$$

then $y(\lambda) \geqq \varepsilon^2$ for all $\lambda \in \Lambda$ and thus y^{-1} exists in $C(\Lambda)$. Thus y is not contained in any maximal ideal (Lemma 1.12) but $y(\mathfrak{M}_0) = 0$ which means that $y \in \mathfrak{M}_0$, a contradiction proving that $\mathscr{M}_\Lambda$ is dense in $\mathscr{M}$. Now $x(\mathfrak{M})$ is a continuous function of $\mathfrak{M}$ (Lemma 8) and is an extension to $\mathscr{M}$ of $x(\mathfrak{M}_\lambda) = x(\lambda)$. Thus, to complete the proof of the theorem, it will suffice to show that the one-to-one map $\delta : \lambda \to \mathfrak{M}_\lambda$ of Λ onto $\mathscr{M}_\Lambda$ is a homeomorphism. The neighborhood

$$(\alpha) \qquad \{\mathfrak{M}_\lambda|\mathfrak{M}_\lambda \in \mathscr{M}_\Lambda, |x_i(\mathfrak{M}_\lambda)-x_i(\mathfrak{M}_{\lambda_0})| < \varepsilon, \qquad i = 1, \ldots, n\}$$

is mapped by δ^{-1} onto the set

$$(\beta) \qquad \{\lambda|\lambda \in \Lambda, |x_i(\lambda)-x_i(\lambda_0)| < \varepsilon, \qquad i = 1, \ldots, n\}$$

which, since x_i is continuous, is open. Thus δ is continuous. To see that δ^{-1} is continuous, i.e., to see that δ maps open sets onto open sets note first that open sets of the form (β) are mapped into sets of the form (α) which are open in $\mathscr{M}_\Lambda$. Now, the complete regularity of Λ enables us to see that every open set in Λ is a union of sets of the form (β), for let G be a neighborhood of λ_0 and let $f \in C(\Lambda)$ be such that $0 \leqq f(\lambda) \leqq 1$; $f(\lambda_0) = 1$; $f(\lambda) = 0$, $\lambda \notin G$. Then the set

$$\{\lambda|\lambda \in \Lambda, |f(\lambda)-f(\lambda_0)| < 1/2\}$$

is a neighborhood of λ_0 which is contained in G. Thus every open set in Λ is a union of sets of type (β) and δ therefore maps open sets onto open sets and is a homeomorphism. Q.E.D.

17 COROLLARY. *If Λ is a compact Hausdorff space then it is homeomorphic with the structure space of $C(\Lambda)$.*

18 COROLLARY. *If Λ_1 and Λ_2 are compact Hausdorff spaces such*

that the algebras $C(\Lambda_1)$ and $C(\Lambda_2)$ are equivalent, then Λ_1 and Λ_2 are homeomorphic.

To see the remarkable nature of Theorem 16, let Λ be the half open interval $0 < \lambda \leqq 1$ of reals. The space Λ is a dense subset of the compact space $0 \leqq \lambda \leqq 1$. However, in the Stone-Čech compactification the bounded continuous function $\sin(1/\lambda)$ has a continuous extension which is not the case in the compactification obtained by adding one point. In fact, no effective construction of the Stone-Čech compactification has ever been given.

3. Commutative B^*-Algebras

We have already observed that the B-spaces of all bounded functions on a set, all bounded continuous functions on a topological space, and all almost periodic functions, if taken with the natural notion of multiplication, are algebras with an involution. The involution in each case is defined by the equation $f^*(s) = \overline{f(s)}$. These algebras are also B^*-algebras according to the following definition.

1 DEFINITION. A *B^*-algebra* is a B-algebra with an involution $*$ which satisfies the identity $|x^*x| = |x|^2$.

Besides the preceding examples, which are all commutative B^*-algebras, there is the algebra $B(\mathfrak{H})$ of all bounded linear operators in Hilbert space $\mathfrak{H}$. In this algebra the involution operation is that of forming Hilbert space adjoints, i.e., T^* is the *Hilbert space adjoint* of T and is defined in terms of the scalar product (x, y) in $\mathfrak{H}$ by the identity

$$(Tx, y) = (x, T^*y), \qquad x, y \in \mathfrak{H}. \tag{i}$$

To verify the identity $|T^*T| = |T|^2$ we note first that

$$\begin{aligned} |T^*T| &= \sup |(T^*Tx, y)| = \sup |(Tx, Ty)| \\ &\geqq \sup |(Tx, Tx)| = \sup |Tx|^2 = |T|^2, \end{aligned}$$

where the suprema are taken over all x, y in $\mathfrak{H}$ with $|x| \leqq 1$, $|y| \leqq 1$. On the other hand

$$|T^*| = \sup |(T^*x, y)| = \sup |(x, Ty)| = |T|,$$

and so

$$|T^*T| \leqq |T^*||T| = |T|^2.$$

Thus $|T^*T| = |T|^2$, which, together with Lemma VI.2.10, proves the following lemma.

2 LEMMA. *The algebra $B(\mathfrak{H})$ of all bounded linear operators in Hilbert space $\mathfrak{H}$ in which the operation $*$ of involution is defined by equation* (i) *is a B^*-algebra.*

Our chief objective in this section is to characterize commutative B^*-algebras. It will be shown that the homomorphism $x \to x(\cdot)$ (see Theorem 2.9) of a commutative B^*-algebra $\mathfrak{X}$ into the algebra $C(\Lambda)$ of all continuous functions on the structure space Λ of $\mathfrak{X}$ is an isometric isomorphism of $\mathfrak{X}$ onto all of $C(\Lambda)$. It will also be shown that this isomorphism is a $*$-isomorphism, i.e., one preserving the operation of involution. This basic result, which is due to Gelfand and Naĭmark, will find many applications in the next two chapters.

3 LEMMA. *If $\mathfrak{X}$ is a commutative B^*-algebra then $|x^2| = |x|^2$, $|x| = |x^*|$, and the unit e satisfies the equation $e = e^*$.*

PROOF. For an arbitrary x in $\mathfrak{X}$, $|x^2|^2 = |(x^2)^*x^2| = |(x^*)^2x^2| = |(xx^*)^*(xx^*)| = |xx^*|^2 = |x|^4$. Thus $|x^2| = |x|^2$. Also $|xx^*| = |x|^2$, $|xx^*| = |x^{**}x^*| = |x^*|^2$, and hence $|x| = |x^*|$. Now $ee^* = e^*$, $ee^* = e^{**}e^* = (e^*e)^* = e$, and so $e^* = e$. Q.E.D.

4 DEFINITION. A *$*$-homomorphism* of a B^*-algebra $\mathfrak{X}$ into a B^*-algebra $\mathfrak{Y}$ is a homomorphic map h of $\mathfrak{X}$ into $\mathfrak{Y}$ which preserves involutions, i.e., $h(x)^* = h(x^*)$. A *$*$-isomorphism* between the B^*-algebras $\mathfrak{X}$ and $\mathfrak{Y}$ is a $*$-homomorphism h of $\mathfrak{X}$ into $\mathfrak{Y}$ which is also an isomorphism (i.e., one-to-one) with $h\mathfrak{X} = \mathfrak{Y}$. When such an isomorphism exists the algebras $\mathfrak{X}$ and $\mathfrak{Y}$ are called *$*$-isomorphic* or *$*$-equivalent*. The symbol $\mathfrak{X} \overset{*}{=} \mathfrak{Y}$ is sometimes used to mean that $\mathfrak{X}$ and $\mathfrak{Y}$ are $*$-equivalent. The structure space (Definition 2.7) of a B^*-algebra $\mathfrak{X}$ is sometimes called the *spectrum* of $\mathfrak{X}$ and is often denoted by the symbol $\sigma(\mathfrak{X})$.

5 LEMMA. (*Arens*) *If Λ is the spectrum of the commutative B^*-algebra $\mathfrak{X}$ then the map $x \to x(\cdot)$ of $\mathfrak{X}$ into $C(\Lambda)$ is a $*$-homomorphism.*

PROOF. It was shown in Theorem 2.9 that the map $x \to x(\cdot)$ is a homomorphism. Thus it suffices to show that for every $\lambda \in \Lambda$ and

$x \in \mathfrak{X}$ we have $x^*(\lambda) = \overline{x(\lambda)}$. Let $x(\lambda) = \alpha + \beta i$ and $x^*(\lambda) = \gamma + \delta i$ where $\alpha, \beta, \gamma, \delta$ are real. We assume that $\beta + \delta \neq 0$ and show that this leads to a contradiction. Let $y = [x + x^* - (\alpha+\gamma)e]/(\beta+\delta)$, then $y^* = y$ and $y(\lambda) = i$. Let N be a real number. Then $(y+Nie)(\lambda) = y(\lambda) + Ni = i(1+N)$, and hence $|1+N| \leq |y+Nie|$. Hence $(1+N)^2 \leq |y+Nie|^2 = |(y+Nie)(y+Nie)^*| = |(y+Nie)(y-Nie)| = |y^2+N^2e| \leq |y^2|+N^2$. Since this inequality must hold for all real N, a contradiction is obtained by placing $N = |y^2|$. Hence $\beta+\delta = 0$ and $x(\lambda) = \alpha + \beta i$, $x^*(\lambda) = \gamma - \beta i$, $(ix)(\lambda) = ix(\lambda) = -\beta + \alpha i$, $(ix)^*(\lambda) = -ix^*(\lambda) = -\beta - \gamma i$. Hence by what has already been proved $\alpha - \gamma = 0$ and so $x^*(\lambda) = \overline{x(\lambda)}$. Q.E.D.

6 COROLLARY. *If $x = x^*$ then $x(\lambda)$ is real.*

→ 7 THEOREM (*Gelfand-Naimark*) *A commutative B^*-algebra is isometrically $*$-isomorphic with the algebra of all complex continuous functions on its spectrum.*

PROOF. Let Λ be the spectrum of the B^*-algebra $\mathfrak{X}$. Since, by Lemma 3, $|x^m| = |x|^m$ if m is a power of 2, it follows from Lemma 2.4 that

$$\sup_{\lambda \in \Lambda} |x(\lambda)| = \lim_n \sqrt[n]{|x^n|} = |x|.$$

Thus the map $x \to x(\cdot)$ of $\mathfrak{X}$ into $C(\Lambda)$ is isometric and $\mathfrak{X}$ has no radical. Lemma 5 shows that this map is a $*$-homomorphism. It remains to be shown that every continuous function on Λ is the correspondent of some x in $\mathfrak{X}$. To do this we apply the general Weierstrass theorem (IV.6.17) to the algebra $C(\Lambda)$ of all continuous functions on Λ. Let C be the subalgebra of $C(\Lambda)$ corresponding to the functions $x(\cdot)$. Since $|x| = \sup |x(\lambda)|$ and $\mathfrak{X}$ is complete, we see that C is closed in $C(\Lambda)$. Now let λ_1, λ_2 be two different maximal ideals in Λ, and let $y \in \lambda_1$, $y \notin \lambda_2$. Then $y(\lambda_1) \neq y(\lambda_2)$, so C distinguishes between points. Lemma 5 shows that the conditions of the Weierstrass theorem hold, and hence we conclude that $C = C(\Lambda)$. Q.E.D.

8 COROLLARY. *If Λ is the spectrum of the commutative B^*-algebra $\mathfrak{X}$, then the homomorphism $x \to x(\cdot)$ of Theorem 2.9 is an isometric $*$-isomorphism of $\mathfrak{X}$ onto $C(\Lambda)$.*

9 COROLLARY. *A commutative B^*-algebra of operators in Hilbert space is isometrically *-equivalent to the B^*-algebra of all continuous functions on its spectrum.*

10 COROLLARY. *Let $\mathfrak{Y}$ be a B^*-subalgebra of the commutative B^*-algebra $\mathfrak{X}$ and suppose that $\mathfrak{X}$ and $\mathfrak{Y}$ have the same unit e. Then an element y in $\mathfrak{Y}$ has an inverse in $\mathfrak{X}$ if and only if it has an inverse in $\mathfrak{Y}$. Consequently the spectrum of y as an element of $\mathfrak{Y}$ is the same as its spectrum as an element of $\mathfrak{X}$.*

PROOF. If y^{-1} exists as an element of $\mathfrak{Y}$ then, since $\mathfrak{X}$ and $\mathfrak{Y}$ have the same unit, y^{-1} exists as an element of $\mathfrak{X}$. Conversely, if y has an inverse in $\mathfrak{X}$ then, since $(y^{-1})^*y^* = (yy^{-1})^* = e^* = e$, y^* has an inverse in $\mathfrak{X}$. Consequently yy^* has an inverse in $\mathfrak{X}$. But, by Theorem 7, the spectrum of yy^* is non-negative and hence the resolvent set $\rho(yy^*)$ is connected. It follows from Corollary 1.10 that yy^* has an inverse in $\mathfrak{Y}$. Thus y has an inverse in $\mathfrak{Y}$. Q.E.D.

11 COROLLARY. *Let x be an element of the commutative B^*-algebra $\mathfrak{X}$ and let $B^*(x)$ be the smallest closed B^*-subalgebra containing x and the unit e in $\mathfrak{X}$. Then $B^*(x)$ is isometrically *-equivalent to the algebra $C(\sigma(x))$.*

PROOF. In view of Corollary 10 the spectrum of x as an element of $B^*(x)$ is the same as its spectrum $\sigma(x)$ as an element of $\mathfrak{X}$. Thus we may and shall assume that polynomials in e, x, and x^* are dense in $\mathfrak{X}$, i.e., that $\mathfrak{X} = B^*(x)$. Let $\Lambda = \sigma(\mathfrak{X})$ so that $\sigma(x) = x(\Lambda)$. Consider the continuous map $\lambda \to x(\lambda)$ of Λ onto $\sigma(x)$. This map is one-to-one. For if $x(\lambda) = x(\lambda')$, then $x^*(\lambda) = \overline{x(\lambda)} = \overline{x(\lambda')} = x^*(\lambda')$ and $y(\lambda) = y(\lambda')$ for every polynomial in e, x, x^*. Since such y are dense in $\mathfrak{X}$ we have $y(\lambda) = y(\lambda')$ for every y in $\mathfrak{X}$. Then, by Corollary 8, every continuous function on Λ has the same value at λ as at λ'. Since a compact Hausdorff space is normal this means that $\lambda = \lambda'$. Thus the map $\lambda \to x(\lambda)$ is a continuous one-to-one map of the compact space Λ onto the compact space $x(\Lambda) = \sigma(x)$. It follows that $\sigma(x)$ and Λ are homeomorphic. Thus, from Theorem 7, it is seen that $\mathfrak{X} = B^*(x) \stackrel{*}{=} C(\sigma(x))$. Q.E.D.

In general there are many isometric *-isomorphisms between $B^*(x)$ and $C(\sigma(x))$, for it is clear that any homeomorphism in $\sigma(x)$ generates an isometric automorphism in $C(\sigma(x))$ and thus transforms

one *-isomorphism of $B^*(x)$ onto $C(\sigma(x))$ into another one. There is one isometric *-isomorphism of $B^*(x)$ onto $C(\sigma(x))$ that we wish to single out. In the notation of the preceding proof the *-isomorphism $y \leftrightarrow y(x^{-1}(\cdot))$ of $B^*(x)$ onto $C(\sigma(x))$ has the property that x corresponds to the function $x(x^{-1}(\mu)) = \mu$, $\mu \in \sigma(x)$. Clearly the requirement that x and $g(\mu) \equiv \mu$ be corresponding elements determines the *-isomorphism uniquely and we are thus led to the following definition.

12 DEFINITION. Let x be an element of a commutative B^*-algebra and let $f \in C(\sigma(x))$. By $f(x)$ will be meant the element in $B^*(x)$ corresponding to the function f in $C(\sigma(x))$ under the *-isomorphism between $B^*(x)$ and $C(\sigma(x))$ which is uniquely defined by the requirement that x and $g(\mu) \equiv \mu$ be corresponding elements.

The notation introduced in Definition 12 is consistent with previous usage. For if f is the polynomial $f(\mu) = \sum \alpha_{nm}\mu^n\bar{\mu}^m$ in μ and $\bar{\mu}$ then $f(x) = \sum \alpha_{nm} x^n x^{*m}$. The symbol $f(x)$ has also been used for the element $(2\pi i)^{-1}\int_C f(\lambda)(\lambda e - x)^{-1} d\lambda$ (see Definition VII.3.9) provided that f is analytic and single valued on an open set containing $\sigma(x)$. The next lemma shows that the two definitions for $f(x)$ coincide.

13 LEMMA. *Let f be a complex function which is single valued and analytic on an open set containing the spectrum of an element x in a commutative B^*-algebra. Then the meanings assigned to the symbol $f(x)$ in Definition* 12 *and Definition* VII.3.9 *coincide.*

PROOF. Let $y \leftrightarrow y(\cdot)$ be the *-isomorphism between $B^*(x)$ and $C(\sigma(x))$ as in Definition 12 and let

$$y = \frac{1}{2\pi i}\int_C f(\lambda)(\lambda e - x)^{-1} d\lambda,$$

where the curve C is as required in Definition VII.3.9. Then

$$\begin{aligned} y(\mu) &= \frac{1}{2\pi i}\int_C f(\lambda)(\lambda e - x)^{-1}(\mu) d\lambda \\ &= \frac{1}{2\pi i}\int_C \frac{f(\lambda)}{\lambda - \mu} d\lambda = f(\mu), \end{aligned}$$

and thus y is the element $f(x)$ of Definition 12. Q.E.D.

It is evident that the preceding corollaries may be applied to an

operator T in a Hilbert space $\mathfrak{H}$ provided that T is an element of a commutative B^*-subalgebra of the B^*-algebra $B(\mathfrak{H})$ of all bounded operators in $\mathfrak{H}$. This leads us to the notion of a normal operator as given in the following definition.

14 DEFINITION. A bounded linear operator T in a Hilbert space is said to be *normal* if $TT^* = T^*T$, and *self adjoint* if $T = T^*$.

It is clear that the smallest closed subalgebra of $B(\mathfrak{H})$ which contains a normal operator T, its adjoint T^*, and the identity I is a commutative B^*-algebra. Thus we may state the following corollary.

15 COROLLARY. *Let T be a normal operator in a Hilbert space $\mathfrak{H}$ and let $B^*(T)$ be the smallest closed subalgebra of $B(\mathfrak{H})$ which contains the elements I, T, and T^*. Then $B^*(T)$ is isometrically *-equivalent to the algebra $C(\sigma(T))$. Furthermore an isometric *-isomorphism is uniquely determined by the requirement that the operator T correspond to the function $g(\mu) \equiv \mu$, $\mu \in \sigma(T)$. If the symbol $f(T)$ is used for the operator corresponding to the scalar function $f \in C(\sigma(T))$ under this unique isomorphism, then for every f which is single valued and analytic on an open set containing the spectrum $\sigma(T)$ we have*

$$f(T) = \frac{1}{2\pi i}\int_C f(\lambda)(\lambda I - T)^{-1} d\lambda,$$

where the curve C satisfies the requirements of Definition VII.3.9.

PROOF. This follows from Corollary 11 and Lemma 13. Q.E.D.

4. Exercises

1 Let S be a compact Hausdorff space. Show that

(i) There is a one-to-one correspondence between closed ideals in $C(S)$ and closed sets $F \subset S$ given by

$$F \leftrightarrow \mathfrak{J}_F = \{f \in C(S) | f(F) = 0\}.$$

(ii) If $\mathfrak{A}$ is a closed subalgebra of $C(S)$ which contains the complex conjugate of each of its elements, then there is a decomposition of S into the union of closed sets F upon each of which the elements of $\mathfrak{A}$ are constant, and such that each continuous function constant on each set F lies in $\mathfrak{A}$.

2 Let S be a compact metric space. Let $\mathfrak{J} \subset C(S)$ be the set of all functions f vanishing on some neighborhood, depending on f, of a point x_0 in S. Then $\mathfrak{J}$ is an ideal and is closed if and only if x_0 is an isolated point of S. Thus $C(S)$ always has ideals which are not closed, provided that S is infinite.

3 Let S be a compact Hausdorff space, and let $n(S)$ be the least number of generators of the algebra $C(S)$. Find $n(S)$ for

(a) The unit circle $|z| = 1$.

(b) The unit sphere in three-dimensional Euclidean space.

(c) The Hilbert cube $\{x \,|\, |x_i| \leq 1/i\}$ in l_2.

(d) The configuration made up of the arcs joining each of the points $(0, 0, 1)$, $(0, 0, 2)$, and $(0, 0, 3)$ in three-dimensional Euclidean space to each of the points $(1, 0, 0)$, $(-1, 0, 0)$, and $(0, 1, 0)$.

4 Let $\mathfrak{X}$ be a Banach algebra whose elements are continuous functions on a compact Hausdorff space S with the ordinary operations of addition and multiplication. Suppose that

(a) if $x(\cdot) \in \mathfrak{X}$ then $\overline{x(\cdot)} \in \mathfrak{X}$.

(b) if $x(t) \in \mathfrak{X}$ and is not zero on S then $1/x(\cdot) \in \mathfrak{X}$.

Then for each maximal ideal $\mathfrak{M} \subset \mathfrak{X}$ there is a point $t_0 \in S$ so that $x(\mathfrak{M}) = x(t_0)$, $x \in \mathfrak{X}$. Conversely, if for each maximal ideal $\mathfrak{M}$ there is a $t_0 \in S$ such that $x(\mathfrak{M}) = x(t_0)$, then if $x(\cdot) \in \mathfrak{X}$ and is not zero on S, $1/x(\cdot) \in \mathfrak{X}$.

5 Let $\mathfrak{X}_1$ be a B-algebra satisfying condition (a) of Exercise 4 and $\mathfrak{X}_2$ a dense subalgebra of $\mathfrak{X}_1$ satisfying conditions (a) and (b). Suppose that $\mathfrak{X}_2$ can be supplied with a norm in such a way as to become a B-algebra. Show that $\mathfrak{X}_1$ also satisfies condition (b).

6 Let C^n be the class of all complex valued functions $x(t)$ on $[0, 1]$ with n continuous derivatives. Show that C^n with the ordinary operations of addition and multiplication is a Banach algebra if

$$|x| = \sup_{0 \leq t \leq 1} \sum_{k=0}^{n} \frac{|x^{(k)}(t)|}{k!}.$$

What are the maximal ideals?

7 Let $CBV[0, 1]$ be the algebra of continuous functions $x(t)$ of bounded variation on $[0, 1]$ with

$$|x| = \sup_{0 \leq t \leq 1} |x(t)| + v(x, [0, 1])$$

and the ordinary operations of addition and multiplication. Show that $CBV[0, 1]$ is a Banach algebra. Identify the maximal ideals.

8 Let $K^{(n)}$ be the algebra of polynomials $x = \sum_{k=0}^{n} a_k\lambda^k$ with complex coefficients and norm $|x| = \sum_{k=0}^{n} |a_k|$. Addition and multiplication are as usual except that in forming products λ^{n+1} is taken to be zero. Show that $K^{(n)}$ is a Banach algebra with a unique maximal ideal.

9 The space l_1 of absolutely convergent series $a = \{\alpha_n, -\infty < n < \infty\}$ is a commutative Banach algebra under the multiplication

$$ab = \sum_{n=-\infty}^{\infty} \alpha_{m-n}\beta_n, \qquad a = \{\alpha_n\}, \qquad b = \{\beta_n\} \in l_1.$$

The space $\mathscr{M}$ of maximal ideals is homeomorphic with the reals modulo 2π (i.e., homeomorphic with a circle) in such a way that if $\mathfrak{M} \in \mathscr{M}$ and θ are corresponding elements then

$$a(\mathfrak{M}) = \sum_{n=-\infty}^{\infty} \alpha_n e^{in\theta}.$$

10 (N. Wiener) If the absolutely convergent trigonometric series $\sum_{n=-\infty}^{\infty} \alpha_n e^{in\theta}$ never vanishes, then its reciprocal is also an absolutely convergent trigonometric series.

11 (N. Wiener-P. Lévy) Prove that if f is analytic and single valued on a neighborhood of the range of the absolutely convergent trigonometric series

$$g(\theta) = \sum_{-\infty}^{\infty} \alpha_n e^{in\theta}$$

then there is an absolutely convergent trigonometric series

$$h(\theta) = \sum_{-\infty}^{\infty} \beta_n e^{in\theta} \text{ with } h(\theta) = f(g(\theta)).$$

12 Let $h : \mathfrak{A}_1 \to \mathfrak{A}_2$ be a continuous homomorphism where $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are Banach algebras. Let $h(a) = 0$ imply $a = 0$, and let $\mathfrak{A}_2$ have no radical. Show that $\mathfrak{A}_1$ has no radical.

13 Let M be the set of all regular countably additive Borel measures μ on the real line R. For μ, λ in M, let $\mu \times \lambda$ be the direct product measure in $R \times R$ determined by μ and λ. Define a measure $\mu * \lambda$ by putting

$$(\mu * \lambda)(E) = (\mu \times \lambda)(E_1)$$

where E is an arbitrary Borel subset of R, and $E_1 = \{(x, y) \in R \times R \,|\, x+y \in E\}$. Show that with the product $\mu * \lambda$ the Banach space M is a commutative Banach algebra.

14 If f is in $L_1(-\infty, \infty)$, and if $\lambda(E) = \int_E f(s)ds$ show that

$$(\lambda * \mu)(E) = \int_E ds \int_{-\infty}^{\infty} f(s-t)\mu(dt).$$

for every μ in the space M of Exercise 13. If $\mu(E) = \int_E g(s)ds$ for some g in $L_1(-\infty, \infty)$ then

$$(\lambda * \mu)(E) = \int_E ds \int_{-\infty}^{\infty} f(s-t)g(t)dt.$$

15 Let ξ be in the resolvent set of the bounded linear operator T and let $d(\xi)$ be the distance from ξ to the spectrum of T. Prove that

$$1 \leqq d(\xi)|R(\xi; T)|.$$

16 Let A and B be commuting bounded operators in a complex Banach space. Show

$$|\sigma(A+B)| \subseteq |\sigma(A)| + |\sigma(B)|$$
$$|\sigma(AB)| \subseteq |\sigma(A)||\sigma(B)|.$$

17 If a (commutative) B-algebra has only trivial ideals, then it is isometrically isomorphic to the complex numbers. Show that there exist non-commutative algebras with no two-sided ideals which satisfy all the axioms for a B-algebra except the commutative law and which are not the complex numbers.

18 Find a non-commutative B-algebra $\mathfrak{A}$ and an element $x \neq 0$ in $\mathfrak{A}$ with $x^2 = 0$ and with x contained in no two-sided ideal in $\mathfrak{A}$.

19 In a commutative B-algebra, the spectral radius is a continuous function.

20 (Kaplansky) Define multiplication in l_1 by

$$(x_1, x_2, x_3, \ldots)(y_1, y_2, y_3, \ldots) = (x_1y_1, x_2y_2, x_3y_3, \ldots).$$

Show that with this multiplication l_1 is a commutative algebra with no unit and that $|xy| < |x| \cdot |y|$.

(i) Show that there is a one-to-one correspondence between closed ideals and subsets of the integers.

(ii) There is a one-to-one correspondence between maximal ideals and integers.

(iii) The topology on the maximal ideals is discrete.

(iv) Characterize the closed subalgebras of l_1.

21 Let $\mathfrak{X}$ be a commutative B-algebra with structure space $\mathscr{M}$ and suppose that for every closed set $\mathscr{M}_1 \subset \mathscr{M}$ and every maximal ideal $\mathfrak{M}_1 \notin \mathscr{M}_1$ there is an x in $\mathfrak{X}$ with $x(\mathfrak{M}_1) \neq 0$ and $x(\mathscr{M}_1) = 0$. Prove the following:

(a) Let $\mathfrak{J}$ be the intersection of all maximal ideals in a given closed set $\mathscr{M}_0 \subset \mathscr{M}$. Then $\sigma(x-\mathfrak{J}) = x(\mathscr{M}_0)$.

(b) If $x(\mathfrak{M}_0) = 0$, $x(\mathfrak{M}_1) \neq 0$ and $y(\mathfrak{M}) = 0$ for every $\mathfrak{M}$ in some neighborhood of $\mathfrak{M}_0$ then there is a z such that $zx-y$ is in the radical.

(c) Let $\mathfrak{X}$ be semi-simple and have the property that for every x in $\mathfrak{X}$ there is an $\bar{x}$ in $\mathfrak{X}$ with $\bar{x}(\mathfrak{M}) = \overline{x(\mathfrak{M})}$ for every $\mathfrak{M}$ in $\mathscr{M}$. Let $E \subset \mathfrak{X}$ and $\mathscr{M}_0 = \{\mathfrak{M} | x(\mathfrak{M}) = 0,\ x \in E\}$. Show that the ideal determined by E contains every element y for which $y(\mathfrak{M})$ vanishes for $\mathfrak{M}$ in a neighborhood of the closure of $\mathscr{M}_0$.

5. Notes and Remarks

The concept of a general normed algebra was introduced first by Michal and Martin [1] and Nagumo [1]. However, since the publication, in 1941, of the fundamental papers of Gelfand [1, 3, 4, 5] and Gelfand and Šilov [1], the study of these algebras has occupied the attention of many authors. Since only the most elementary aspects of the theory of B-algebras are required for our purposes, we refer the reader to Hille [1] (or the revised edition with R. S. Phillips), Loomis [1], Naĭmark (*Normed rings*, Noordhoff, Groningen, 1959), and Rickart [10] for other topics.

Preliminary notions. Many of the concepts in this section have direct analogues in the B-algebra of bounded linear operators on a B-space, which have already been discussed in Section VII.11. The notion of a topological divisor of zero is due to Šilov [2]. The singular elements of a B-algebra have been studied in detail by Rickart [4] to whom Lemma 4 is due, although a form of this lemma was proved earlier by Bochner and Phillips [2]. It was announced by Mazur [3] and a proof given by Gelfand [1], that if a B-algebra is a field it is isometrically isomorphic to the complex numbers. This is an extension of a theorem of Frobenius; similar conclusions have been obtained by

Arens [8], Edwards [2], Lorch [10], Ramaswami [1], Šilov [5], Stone [1], Tornheim [1], and Wright [1] under a wide variety of hypotheses. (Compare Theorem 6 and Corollary 7.) Lemma 9 is due to Lorch [9]. The study of ideal theory in B-algebra was inaugurated by Gelfand [1] to whom most of the results given in Section 1 are due.

B- and B^*-algebras. The results of Section 2 are due to Gelfand [1]. The fundamental Theorem 3.7 was proved by Gelfand and Naĭmark [1]. In their proof, they proved Lemma 3.5 by using a fairly deep result of Šilov that was not generally available. The proof of this lemma given here is that given by Arens [6], who has also (Arens [7]) obtained this result in greater generality. A simple direct proof of Corollary 3.6 was given by Fukamiya [2], and can be used to prove Lemma 3.5. Corollary 3.10 is due to Rickart [6] who has also proved stronger results on spectral permanence (see Rickart [9]).

Non-commutative B^*-algebras. Although our attention has been directed towards commutative B-algebras, much is known in the non-commutative case. In analogy with Theorem 3.7, we remark that Gelfand and Naĭmark [1] have given the following characterization of non-commutative B^*-algebras:

THEOREM. *Every B^*-algebra is isometrically $*$-isomorphic with a subalgebra of the algebra of all bounded linear operators on some complex Hilbert space.*

It may be remarked that in their proof they assumed that the B^*-algebra $\mathfrak{X}$ satisfied the additional condition that for every x in $\mathfrak{X}$, the element $e+x^*x$ possesses an inverse, in which case it was called a C^*-algebra. Only recently Kaplansky has proved, using some results of Fukamiya [3] and Kelley and Vaught [1], that this condition is a consequence of the other properties and hence is superfluous (cf. Math. Rev. 14, 884 (1953)). T. Ono (see Rickart [10; p. 248] for reference and comments) showed that the condition $|x^*x| = |x|^2$ in Definition 3.1 can be replaced by the condition $|xx^*| = |x||x^*|$.

The theory of weakly closed non-commutative B^*-algebras of operators in a Hilbert space has been extensively developed by Murray and von Neumann and many others. Since we will not require the results of this theory we present only a single result, due to von Neumann [2], in this direction. If $\mathfrak{A}$ is a collection of bounded linear opera-

tors in a Hilbert space $\mathfrak{H}$, the *centralizer* (or commutant) $\mathfrak{A}^c$ of $\mathfrak{A}$ is the set of all bounded linear operators in $\mathfrak{H}$ which commute with every operator in $\mathfrak{A}$.

LEMMA. *The centralizer of a B^*-algebra of operators in Hilbert space is a B^*-algebra of operators and is closed in the weak operator topology.*

PROOF. It is easily seen that the centralizer $\mathfrak{A}^c$ of a B^*-algebra $\mathfrak{A}$ is a B^*-algebra. If (B_α) is a generalized sequence in $\mathfrak{A}^c$ converging in the weak operator topology to an operator B, then for every $A \in \mathfrak{A}$ and $x, y \in \mathfrak{H}$ we have $(BAx, y) = \lim (B_\alpha Ax, y) = \lim (AB_\alpha x, y) = (ABx, y)$, and so $B \in \mathfrak{A}^c$. Q.E.D.

THEOREM. *A B^*-algebra of operators in Hilbert space is equal to the centralizer of its centralizer if and only if it is closed in the weak operator topology.*

PROOF. It is clear from the definition of the centralizer that $\mathfrak{A} \subseteq (\mathfrak{A}^c)^c$ and from the lemma that $\mathfrak{A}$ is closed in the weak operator topology if $\mathfrak{A} = (\mathfrak{A}^c)^c$. To prove the theorem it suffices to show that if $\mathfrak{A}$ is closed in the weak operator topology, then every neighborhood of a point B in $(\mathfrak{A}^c)^c$ contains an element of $\mathfrak{A}$. In order to illustrate the idea of the proof this statement will be demonstrated first for a neighborhood of the form

$$N_1 = \{A \in B(\mathfrak{H}) \mid |(x_1, (A - B)y_1)| < \varepsilon\},$$

where $\varepsilon > 0$ and x_1 and y_1 are fixed non-zero elements in $\mathfrak{H}$. Let E be the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{M} = \overline{\text{sp}}\,\{Ay_1 \mid A \in \mathfrak{A}\}$. For every $A \in \mathfrak{A}$ we have $A\mathfrak{M} \subseteq \mathfrak{M}$, and $A^*\mathfrak{M} \subseteq \mathfrak{M}$; hence $EAE = AE$, and $EA^*E = A^*E$. Taking adjoints in this last equation it follows that $EA = EAE = AE$, proving that $E \in \mathfrak{A}^c$. Hence $By_1 = BEy_1 = EBy_1 \in \mathfrak{M}$ and so there is an A in $\mathfrak{A}$ with $|Ay_1 - By_1| < \varepsilon/|x_1|$, which implies that A is in N_1, as desired.

We now consider an arbitrary neighborhood

$$N = \{A \in B(\mathfrak{H}) \mid |(x_i, (A - B)y_i)| < \varepsilon,\ i = 1, \ldots, n\}$$

of B, and we may assume that $|x_i| \leq 1$. Let $\mathfrak{H}^+$ be the direct sum (cf. IV. 4. 17) of $\mathfrak{H}$ with itself n times, and for each operator A defined on $\mathfrak{H}$ let A^+ be the operator defined on $\mathfrak{H}^+ = \mathfrak{H} \oplus \ldots \oplus \mathfrak{H}$ by the

equation $A^+[z_1, \ldots, z_n] = [Az_1, \ldots, Az_n]$. The set of A^+ with A in $\mathfrak{A}$ will be denoted by $\mathfrak{A}^+$. Now the general bounded linear operator C in $\mathfrak{H}^+$ has the form

$$C[z_1, \ldots, z_n] = [\sum_{i=1}^{n} C_{1i}z_i, \ldots, \sum_{i=1}^{n} C_{ni}z_i],$$

and therefore $(\mathfrak{A}^+)^c$ consists of those C for which all the operators C_{ij} are in $\mathfrak{A}^c$. From this is is seen that $((\mathfrak{A}^+)^c)^c$ consists of the operators A^+ with A in $(\mathfrak{A}^c)^c$. In view of this representation for the general element in $((\mathfrak{A}^+)^c)^c$, the argument used in the case $n = 1$ may be applied to obtain an A^+ in $\mathfrak{A}^+$ with

$$|(A^+ - B^+)[y_1, \ldots, y_n]| = \sum_{i=1}^{n} |(A-B)y_i| < \varepsilon,$$

from which it follows that A is in N. Q.E.D.

This theorem is of importance since it characterizes the weakly closed B^*-algebras of operators in terms of an algebraic property, and suggests that they are of particular interest.

The foundations of the theory of these algebras are to be found in the papers of von Neumann [2, 13, 14, 15] and Murray and von Neumann [1]. An expository account is given by Naĭmark [2], and a more comprehensive treatment is given in the books of Dixmier [5] and Rickart [10], where references are given.

Generalizations. We conclude by mentioning that many of the results of B-algebras have been extended to more general topological algebras by Arens [2, 9] and Michael [2]. For an expository account of the theory of topological rings and algebras, we recommend Kaplansky [4].

CHAPTER X

Bounded Normal Operators in Hilbert Space

1. Terminology and Preliminary Notions

The spectral theorem to be proved in this chapter will introduce a theory which parallels in Hilbert space the theory in n-dimensional unitary space associated with the classical reduction of a finite normal matrix of complex numbers. Throughout the chapter the symbol T^* will be used for the Hilbert space adjoint of the operator T in Hilbert space $\mathfrak{H}$. The symbol (x, y) will be used for the scalar product of the vectors x and y in $\mathfrak{H}$. By definition then, $(Tx, y) = (x, T^*y)$, and T is *normal* if and only if $TT^* = T^*T$. The Gelfand-Naĭmark representation theorem (in particular Corollaries IX.3.9 and IX.3.15) will yield, for normal operators in Hilbert space, a reduction theory which is more complete than that developed in Chapter VII for general operators in a complex B-space. Although the present chapter is independent of Chapter VII, it may help in motivating the study of normal operators if we interpret the reduction problem in the light of the general results of Chapter VII. There we associated with an operator T in a complex B-space a Boolean algebra of sets in the complex plane which were called spectral sets. A spectral set was defined as any subset of the spectrum $\sigma(T)$ which is both open and closed in the relative topology of $\sigma(T)$. With each spectral set σ was associated the projection operator

$$\text{(i)} \qquad E(\sigma) = \frac{1}{2\pi i}\int_{C(\sigma)} (\lambda I - T)^{-1} d\lambda$$

where $C(\sigma)$ is any rectifiable Jordan curve in $\rho(T)$ surrounding σ but no other point of $\sigma(T)$. What concerns us at present is not so much the definition (i) of the projections $E(\sigma)$ but rather some of their properties. These projections satisfy the identities

$$\text{(ii)} \qquad \begin{aligned} &E(\sigma \cap \delta) = E(\sigma) \wedge E(\delta), \qquad &E(\sigma \cup \delta) = E(\sigma) \vee E(\delta), \\ &E(\sigma(T)) = I, \qquad &E(\phi) = 0, \end{aligned}$$

where σ, δ are arbitrary spectral sets and where ϕ is the void set. Here we have used the notations $A \wedge B$ and $A \vee B$ for the *intersection* and *union* of two commuting projections A and B. We recall that these operators are defined by the equations $A \wedge B = AB$, $A \vee B = A+B-AB$ and that the intersection and union of two commuting projections are again projection operators. Also the ranges of the intersection and union of two commuting projection operators are given by the equations $(A \wedge B)\mathfrak{X} = (A\mathfrak{X}) \cap (B\mathfrak{X})$, and $(A \vee B)\mathfrak{X} = (A\mathfrak{X})+(B\mathfrak{X}) = \text{sp}\,(A\mathfrak{X}, B\mathfrak{X})$, respectively. Thus, in a Boolean algebra of projections, the *order relation* $A \leqq B$, which is defined by the equation $AB = A$, may be interpreted geometrically as meaning $A\mathfrak{X} \subseteq B\mathfrak{X}$. Stated otherwise the equations (ii) assert that the function $\sigma \to E(\sigma)$ is a *homomorphic map* of the Boolean algebra of spectral sets onto a Boolean algebra of projection operators in $\mathfrak{X}$ and that furthermore this homomorphism takes the unit $\sigma(T)$ in the algebra of spectral sets into the unit I of the algebra of projections. This observation leads to the notion of a spectral measure in a B-space $\mathfrak{X}$. A *spectral measure* in $\mathfrak{X}$ is a homomorphic map of a Boolean algebra of sets into a Boolean algebra of projection operators in $\mathfrak{X}$ which has the additional property that it maps the unit in its domain into the identity operator I in its range. Thus with every bounded operator T in a complex B-space is associated, by means of equation (i), a spectral measure E defined on the family of spectral sets of T. This spectral measure is also related (VII.3.20) to T by the equations

$$\text{(iii)} \qquad E(\delta)T = TE(\delta), \qquad \sigma(T_\delta) = \delta$$

where δ is an arbitrary spectral set of T and where $\sigma(T_\delta)$ is the spectrum of the restriction T_δ of T to the manifold $\mathfrak{X}_\delta = E(\delta)\mathfrak{X}$. Thus, if $\delta_1, \ldots, \delta_n$ are disjoint spectral sets of T whose union is the whole spectrum $\sigma(T)$, the space $\mathfrak{X}$ may be decomposed into a direct sum $\mathfrak{X} = \mathfrak{X}_{\delta_1} \oplus \ldots \oplus \mathfrak{X}_{\delta_n}$ where T maps each of the spaces $\mathfrak{X}_{\delta_i} = E(\delta_i)\mathfrak{X}$ into itself and the spectrum $\sigma(T_{\delta_i})$ of T as an operator in $\mathfrak{X}_{\delta_i}$ is δ_i. This shows that the study of T may be reduced to the study of its restrictions to the invariant subspaces $\mathfrak{X}_{\delta_i}$. Evidently it is desirable to know whether or not the operator T may be reduced further, i.e., whether or not the domain of definition of the spectral measure may be extended to a larger Boolean algebra of sets in the plane in such a

manner that the properties (ii) and (iii) are preserved. Without relaxing the condition (iii) this is clearly impossible, for if $\delta = \sigma(T_\delta)$ then, since the spectrum of an operator is always closed (IX.1.5), every set in the domain of a spectral measure satisfying (iii) is necessarily an open and closed subset of $\sigma(T)$ and thus a spectral set. However, in order to reduce the study of T to its study on invariant subspaces in which it has a smaller spectrum it is quite sufficient to find a spectral measure which satisfies, instead of (iii), the condition

$$\text{(iv)} \qquad E(\delta)T = TE(\delta), \qquad \sigma(T_\delta) \subseteq \bar{\delta},$$

where $\bar{\delta}$ is the closure of δ. As will be seen in the next section, a normal operator T in Hilbert space $\mathfrak{H}$ determines a spectral measure which is defined on the Boolean algebra $\mathscr{B}$ of all Borel sets in the plane and which satisfies (iv) for every $\delta \in \mathscr{B}$. This spectral measure associated with a normal operator is also countably additive on $\mathscr{B}$ in the strong operator topology. This means that for every sequence $\{\delta_i\}$ of disjoint Borel sets

$$\text{(v)} \qquad \sum_{i=1}^{\infty} E(\delta_i)x = E\Big(\bigcup_{i=1}^{\infty} \delta_i\Big)x, \qquad x \in \mathfrak{H}.$$

A spectral measure E defined on the Borel sets in the plane and satisfying (iv) for every Borel set δ and (v) for every sequence $\{\delta_i\}$ of disjoint Borel sets is called a *resolution of the identity for* T. With this terminology the spectral theorem for bounded normal operators in Hilbert space asserts that every such operator has a uniquely determined resolution of the identity.

One of the many uses of this theorem is to generalize the operational calculus for normal operators similar to the operational calculus established in Section VII.1 for finite matrices. Before describing the operational calculus for normal operators let us recall the form this calculus assumes for a normal matrix in finite dimensional space. If T is a normal operator in the finite dimensional Hilbert space $\mathfrak{H}$ then its minimal equation has only simple roots and the indices $\nu_1, \ldots, \nu_k$ associated with the eigenvalues $\lambda_1, \ldots, \lambda_k$ are all equal to 1. Thus, for a finite normal matrix T, the operational calculus (VII.1.8) is given by the formula $f(T) = \sum_{i=1}^{k} f(\lambda_i)E(\lambda_i)$, where f is an arbitrary polynomial and where $E(\lambda_i)$ is the operator which projects $\mathfrak{H}$ onto the manifold $\{x \mid x \in \mathfrak{H}, (T - \lambda_i I)x = 0\}$ of eigenvectors

associated with λ_i. If we define, for every Borel set δ, the operator $E(\delta)$ to be 0 if δ contains none of the spectrum $\{\lambda_1, \ldots, \lambda_k\}$ of T and otherwise let $E(\delta)$ be the sum of all the projections $E(\lambda_i)$ for which $\lambda_i \in \delta$, then the function E is a resolution of the identity for T and the operational calculus is given by the formula

$$\text{(vi)} \qquad f(T) = \int_{\sigma(T)} f(\lambda)E(d\lambda),$$

where the integral is defined as the finite sum $\sum_{i=1}^{k} f(\lambda_i)E(\lambda_i)$. If the Hilbert space $\mathfrak{H}$ is infinite dimensional there is still an operational calculus for a normal operator T with resolution of the identity E which is given by the formula (vi), but in this situation it is necessary to define the integral appearing in (vi) and to define the algebra of scalar functions f to which the formula may be applied. One class of scalar functions f, other than polynomials, for which the operator $f(T)$ has already been defined is the class $C(\sigma(T))$ of all complex continuous functions on the spectrum. For, in Corollary IX.8.15, it was shown that there is one and only one isometric *-isomorphism between $C(\sigma(T))$ and the B^*-algebra $B^*(T)$ determined by T provided that we require the scalar function $g(\lambda) = \lambda$ and the operator T be corresponding elements. The operator $f(T)$ is, by definition, the uniquely determined operator in Hilbert space which corresponds to the continuous scalar function f under this *-isomorphism. This abstract *-isomorphism $f \leftrightarrow f(T)$ between $C(\sigma(T))$ and $B^*(T)$ is given a concrete analytical representation in equation (vi) and thus equation (vi) determines an operational calculus. The term operational calculus is used here and elsewhere in the following sense. Let $f \to T(f)$ be a homomorphic map of a B-algebra of scalar functions into a B-algebra of operators in a Banach space. Then any method of calculating the operator $T(f)$ from the scalar function f is called an *operational calculus*. An operational calculus usually assumes the form of an analytical formula which gives a concrete representation of the abstract homomorphism $f \to T(f)$. In the present chapter all of the homomorphisms $f \to T(f)$ will be *-homomorphisms between B^*-algebras.

In the study of normal operators we shall find considerable use for calculi defined in terms of various general forms of equation (vi). Thus it becomes necessary to define the integral $\int f(\lambda)E(d\lambda)$ of a

scalar function f with respect to the operator valued set function E. In the present chapter we shall only integrate bounded functions f and so the following discussion of the integral will be restricted to that case. Let Σ be a field of subsets of a set S and let E be a function which maps Σ into the algebra of bounded linear operators on the B-space $\mathfrak{X}$. It is assumed that E is additive and bounded, i.e., there is a constant K such that for every pair δ, σ of disjoint sets in Σ,

$$\text{(vii)} \qquad E(\delta \cup \sigma) = E(\delta) + E(\sigma), \qquad |E(\sigma)| \leq K.$$

It will not be necessary in defining the integral to assume that E is a spectral measure, or that $E(\delta)$ is a projection. Thus the basis for the integral is simply a bounded additive operator valued function E on a field Σ of subsets of an abstract set S. The functions we shall integrate are the bounded Σ-measurable functions. A *Σ-measurable* function (cf. IV.2.12) is a function f with $f^{-1}(A) \in \Sigma$ for every Borel set A in the range of f. If Σ is the family of Borel sets in a topological space S then a Σ-measurable function is sometimes called a *Borel measurable function* or simply a *Borel function*. The class $B(S, \Sigma)$ (cf. IV.2.12) is the closed linear manifold, in the space of all bounded functions on S, which is determined by the characteristic functions of sets in Σ. The norm in the space of all bounded functions on S, and thus the norm in $B(S, \Sigma)$, is $|f| = \sup_{s \in S} |f(s)|$. We shall use the term *Σ-simple* function for any function on S having the form

$$\text{(viii)} \qquad f = \sum_{i=1}^{n} \alpha_i \chi_{\delta_i}$$

where χ_{δ_i} is the characteristic function of a set δ_i in Σ. It is readily verified that if $\sum_{i=1}^{n} \alpha_i \chi_{\delta_i} = \sum_{j=1}^{m} \beta_j \chi_{\sigma_j}$ then $\sum_{i=1}^{n} \alpha_i E(\delta_i) = \sum_{j=1}^{m} \beta_j E(\sigma_j)$, and we may thus define the *integral* of the Σ-simple function (viii) by the equation

$$\int_S f(s) E(ds) = \sum_{i=1}^{n} \alpha_i E(\delta_i).$$

Now, since the total variation of a scalar valued additive set function μ on Σ is at most $4 \sup_{\delta \in \Sigma} |\mu(\delta)|$ (III.1.5) we have, for a Σ-simple function f,

$$\left|x^*\left[\int_S f(s)E(ds)\right]x\right| = \left|\int_S f(s)x^*E(ds)x\right|$$
$$\leqq 4\sup_{s\in S}|f(s)|\sup_{\delta\in\Sigma}|E(\delta)||x||x^*|, \qquad x\in\mathfrak{X}, \qquad x^*\in\mathfrak{X}^*,$$

and thus (II.3.15)

$$\left|\int_S f(s)E(ds)\right| \leqq 4K\sup_{s\in S}|f(s)|,$$

where K is the constant in (vii). This shows that if the sequence $\{f_n\}$ of Σ-simple functions converges in $B(S,\Sigma)$ to f then the sequence $\{\int_S f_n(s)E(ds)\}$ of integrals converges, and its limit depends only upon f and not upon the particular sequence $\{f_n\}$ used to approximate f. Thus we may define the *integral* of f by the equation

$$\int_S f(s)E(ds) = \lim_n \int_S f_n(s)E(ds).$$

For a set δ in Σ the integral $\int_\delta f(s)E(ds)$ is defined as $\int_S f(s)\chi_\delta(s)E(ds)$. It is clear that the map $f \to \int_S f(s)E(ds)$ is a continuous linear map of $B(S,\Sigma)$ into the algebra of bounded operators in $\mathfrak{X}$. If the set function E is a spectral measure, the map $f \to \int_S f(s)E(ds)$ is also a homomorphism. To see this let $f\in B(S,\Sigma)$ and note that the operators $\int_S f(s)g(s)E(ds)$ and $[\int_S f(s)E(ds)][\int_S g(s)E(ds)]$ both depend linearly and continuously upon g, and that if g is the characteristic function of the set δ in Σ, then

$$\int_S f(s)g(s)E(ds) = \int_S f(s)E(ds\cap\delta)$$
$$= \int_S f(s)E(ds)E(\delta) = \left[\int_S f(s)E(ds)\right]\left[\int_S g(s)E(ds)\right].$$

Thus for f in $B(S,\Sigma)$ the equation

$$\int_S f(s)g(s)E(ds) = \left[\int_S f(s)E(ds)\right]\left[\int_S g(s)E(ds)\right]$$

holds for all Σ-simple functions g, and hence, by a continuity argument, this equation holds for all f and g in $B(S,\Sigma)$. If, furthermore, E is a *self adjoint spectral measure* in Hilbert space, by which is meant that $E(\delta) = E(\delta)^*$ for δ in Σ, then the map $f \to \int_S f(s)E(ds)$ is a $*$-homomorphism of the B^*-algebra $B(S,\Sigma)$ into the B^*-algebra of bounded operators in Hilbert space. To see this let f be the Σ-simple function (viii). Then

$$\left[\int_S f(s)E(ds)\right]^* = \sum_{i=1}^{n} \overline{\alpha_i} E(\delta_i) = \int_S \bar{f}(s)E(ds),$$

and since simple functions are dense in $B(S, \Sigma)$, we have

$$\left[\int_S f(s)E(ds)\right]^* = \int_S \bar{f}(s)E(ds) \text{ for all } f \text{ in } B(S, \Sigma).$$

In summary we state the following theorem.

1 THEOREM. *Let E be a bounded self adjoint spectral measure in Hilbert space defined on a field Σ of subsets of a set S. Then the map $f \to T(f)$ defined by the equation*

$$T(f) = \int_S f(s)E(ds), \qquad f \in B(S, \Sigma),$$

*is a continuous *-homomorphic map of the B^*-algebra $B(S, \Sigma)$ of bounded Σ-measurable functions on S into the B^*-algebra of bounded operators on Hilbert space.*

Returning now to the general integral $\int f(s)E(ds)$ where E is merely a bounded additive operator valued set function, we observe that the integral has been defined in terms of the uniform operator topology. It is clear that if v is a bounded additive vector valued set function on Σ, the integral $\int f(s)v(ds)$ of a bounded Σ-measurable function f may be defined similarly. Thus if E is a bounded additive set function on Σ whose values $E(\delta)$ are bounded operators in the B-space $\mathfrak{X}$, and if $x \in \mathfrak{X}$, then the integral $\int f(s)E(ds)x$ is defined for every bounded Σ-measurable function f on S. It follows immediately that

$$\left[\int_S f(s)E(ds)\right] x = \int_S f(s)E(ds)x.$$

Similarly if $x \in \mathfrak{X}$ and $x^* \in \mathfrak{X}^*$, then every bounded Σ-measurable function f on S is integrable with respect to the bounded additive scalar set function $x^*E(\delta)x$, $\delta \in \Sigma$, and

$$x^* \left[\int_S f(s)E(ds)\right] x = \int_S f(s)x^*E(ds)x.$$

In Hilbert space this identity takes the form

$$\left(\left[\int_S f(s)E(ds)\right] x, y\right) = \int_S f(s)(E(ds)x, y), \qquad x, y \in \mathfrak{H}.$$

The notation $\int_S f(s)E(ds)$ is self-explanatory, but perhaps it will help if we mention explicitly that the symbols

$$\int_S f(s)\int_{ds} g(t)E(dt), \qquad \int_S f(s)E(ds\cap\delta),$$

refer to the integral of f with respect to the additive operator valued set functions whose values on a set $\sigma\in\Sigma$ are

$$\int_\sigma g(t)E(dt), \qquad E(\sigma\cap\delta),$$

respectively. The integral $\int_S f(s)E(ds\cap\delta)$ is itself a bounded additive set function for δ in Σ, and thus the integral of a function g in $B(S,\Sigma)$ with respect to this set function is written as

$$\int_S g(t)\int_S f(s)E(ds\cap dt).$$

If f is the characteristic function of a set $\delta\in\Sigma$, then

$$\text{(ix)}\qquad \int_S g(t)\int_S f(s)E(ds\cap dt) = \int_S g(t)f(t)E(dt),$$

and since both terms of this equation are linear and continuous for f in $B(S,\Sigma)$, we see that (ix) holds for all $f, g\in B(S,\Sigma)$.

Let S be a normal topological space, and let E be regular in the sense that the scalar measure $x^*E(\cdot)x$ is regular for each x in $\mathfrak{X}$ and x^* in $\mathfrak{X}^*$. Then the vanishing of the integral $\int_S f(s)E(ds)$ for every bounded continuous function f on S implies that

$$x^*\left[\int_S f(s)E(ds)\right]x = \int_S f(s)x^*E(ds)x = 0, \qquad f\in C(S),$$

and hence (IV. 6.2) that $x^*E(\delta)x = 0$ for every Borel set δ in S and every pair x, x^* with $x\in\mathfrak{X}, x^*\in\mathfrak{X}^*$. It follows (II.3.15) that $E(\delta) = 0$. Thus if E and A are bounded additive regular operator valued set functions defined on the Borel sets of a normal topological space S, and if $\int_S f(s)E(ds) = \int_S f(s)A(ds)$ for every bounded continuous function f on S, then $E(\delta) = A(\delta)$ for every Borel set δ in S.

Another property of the integral which will be used frequently is the *change of measure principle.* In the statement of this principle we have two spectral measures E and E_1 on Σ which are related by the equation

$$E(\delta) = E_1(h^{-1}(\delta)), \qquad \delta\in\Sigma,$$

where h is a map of S into itself with the property that for every δ in Σ the set $h^{-1}(\delta) = \{s|h(s)\in\delta\}$ is in Σ. If f is the characteristic

function of a set δ in Σ then the equation relating E and E_1 may be written as

$$\int_S f(s)E(ds) = \int_S f(h(s))E_1(ds).$$

Since both sides of this equation are linear and continuous for f in $B(S, \Sigma)$, it is seen that the equation holds for every bounded Σ-measurable function f.

The elementary properties of the integral $\int_S f(s)E(ds)$ which have been discussed in this section will be used repeatedly throughout this chapter, usually without explicit reference to the property in question.

2. The Spectral Theorem for Bounded Normal Operators

Before proving that a bounded normal operator in Hilbert space has a resolution of the identity, we will prove the following more general theorem which will be used frequently when its special case is not readily applicable. This more general theorem gives a spectral measure which reduces simultaneously each member of an arbitrary family of commuting normal operators. This general theorem and the Gelfand-Naimark theorem (IX.3.7) together provide the key to the whole theory of normal operators in Hilbert space. This chapter is, in a sense, an enumeration of corollaries and special cases of these two results. In the next chapter will be found applications of these two theorems to diverse problems in analysis.

→ 1 Theorem. (*General spectral theorem*) *Every commutative B^*-algebra $\mathfrak{A}$ of operators in Hilbert space $\mathfrak{H}$ is isometrically *-equivalent to the algebra $C(\Lambda)$ of all complex continuous functions on the spectrum Λ of $\mathfrak{A}$. Furthermore every isometric *-isomorphism $f \leftrightarrow T(f)$ between these algebras determines uniquely a spectral measure E defined on the Borel sets $\mathscr{B}$ in Λ which has the following properties:*

(i) *for every $x, y \in \mathfrak{H}$ the set function $(E(\sigma)x, y)$, $\sigma \in \mathscr{B}$, is a regular countably additive set function on $\mathscr{B}$,*

(ii) $E(\delta)T = TE(\delta)$, $\quad E(\delta) = E(\delta)^*$, $\quad \delta \in \mathscr{B}$, $\quad T \in \mathfrak{A}$,

(iii) $T(f) = \int_\Lambda f(\lambda)E(d\lambda)$, $\quad f \in C(\Lambda)$.

Proof. The first statement has been proved in Corollary IX.3.9. The proof of the second will require the following lemma.

2 LEMMA. *A bounded bilinear form which satisfies the identity* $[x, y] = \overline{[y, x]}$ *uniquely determines a bounded self adjoint operator* A *which satisfies the identity* $[x, y] = (Ax, y)$.

PROOF. For fixed y the number $[x, y]$ depends linearly and continuously upon x, and hence (IV.4.5) there is a point $Ay \in \mathfrak{H}$ with $[x, y] = (x, Ay)$. Since $[x, y]$ is bounded and bilinear, A is bounded and linear, and the formula $[x, y] = \overline{[y, x]}$ shows that A is self adjoint. Q.E.D.

Proceeding now to the proof of the theorem we observe that for each pair $x, y \in \mathfrak{H}$ the number $(T(f)x, y)$ is linear in f and $|(T(f)x, y)| \leq |f||x||y|$. Thus, by the Riesz representation theorem (IV.6.3), there is a uniquely determined regular measure $\mu(\cdot, x, y)$ on $\mathscr{B}$ with

$$\text{(a)} \qquad (T(f)x, y) = \int_\Lambda f(\lambda)\mu(d\lambda, x, y), \qquad f \in C(\Lambda),$$

$$\text{(b)} \qquad |\mu(e, x, y)| \leq v(\mu(\cdot, x, y), e) \leq |x||y|, \qquad e \in \mathscr{B}.$$

Since $(T(f)\alpha x, y) = \alpha(T(f)x, y)$ we have from (a),

$$\int_\Lambda f(\lambda)\mu(d\lambda, \alpha x, y) = \int_\Lambda f(\lambda)\alpha\mu(d\lambda, x, y), \qquad f \in C(\Lambda),$$

and, since the regular measure is uniquely determined by the functional, it is seen that $\mu(\delta, \alpha x, y) = \alpha\mu(\delta, x, y)$. Similarly, it may be shown that $\mu(\delta, x, y)$ is bilinear in x and y. Now if f is real then $T(f) = T(\bar{f}) = T(f)^*$ and so $(T(f)x, y) = \overline{(T(f)y, x)}$. It follows from (a) that

$$\int_\Lambda f(\lambda)\mu(d\lambda, x, y) = \int_\Lambda f(\lambda)\overline{\mu(d\lambda, y, x)}, \qquad f \in C(\Lambda),$$

and thus, by the uniqueness argument, that $\mu(\delta, x, y) = \overline{\mu(\delta, y, x)}$. Lemma 2 and (b) show that for each δ in $\mathscr{B}$ there is a uniquely defined bounded self adjoint operator $E(\delta)$ such that $\mu(\delta, x, y) = (E(\delta)x, y)$. It is clear that $E(\delta)$ is additive on $\mathscr{B}$ and from (b) it is seen that $|E(\delta)| \leq 1$. Thus (iii) follows from (a). To see that $E(\delta)E(\sigma) = E(\delta \cap \sigma)$ we note that for every pair f, g in $C(\Lambda)$ we have

$$\begin{aligned}\int_\Lambda f(\lambda)\int_\Lambda g(\mu)E(d\mu \cap d\lambda) = \int_\Lambda f(\lambda)\int_{e\lambda} g(\mu)E(d\mu) \\ = \int_\Lambda f(\lambda)g(\lambda)E(d\lambda) = T(fg) \\ = T(f)T(g) = \int_\Lambda f(\lambda)T(g)E(d\lambda) \\ = \int_\Lambda f(\lambda)\int_\Lambda g(\mu)E(d\mu)E(d\lambda).\end{aligned}$$

Thus

$$\int_\Lambda g(\mu)E(d\mu \cap \delta) = \int_\Lambda g(\mu)E(d\mu)E(\delta), \qquad \delta \in \mathscr{B}, \qquad g \in C(\Lambda),$$

and hence $E(\sigma \cap \delta) = E(\sigma)E(\delta)$ for every pair σ, δ in $\mathscr{B}$. Hence E is a spectral measure and, in particular, all of the projections $E(\delta)$ commute. It follows then from (iii) that the projections $E(\delta)$ also commute with $T(f)$ and this completes the proof of the theorem. Q.E.D.

3 COROLLARY. *The spectral measure is countably additive in the strong operator topology.*

PROOF. If $\{\delta_n\}$ is a sequence of Borel sets decreasing to the void set then $|E(\delta_n)x|^2 = (E(\delta_n)x, E(\delta_n)x) = (E(\delta_n)x, x) \to 0$ by (i). Q.E.D.

The preceding calculation also shows that the regularity of the scalar measure $(E(\delta)x, y)$ for every x, y in $\mathfrak{H}$ implies that the vector measure $E(\delta)x$ is regular in the strong topology of $\mathfrak{H}$, i.e., for every $\delta \in \mathscr{B}$ and $\varepsilon > 0$ there is a closed set $F \subseteq \delta$ and an open set $G \supseteq \delta$ such that $|E(\sigma)x| < \varepsilon$ for every σ in $\mathscr{B}$ with $\sigma \subseteq G - F$. This follows since $|E(\sigma)x|^2 = (E(\sigma)x, x)$. Thus, for a *self adjoint* spectral measure (i.e., a spectral measure E with $E(\delta) = E(\delta)^*$) *the notions of countable additivity and regularity are the same in the weak operator topology as in the strong operator topology.* These notions will never be used in the uniform operator topology since, in any B-space, every projection $E \neq 0$ has $|E| \geqq 1$. Thus we may use expressions such as "regular, countably additive self adjoint spectral measure" without ambiguity.

4 COROLLARY. *A bounded normal operator T uniquely determines a regular countably additive self adjoint spectral measure E on the Borel sets of the complex plane which vanishes on $\rho(T)$ and has the property that*

$$f(T) = \int_{\sigma(T)} f(\lambda)E(d\lambda), \qquad f \in C(\sigma(T)).$$

PROOF. If we put $E(\delta) = 0$ when $\delta \cap \sigma(T)$ is void, then Corollary 4 follows immediately from Theorem 1 and Corollary IX.3.15. Q.E.D.

5 DEFINITION. The uniquely defined spectral measure associated, in Corollary 4, with the normal operator T is called the *resolution of the identity* for T.

In order to relate this notion of the resolution of the identity with that given in Section 1 we state the following corollary.

6 COROLLARY. *If E is the resolution of the identity for the normal operator T and if δ is a Borel set of complex numbers, then*

$$E(\delta)T = TE(\delta), \qquad \sigma(T_\delta) \subseteq \bar{\delta},$$

where T_δ is the restriction of T to $E(\delta)\mathfrak{H}$.

PROOF. The first statement follows from Theorem 1(ii). Now for $\xi \notin \bar{\delta}$ it is seen from Theorem 1.1 and Corollary 4 that the operator $R = \int(\xi - \lambda)^{-1}\chi_\delta(\lambda)E(d\lambda)$ satisfies the equation $R(\xi I - T) = E(\delta)$. Q.E.D.

7 COROLLARY. *Let E be a regular countably additive self adjoint spectral measure defined on the Borel sets in the complex plane. Then E is the resolution of the identity for the normal operator T if and only if $T = \int_{\sigma(T)} \lambda E(d\lambda)$.*

PROOF. If $T = \int_{\sigma(T)} \lambda E(d\lambda)$ and E is self adjoint then $T^* = \int_{\sigma(T)} \bar{\lambda} E(d\lambda)$. Thus, by Theorem 1.1, for any polynomial $p(\lambda, \bar{\lambda})$ in λ and $\bar{\lambda}$ we have $p(T, T^*) = \int_{\sigma(T)} p(\lambda, \bar{\lambda})E(d\lambda)$. It follows from the Weierstrass approximation theorem that $f(T) = \int_{\sigma(T)} f(\lambda)E(d\lambda)$ for every f in $C(\sigma(T))$. Thus, by Corollary 4, E is the resolution of the identity for T. The converse statement follows directly from Corollary 4. Q.E.D.

→ 8 COROLLARY. *Let E be the resolution of the identity for the bounded normal operator T and for each complex bounded Borel function on the spectrum $\sigma(T)$ let*

$$\text{(i)} \qquad f(T) = \int_{\sigma(T)} f(\lambda)E(d\lambda).$$

*Then the map $f \to f(T)$ is a continuous *-homomorphism of the B^*-*

algebra of bounded Borel functions on $\sigma(T)$ into the B^*-algebra of bounded operators in Hilbert space with the property that the functions $f(\lambda) \equiv \lambda$ and $f(\lambda) \equiv 1$ map into the operators T and I respectively. This homomorphism has the further properties

$$\text{(ii)} \qquad |f(T)x|^2 = \int_{\sigma(T)} |f(\lambda)|^2 (E(d\lambda)x, x), \qquad x \in \mathfrak{H};$$

(iii) *if the uniformly bounded sequence $\{f_n\}$ of complex Borel functions converges at each point of $\sigma(T)$ to the function f then $f_n(T) \to f(T)$ in the strong operator topology.*

Proof. By Theorem 1.1 the map $f \to f(T)$ is a continuous $*$-homomorphism. By Corollary 4 the functions λ, 1 map into the operators T, I respectively. To prove (ii) we have

$$\begin{aligned} |f(T)x|^2 &= (f(T)x, f(T)x) = (f(T)^*f(T)x, x) \\ &= (\bar{f}(T)f(T)x, x) = \int_{\sigma(T)} |f(\lambda)|^2 (E(d\lambda)x, x). \end{aligned}$$

Statement (iii) follows from (ii) since

$$|f_n(T)x - f(T)x|^2 = \int_{\sigma(T)} |f_n(\lambda) - f(\lambda)|^2 (E(d\lambda)x, x) \to 0$$

by the dominated convergence theorem of Lebesgue. Q.E.D.

It is desirable at times to have an operational calculus similar to that given in Theorem 1.1, but which represents an isometric $*$ isomorphism (rather than just a continuous $*$-homomorphism) between a commutative B^*-algebra of operators in Hilbert space and a B^*-algebra whose elements are equivalence classes of functions. In using such a calculus, which we shall now describe, one may form bounded operators from functions which may not be bounded. Let E be a countably additive self adjoint spectral measure defined on the σ field Σ of subsets of a set S. A scalar or vector valued function f on S is said to be *E-essentially bounded* if the number

$$E\text{-ess}\sup_{s \in S} |f(s)| = \inf_{E(\delta) = I} \sup_{s \in \delta} |f(s)|$$

is finite. Since E is countably additive there is a set δ_0 in Σ with $E(\delta_0) = I$ and

$$E\text{-ess}\sup_{s \in S} |f(s)| = \sup_{s \in \delta_0} |f(s)|,$$

and thus there is a bounded function f_0 on S with $f(s) = f_0(s)$ except for s in a set having E measure zero. If f is Σ-measurable then f_0 is a bounded Σ-measurable function, i.e., an element of the B^*-algebra $B(S, \Sigma)$. The algebra $EB(S, \Sigma)$ of E-essentially bounded Σ-measurable functions on S is the B^*-algebra whose elements are equivalence classes of Σ-measurable scalar functions on S determined by the bounded Σ-measurable scalar functions on S in such a way that each equivalence class consists of all Σ-measurable functions which differ from some bounded Σ-measurable function only on a set of E measure zero. That is, $EB(S, \Sigma)$ is $B(S, \Sigma)$ reduced modulo E-null sets. The norm in $EB(S, \Sigma)$ is

$$|f| = E\text{-ess sup } |f(s)|.$$

Even though the elements of $EB(S, \Sigma)$ are equivalence classes we shall, for convenience, speak of these elements as functions on S. The situation is quite analogous to that encountered in the space $L_\infty(S, \Sigma, \mu)$. The algebraic operations in $EB(S, \Sigma)$ are the natural ones, and in particular the involution operation is defined as $f^* = \bar{f}$ where, as usual, $\bar{f}(\lambda) = \overline{f(\lambda)}$. For a bounded Σ-measurable function g on S we have $\int_\delta g(s)E(ds) = \int_S g(s)E(ds)$ if $E(\delta) = I$, and thus we may define the *integral* of an E-essentially bounded function f as the integral of any bounded Σ-measurable function g which differs from f only on a set of E measure zero. It is seen from Theorem 1.1 that the map $f \to \int_S f(s)E(ds)$ of $EB(S, \Sigma)$ into the algebra of operators in Hilbert space is a continuous $*$-homomorphism. The next result shows more, namely that this map is an isometric $*$-isomorphism.

9 COROLLARY. *Let E be a countably additive self adjoint spectral measure on the σ-field Σ of subsets of a set S. Then the map $f \to T(f)$, of $EB(S, \Sigma)$ into the algebra of operators in Hilbert space, which is defined by the equation*

$$\text{(i)} \qquad T(f) = \int_S f(s)E(ds), \qquad f \in EB(S, \Sigma),$$

is an isometric $*$-isomorphism with the following properties:

(ii) *the inverse $T(f)^{-1}$ exists as a bounded operator if and only if $1/f$ is E-essentially bounded on S;*

$$\text{(iii)} \qquad \sigma(T(f)) = \bigcap_{E(\delta)=I} \overline{f(\delta)}, \qquad f \in EB(S, \Sigma);$$

$$\text{(iv)} \qquad |T(f)x|^2 = \int_S |f(\lambda)|^2 (E(d\lambda)x, x), \qquad f \in EB(S, \Sigma);$$

(v) *if* $\{f_n\}$ *is a bounded sequence in* $EB(S, \Sigma)$ *and if* $\lim_n f_n(s) = f(s)$ *except for those* s *in a set of* E*-measure zero, then* $T(f_n)x \to T(f)x$ *for every* x *in Hilbert space.*

Proof. Statements (iv) and (v) may be proved just as the corresponding statements in Corollary 8 were proved. To see that the map $f \to T(f)$ is isometric let $\delta \in \Sigma$ and $E(\delta) = I$. Then

$$|T(f)| = \left|\int_\delta f(\lambda)E(d\lambda)\right| \leqq \sup_{\lambda \in \delta} |f(\lambda)|,$$

and hence $|T(f)| \leq E\text{-ess}_{s \in S} \sup |f(s)|$. Conversely, let $r < E\text{-ess}_{s \in S} \sup |f(s)|$ and $\delta_r = \{s \mid |f(s)| > r\}$, so that $E(\delta_r) \neq 0$ and, for some vector x, $0 \neq x = E(\delta_r)x$. Thus, by (iv),

$$\begin{aligned} |T(f)x|^2 &= \int_S |f(\lambda)|^2 (E(d\lambda)E(\delta_r)x, x) \\ &= \int_S |f(\lambda)|^2 (E(d\lambda \cap \delta_r)x, x) \\ &= \int_{\delta_r} |f(\lambda)|^2 (E(d\lambda)x, x) \geq r^2|x|^2, \end{aligned}$$

which, since $x \neq 0$, shows that $|T(f)| \geqq r$. Since r was an arbitrary number less than E-ess sup $|f(s)|$ we have $|T(f)| \geq E\text{-ess}_{s \in S} \sup |f(s)|$, which shows that $|T(f)| = |f|$ and proves that the map $f \to T(f)$ is an isometric *-isomorphism. To prove (ii) we note that if $1/f$ is E-essentially bounded on S then $T(1/f) \cdot T(f) = T(1) = I$ and $T(1/f)$ exists. Conversely if $T(f)^{-1}$ exists as a bounded operator in Hilbert space, let $r > |T(f)^{-1}|$. We wish to show that $E(\delta_r) = 0$, where $\delta_r = \{s \mid |1/f(s)| > r\} = \{s \mid |f(s)| < r^{-1}\}$. If not, there is a vector x with $0 \neq x = E(\delta_r)x$, and thus, by (iv),

$$|T(f)x|^2 = \int_{\delta_r} |f(s)|^2 (E(ds)x, x) \leq r^{-2}|x|^2.$$

Therefore $|x| = |T(f)^{-1}T(f)x| < r|T(f)x| \leq |x|$ which is a contradiction that completes the proof of (ii). Finally, to prove (iii) suppose that $\lambda_0 \notin \overline{f(\delta)}$, where $E(\delta) = I$. Then $1/(\lambda_0 - f)$ is an E-essentially bounded function and thus $\lambda_0 \in \rho(T(f))$. This proves that $\overline{f(\delta)} \supseteqq \sigma(T(f))$ and thus that

$$\bigcap_{E(\delta)=I} \overline{f(\delta)} \supseteqq \sigma(T(f)).$$

Conversely, if $\lambda_0 \in \rho(T(f))$ then, by (ii), $1/(\lambda_0-f)$ is E-essentially bounded and thus there is a set $\delta \in \Sigma$ with $E(\delta) = I$ and $\lambda_0 \notin \overline{f(\delta)}$. This shows that $\sigma(T(f)) \supseteq \bigcap_{E(\delta)=I} \overline{f(\delta)}$ and completes the proof of the corollary. Q.E.D.

10 COROLLARY. *Let $f \in EB(S, \Sigma)$ where E is the spectral measure of Corollary 9, and let E_f be the resolution of the identity for the operator $T(f)$. Then, for every Borel set δ of complex numbers, $E_f(\delta) = E(f^{-1}(\delta))$.*

PROOF. Since f is essentially bounded there is a constant M and a set $\sigma_0 \in \Sigma$ with $E(\sigma_0) = I$ and $|f(s)| \leq M$ for s in σ_0. Let $T = T(f)$. Then, by 9(ii), for $\lambda_1 \in \rho(T)$ the function $1/(\lambda_1 - f)$ is E-essentially bounded and thus there is a neighborhood N_1 of λ_1 with $E(f^{-1}(N_1)) = 0$. A finite number of these neighborhoods cover the set δ_n of those complex numbers λ with $|\lambda| \leq M$ and whose distance from $\sigma(T)$ is at least $1/n$. Thus $E(f^{-1}(\delta_n)) = 0$. Since E is countably additive $E(f^{-1}(\rho(T))) = 0$. Hence $E(f^{-1}(\delta \cap \sigma(T))) = E(f^{-1}(\delta))$ for every Borel set δ in the plane. Thus by the change of measure principle

$$\int_{\sigma(T)} \lambda E(f^{-1}(d\lambda)) = \int_S f(s) E(ds) = T,$$

and hence, by Corollary 7, $E_f(\delta) = E(f^{-1}(\delta))$. Q.E.D.

3. Eigenvalues and Eigenvectors

One way in which the spectral theory in a general B-space (or Hilbert space) differs from the theory in a finite dimensional space is that, in the general case, a number μ may be in the spectrum $\sigma(T)$ of the operator T and yet $\mu I - T$ may have an inverse. A number of such examples were encountered in Section VII.5 where also a general classification of spectral points was given. We shall repeat here this general classification as applied to operators in Hilbert space and then study some of its connections with the theory of normal operators.

1 DEFINITION. Let $\sigma(T)$ be the spectrum of the bounded linear operator T in the Hilbert space $\mathfrak{H}$. The set of those complex numbers λ in $\sigma(T)$ for which $\lambda I - T$ is not one-to-one is called the *point spectrum* of T and is denoted by $\sigma_p(T)$. Every number μ in $\sigma_p(T)$ is called an *eigenvalue* of T, and every vector $x \neq 0$ with $(\mu I - T)x = 0$ for some

$\mu \in \sigma_p(T)$ is called an ***eigenvector of T associated with the eigenvalue*** μ, or simply an ***eigenvector*** of T. The set of all $\mu \in \sigma(T)$ for which $\mu I - T$ is one-to-one and for which the manifold $(\mu I - T)\mathfrak{H}$ is dense in but not equal to $\mathfrak{H}$ is called the ***continuous spectrum*** of T and is denoted by $\sigma_c(T)$. The set of all $\mu \in \sigma(T)$ for which $\mu I - T$ is one-to-one and for which the manifold $(\mu I - T)\mathfrak{H}$ is not dense in $\mathfrak{H}$ is called the ***residual spectrum*** of T and is denoted by $\sigma_r(T)$.

It should be observed that if $\lambda I - T$ is one-to-one and if $(\lambda I - T)\mathfrak{H} = \mathfrak{H}$ then (II.2.2) $(\lambda I - T)^{-1}$ is a bounded linear operator defined on $\mathfrak{H}$ and therefore λ is not in $\sigma(T)$. This shows that every point in the complex plane is in one and only one of the sets $\rho(T)$, $\sigma_p(T)$, $\sigma_c(T)$, $\sigma_r(T)$. It should also be recalled that $\sigma(T)$ is a non-void closed set (IX.1.5) contained in the disk $\{\lambda \mid |\lambda| \leq |T|\}$, (VII.3.2). These facts are summarized in the following lemma.

2 Lemma. *Let T be a bounded operator in Hilbert space. Then the resolvent set $\rho(T)$ is open and the spectrum is a closed non-void subset of the set $\{\lambda \mid |\lambda| \leq |T|\}$. Furthermore the sets $\rho(T)$, $\sigma_p(T)$, $\sigma_c(T)$, and $\sigma_r(T)$ are disjoint and their union is the whole plane.*

To illustrate these notions consider the Hilbert space l_2 of all sequences $x = \{\alpha_i\}$ of complex numbers for which $|x| = (\sum |\alpha_i|^2)^{1/2} < \infty$. The scalar product in l_2 is $(x, y) = \sum \alpha_i \bar{\beta}_i$ where $x = \{\alpha_i\}$ and $y = \{\beta_i\}$. Let $y = Tx$ be the shift operator in l_2 defined by the relation $y = \{\alpha_2, \alpha_3, \ldots\}$ where $x = \{\alpha_1, \alpha_2, \alpha_3, \ldots\}$. Since $|T| \leq 1$ we see, from Lemma 2, that the spectrum $\sigma(T)$ of T is contained in the unit disk $\{\lambda \mid |\lambda| \leq 1\}$. If $|\lambda| < 1$ then the sequence $x = \{1, \lambda, \lambda^2, \ldots\}$ is in l_2 and $Tx = \lambda x$ so that every λ with $|\lambda| < 1$ is an eigenvalue of T. Since $\sigma(T)$ is closed and contained in the unit disk we must have $\sigma(T) = \{\lambda \mid |\lambda| \leq 1\}$. It is clear that the only eigenvectors associated with the eigenvalue λ where $|\lambda| < 1$ are constant multiples of $\{1, \lambda, \lambda^2, \ldots\}$ so that the manifold $\{x \mid x \in l_2, (T - \lambda I)x = 0\}$ is one dimensional. If $|\lambda| = 1$ then, since $\{1, \lambda, \lambda^2, \ldots\}$ is not in l_2, we have $\lambda \notin \sigma_p(T)$. Such a λ is in $\sigma_c(T)$. To see this let $y = \{\beta_i\}$ be an arbitrary vector in l_2 and let k be an integer with $\sum_{i=k}^{\infty} |\beta_i|^2 < \varepsilon^2$ where ε is a given positive number. Then clearly we may find a sequence $x = \{\alpha_n\}$ with $\alpha_{n+1} = 0$ for $n > k$ and with $\lambda\alpha_n - \alpha_{n+1} = \beta_n$ for $n \leq k$. Thus

$$|(\lambda I - T)x - y| = \Big(\sum_{n=k+1}^{\infty} |\beta_n|^2\Big)^{1/2} < \varepsilon,$$

which shows that $\sigma_c(T) = \{\lambda \mid |\lambda| = 1\}$ and that $\sigma_r(T)$ is void. Examples may also be given of operators A with $\sigma_r(A) = \sigma(A)$; for such operators the point spectrum and the continuous spectrum are void.

The following lemma shows that, for a normal operator, the nature of a spectral point is determined by the resolution of the identity.

3 Lemma. *If E is the resolution of the identity for the normal operator T, then*

(i) *if δ is non-void and open in the relative topology of $\sigma(T)$ then $E(\delta) \neq 0$;*

(ii) *the point spectrum $\sigma_p(T)$ of T consists of all complex numbers μ for which $E(\{\mu\}) \neq 0$;*

(iii) *the residual spectrum $\sigma_r(T)$ of T is void.*

Proof. Let δ be open, $E(\delta) = 0$, and $\lambda_0 \in \delta \cap \sigma(T)$. Then $(\lambda_0 - \lambda)^{-1}$ is E-essentially bounded on $\sigma(T)$ and, by Corollary 2.9(ii), $\lambda_0 \in \rho(T)$, a contradiction which proves (i). To prove (ii) let $E(\{\mu\})x = x \neq 0$. Then

$$
\begin{aligned}
Tx &= \int_{\sigma(T)} \lambda E(d\lambda) E(\{\mu\}) x \\
&= \int_{\sigma(T)} \lambda E(d\lambda \cap \{\mu\}) x = \mu x,
\end{aligned}
$$

so $\mu \in \sigma_p(T)$. Conversely, if $x \neq 0$ and $Tx = \mu x$, let

$$
\begin{aligned}
f_n(\lambda) &= \frac{1}{\mu - \lambda}, \qquad |\mu - \lambda| > \frac{1}{n}, \\
&= 0, \qquad\qquad |\mu - \lambda| < \frac{1}{n}.
\end{aligned}
$$

Then $f_n(T)(\mu I - T) = E(\{\lambda \mid |\lambda - \mu| > 1/n\})$ and, since $(\mu I - T)x = 0$, we have $E(\{\lambda \mid |\lambda - \mu| > 1/n\})x = 0$. By letting $n \to \infty$ it is seen from Corollary 2.8(iii) that $E(\{\lambda \mid \lambda \neq \mu\})x = 0$ and thus $E(\{\mu\})x = x \neq 0$. To prove (iii) it will suffice to show that if $(\mu I - T)\mathfrak{H}$ is not dense in $\mathfrak{H}$ then $\mu \in \sigma_p(T)$. For such a μ there is, by Corollary II.3.13 and Theorem IV.4.5, an $x \neq 0$ orthogonal to $(\mu I - T)\mathfrak{H}$, and hence $(\bar{\mu} I - T^*)x = 0$. By (ii), the resolution of the identity A for T^* has $A(\{\bar{\mu}\}) \neq 0$. Thus by Corollary 2.10, $0 \neq A(\{\bar{\mu}\}) = E(\{\mu\})$ and, by (ii), $\mu \in \sigma_p(T)$. Q.E.D.

4 THEOREM. *If the spectrum of the bounded normal operator T in $\mathfrak{H}$ is countable then there is an orthonormal basis B for $\mathfrak{H}$ consisting of eigenvectors of T. Furthermore,*

$$x = \sum_{y \in B} (x, y) y, \qquad x \in \mathfrak{H},$$

and, for each x, all but a countable number of the coefficients (x, y) are zero.

PROOF. Let $\sigma(T) = \{\mu_1, \mu_2, \ldots\}$ and $\mathfrak{H}_n = E(\mu_n)\mathfrak{H}$ where E is the resolution of the identity for T. If $\mathfrak{H}_n \neq 0$ it follows from Lemma 3 that it consists entirely of eigenvectors of T. By Theorem IV.4.12 each $\mathfrak{H}_n$ has an orthonormal basis B_n. If we let $B = \bigcup_{n=1}^{\infty} B_n$, then every element of B is an eigenvector of T. Since $E(\mu_n)x = x$ for x in B_n and $E(\mu_n)E(\mu_m) = 0$ if $n \neq m$, we see that B is an orthonormal set. Also, by Corollary 2.3,

$$x = \int_{\sigma(T)} E(d\lambda)x = \sum_{n=1}^{\infty} E(\mu_n)x, \qquad x \in \mathfrak{H},$$

and thus no non-zero vector is orthogonal to every element of B. Hence B is complete and, by Theorem IV.4.13, B is an orthonormal basis for $\mathfrak{H}$. The remaining two assertions follow from Definition IV.4.11 and Theorem IV.4.10. Q.E.D.

→ 5 COROLLARY. *The spectrum of a compact normal operator T in $\mathfrak{H}$ is countable and has no point of accumulation in the complex plane except possibly $\mu = 0$. Every non-zero spectral point is an eigenvalue and the number of linearly independent eigenvectors associated with a non-zero spectral point is finite. There is an orthonormal basis for $\mathfrak{H}$ consisting of eigenvectors of T.*

PROOF. The first statements follow from Theorem VII.4.5 and the last from Theorem 4. Q.E.D.

4. Unitary, Self Adjoint, and Positive Operators

There are several special types of normal operators which occur frequently in mathematical analysis and which will be studied briefly in this section. These special types are described in the following definition.

1 DEFINITION. A bounded operator T in Hilbert space $\mathfrak{H}$ is called *unitary* if $TT^* = T^*T = I$; it is called *self adjoint*, *symmetric* or *Hermitian* if $T = T^*$; *positive* if it is self adjoint and if $(Tx, x) \geq 0$ for every x in $\mathfrak{H}$; and *positive definite* if it is positive and $(Tx, x) > 0$ for every $x \neq 0$ in $\mathfrak{H}$.

It is clear that all of these classes of operators are normal.

Unitary operators have a number of other characteristic properties. For example, if U is unitary then $(x, y) = (U^*Ux, y) = (Ux, U^{**}y) = (Ux, Uy)$. Conversely, let U be an operator satisfying the identity $(x, y) = (Ux, Uy)$. Then $(x, y) = (U^*Ux, y)$ and thus, if U has an inverse, $U^{-1} = U^*$, which shows that U is unitary. In other words a unitary operator is an isomorphism of $\mathfrak{H}$ with itself which preserves the inner product (and consequently preserves all the properties of $\mathfrak{H}$). For this reason two operators A and B in $\mathfrak{H}$ which are related by the equation $A = U^*BU$, where U is unitary, have identical properties as operators in $\mathfrak{H}$. Two such operators are called *unitarily equivalent.*

The Hermitian operators are a subclass of $B(\mathfrak{H})$ which play a role in $B(\mathfrak{H})$ much resembling the role of the real numbers as a subclass of the class of all complex numbers. In particular, every $T \in B(\mathfrak{H})$ can be written uniquely in the form $T = A+iB$, where A and B are Hermitian operators. Clearly, A and B must be given by the formulae

$$A = \frac{T+T^*}{2}, \qquad B = \frac{T-T^*}{2i}.$$

It is clear that T is normal if and only if its "real" and "imaginary" parts A and B commute.

The notion of a positive operator allows us to introduce a notion of order into the space $B(\mathfrak{H})$: we write $S \leq T$ if $T - S$ is a positive operator. With this notion of order, $B(\mathfrak{H})$ becomes a partially ordered vector space with many interesting properties, some of which are given in the exercises at the end of the chapter.

The relation between unitary operators and complex numbers of unit modulus, Hermitian operators and real numbers, and positive operators and positive numbers is illustrated in the following theorem.

2 THEOREM. *A bounded normal operator is unitary, Hermitian, or*

positive if and only if its spectrum lies on the unit circle, the real axis, or the non-negative real axis respectively.

Proof. If N is a bounded normal operator then, by Corollary IX.3.15, $NN^* = N^*N = I$ if and only if $\lambda\bar{\lambda} = 1$ for every spectral point λ of N, and $N = N^*$ if and only if $\lambda = \bar{\lambda}$ for every spectral point λ. Now let E be the resolution of the Identity for the self adjoint operator A. If the spectrum of A is non-negative then, by Corollary 2.7, $(Ax, x) = \int_{\sigma(A)} \lambda(E(d\lambda)x, x)$, which shows that the operator A is positive. Conversely, if an open interval δ of negative numbers intersects the spectrum then, by Lemma 3.3(i), $E(\delta) \neq 0$. If $0 \neq x = E(\delta)x$, then $(Ax, x) = \int_{\sigma(A)} \lambda(E(d\lambda)E(\delta)x, x) = \int_{\sigma(A)} \lambda(E(d\lambda \cap \delta)x,x) = \int_{\sigma(A) \cap \delta} \lambda(E(d\lambda)x, x) < 0$, which shows that A is not positive. Q.E.D.

The relations, suggested in the preceding lemma, between the spectrum of a normal operator N and the values (Nx, x) can be amplified considerably (cf. Exercise 8.8). Here, we shall investigate these relations in the particular case in which N is compact and self adjoint. In this case the spectrum $\sigma(N)$, by Corollary 3.5, consists of a sequence $\{\lambda_n^+\}$ (possibly finite or void) of positive numbers, a sequence $\{\lambda_n^-\}$ (possibly finite or void) of negative numbers and of zero (provided that $\mathfrak{H}$ is infinite dimensional). Moreover, it is seen from Corollary 3.5 that the set of eigenvectors belonging to any non-zero eigenvalue form a finite dimensional space. The dimension of this space is known as the *multiplicity* of the corresponding eigenvalue. Let us suppose the positive eigenvalues to be enumerated in decreasing order, each eigenvalue being repeated a number of times equal to its multiplicity: $\lambda_1 \geqq \lambda_2 \geqq \ldots$. The sequence $\{\lambda_n\}$ is either void (which we exclude for the time being), finite, or infinite; in the latter case, $\lambda_n \to 0$. Then

$$(Nx, x) = \int_{\sigma(T)} \lambda(E(d\lambda)x, x) \leq \lambda_1|x|^2,$$

so that $(Nx, x)/|x|^2 \leq \lambda_1$. Consequently λ_1, the largest positive eigenvalue, can be characterized by *the Rayleigh equation*

(i) $$\lambda_1 = \max_x\, (Nx, x)/|x|^2.$$

Having characterized λ_1, how can we characterize λ_2? It is easy to see that if x_1 is a non-zero eigenvector associated with λ_1 then

$$[*] \qquad \lambda_2 = \max_{(x, x_1)=0} (Nx, x)/|x|^2.$$

Since this characterization involves explicit knowledge of an eigenvector x_1, it is unsuitable for many purposes. A more satisfactory characterization is the following. If y is an arbitrary vector, we can find a non-zero linear combination $\alpha_1 x_1 + \alpha_2 x_2$ of x_1 and of an eigenvector x_2 belonging to the eigenvalue λ_2 and orthogonal to x_1 such that $(\alpha_1 x_1 + \alpha_2 x_2, y) = 0$. Since $|\alpha_1 x_1 + \alpha_2 x_2|^2 = |\alpha_1|^2 |x_1|^2 + |\alpha_2|^2 |x_2|^2$, we have

$$\begin{aligned}(N(\alpha_1 x_1 + \alpha_2 x_2), \alpha_1 x_1 + \alpha_2 x_2) &= \lambda_1 |\alpha_1|^2 |x_1|^2 + \lambda_2 |\alpha_2|^2 |x_2|^2 \\ &\geq \lambda_2 |\alpha_1 x_1 + \alpha_2 x_2|^2.\end{aligned}$$

Consequently,

$$\lambda_2 \leq \max_{(x, y)=0} (Nx, x)/|x|^2.$$

On the other hand, [*] shows that if we put $y = x_1$, our maximum is precisely λ_2. Thus, we can write

$$\lambda_2 = \min_{y} \max_{(x, y)=0} (Nx, x)/|x|^2.$$

In the same way, it may be shown that

$$\lambda_3 = \min_{y_1, y_2} \max_{\substack{(x, y_1)=0 \\ (x, y_2)=0}} (Nx, x)/|x|^2,$$

and, in general

$$\text{(ii)} \qquad \lambda_{k+1} = \min_{y_1, \ldots, y_k} \max_{\substack{(x, y_i)=0 \\ i=1, \ldots, k}} (Nx, x)/|x|^2, \qquad k \geq 1,$$

The preceding discussion is summarized in the following theorem.

→ 8 THEOREM. *Let $\lambda_1 \geq \lambda_2 \ldots$ be the positive eigenvalues of the compact self adjoint operator N, each repeated a number of times equal to its multiplicity. Then the eigenvalues $\lambda_1, \lambda_2, \ldots$, are given by equations* (i) *and* (ii).

The characterization, given in Theorem 8, of the kth largest eigenvalue has, besides many theoretical applications, numerous applications to the approximate numerical calculation of eigenvalues. At present we shall only illustrate its usefulness by the following example. Let L and M be two compact self adjoint operators, and let

$\{\lambda_n\}$, $\{\mu_n\}$ be the corresponding sequences of positive eigenvalues each arranged in decreasing order with repetitions according to multiplicity. If $L \leqq M$ then $\lambda_n \leqq \mu_n$ for $n = 1, 2, \ldots$. The proof follows immediately, for since $L \leqq M$, we have $(Lx, x) \leq (Mx, x)$ for every x in $\mathfrak{H}$. Hence the characterization of λ_n, μ_n given in Theorem 3 shows that $\lambda_n \leq \mu_n$ for all $n = 1, 2, \ldots$.

5. Spectral Representation

Let μ be a finite positive measure defined on the Borel sets $\mathscr{B}$ of the complex plane and vanishing on the complement of a bounded set S. One of the simplest examples of a bounded normal operator is the operator T defined by the formula $(Tx)(\lambda) = \lambda x(\lambda), x \in L_2(S, \mathscr{B}, \mu)$. It is easily seen that the spectrum $\sigma(T)$ is the support of the measure μ, i.e., the complement of the largest open set δ for which $\mu(\delta) = 0$. The spectral resolution of T is defined for any Borel set e by the formula $(E(e)x)(\lambda) = \chi_e(\lambda)x(\lambda)$, and thus for any bounded Borel measurable function F the operator $F(T)$ is given by the formula $(F(T)x)(\lambda) = F(\lambda)x(\lambda)$. Our first purpose in this section is to show that, in a sense, the example just given is typical of the structure of every normal operator. More explicitly, if T is a normal operator in $\mathfrak{H}$ there exists a unitary mapping U of $\mathfrak{H}$ onto a suitable function space $L_2(S, \mathscr{B}, \mu)$ or a direct sum of such spaces such that UTU^{-1} has the form of "multiplication by λ." The precise meaning of this phrase is given in the following definition. Here and elsewhere the symbol $\sum_\alpha \mathfrak{H}_\alpha$ will be used for the direct sum of the Hilbert spaces $\mathfrak{H}_\alpha$ (cf. IV.4.18 and IV.4.19). The αth component of an element x in $\sum_\alpha \mathfrak{H}_\alpha$ will be denoted by x_α.

1 Definition. Let T be a normal operator in a Hilbert space $\mathfrak{H}$. Let $\{\mu_\alpha\}$ be a family of finite positive regular measures on the Borel sets of the complex plane. A map U of $\mathfrak{H}$ onto $\sum_\alpha L_2(\mu_\alpha)$ is a *spectral representation of* $\mathfrak{H}$ *onto* $\sum_\alpha L_2(\mu_\alpha)$ *relative to* T if the following conditions are satisfied:

(a) each measure μ_α vanishes on the complement of the spectrum of T;

(b) the operator U is a linear map of $\mathfrak{H}$ onto all of $\sum_\alpha L_2(\mu_\alpha)$ which preserves inner products;

(c) for every Borel function f which is bounded on the spectrum of T we have, for every x in $\mathfrak{H}$ and every α,

$$\big(U(f(T)x)\big)_\alpha(\lambda) = f(\lambda)(Ux)_\alpha(\lambda),$$

for μ_α-almost all λ.

We shall first discuss, in the following theorem, the situation where $\mathfrak{H}$ has a spectral representation onto a single space $L_2(\mu)$.

2 THEOREM. *Let T be a normal operator in a Hilbert space $\mathfrak{H}$. Suppose that for some vector x in $\mathfrak{H}$, linear combinations of the vectors $T^m T^{*n} x$, $m, n \geqq 0$, are dense in $\mathfrak{H}$. Then $\mathfrak{H}$ admits a spectral representation relative to T onto a Hilbert space $L_2(\mu)$.*

PROOF. Let E be the resolution of the identity for T, let $\mu = (E(\cdot)x, x)$, and let $\mathfrak{D}_1$ be the linear manifold in $\mathfrak{H}$ consisting of all vectors of the form $f(T)x$ where f is a bounded Borel function on $\sigma(T)$. By hypothesis $\mathfrak{D}_1$ is dense in $\mathfrak{H}$. We note that if $f(T)x = g(T)x$, then

$$0 = |\{f(T)-g(T)\}x|^2 = \int |f(\lambda)-g(\lambda)|^2 \mu(d\lambda),$$

which shows that $f = g$ μ-almost everywhere. Hence we may define the operator U_1 from $\mathfrak{D}_1$ to $L_2(\mu)$ by placing $U_1 f(T)x = f$. Clearly U_1 is linear and for $y = f(T)x$, $z = g(T)x$ we have, from Corollary 2.8,

$$(y, z) = \int_{\sigma(T)} f(\lambda)\overline{g(\lambda)}\mu(d\lambda) = (U_1 y, U_1 z).$$

This shows that U_1 preserves inner products and is thus one-to-one and continuous. It therefore has a unique extension U from $\bar{\mathfrak{D}}_1 = \mathfrak{H}$ to the L_2-closure of the set of bounded Borel functions, i.e., to $L_2(\mu)$. An elementary continuity argument shows that U is an isometric isomorphism between $\mathfrak{H}$ and $L_2(\mu)$. Now if $f_n(T)x \to y$, then $f_n \to Uy$ in $L_2(\mu)$ and, since $(Uf(T)f_n(T)x)(\lambda) = f(\lambda)f_n(\lambda)$, we have $(Uf(T)y)(\lambda) = f(\lambda)(Uy)(\lambda)$. Q.E.D.

Consider now the case that T is an arbitrary bounded normal operator on a Hilbert space $\mathfrak{H}$. Let us call a subspace $\mathfrak{H}_1 \subseteq \mathfrak{H}$ admissible if $T\mathfrak{H}_1 \subseteq \mathfrak{H}_1$, $T^*\mathfrak{H}_1 \subseteq \mathfrak{H}_1$, and there is an $x_1 \in \mathfrak{H}_1$, such that linear combinations of the vectors $T^n T^{*m} x_1$ are dense in $\mathfrak{H}_1$. An application of Zorn's lemma shows that there is a maximal family $\{\mathfrak{H}_\alpha\}$ of mutually orthogonal admissible subspaces. It is clear that both T and T^* map the orthocomplement $\mathfrak{N}$ of the subspace spanned by

all the spaces $\mathfrak{H}_\alpha$ into itself. Thus, if $\mathfrak{N} \neq 0$, it contains a non-zero admissible subspace which contradicts the maximality of the family $\{\mathfrak{H}_\alpha\}$. This shows that the subspaces $\mathfrak{H}_\alpha$ span $\mathfrak{H}$. It is clear that we may regard $\mathfrak{H}$ as the direct sum $\mathfrak{H} = \sum \mathfrak{H}_\alpha$ of the Hilbert spaces $\mathfrak{H}_\alpha$ (cf. Lemma IV.4.19). Theorem 1 may now be applied to the restriction T_α of T to the space $\mathfrak{H}_\alpha$ to yield a regular positive measure μ_α vanishing on the complement of $\sigma(T_\alpha)$ (and thus on the complement of $\sigma(T)$) and a unitary mapping U_α between $\mathfrak{H}_\alpha$ and $L_2(\mu_\alpha)$ such that $(U_\alpha f(T)x)(\lambda) = f(\lambda)(U_\alpha x)(\lambda)$ for each bounded Borel function f and each x in $\mathfrak{H}_\alpha$. We may summarize the above remarks in the following theorem.

3 Theorem. *Every Hilbert space admits a spectral representation relative to an arbitrary bounded normal operator defined in it.*

We have seen that Theorem 3 is a consequence of the spectral theorem for normal operators. It is worth remarking that it is actually equivalent to the spectral theorem. For if UTU^{-1} has the desired form in the space $\sum_\alpha L_2(\sigma(T), \mathscr{B}, \mu_\alpha)$, and if for each Borel set $e \subsetneq \sigma(T)$, $P(e)$ is the projection defined by the relations $(P(e)Uy)_\alpha(\lambda) = \chi_e(\lambda)(Uy)_\alpha(\lambda)$, $y \in \mathfrak{H}$, then the spectral measure E defined by the equation $E(e) = U^{-1}P(e)U$ is a resolution of the identity for T.

It is often convenient to express the result of Theorem 3 somewhat differently.

4 Corollary. *Let T be a normal operator in the Hilbert space $\mathfrak{H}$. Then there exists a regular positive measure space (S, Σ, μ), a bounded μ-measurable scalar funtion f on S, and an isometric isomorphism U mapping $\mathfrak{H}$ onto $L_2(S, \Sigma, \mu)$ which preserves inner products and is such that for each y in $\mathfrak{H}$,*

$$(UTy)(s) = f(s)(Uy)(s),$$

for μ-almost all s in S.

Proof. Using the notations of the preceding theorem and its proof let us regard the sets $S_\alpha = \sigma(T_\alpha)$ as subsets of distinct replicas of the complex plane so that for distinct subscripts α, β, the sets S_α, S_β are disjoint. Let $S = \bigcup_\alpha S_\alpha$ and let Σ consist of all sets of the form $e = \bigcup_\alpha e_\alpha$ where e_α is a Borel subset of S_α. For such e define $\mu(e) = \sum_\alpha \mu_\alpha(e_\alpha)$ if the series has only a countable number of non-zero terms

and converges; otherwise let $\mu(e) = \infty$. Let the distance between two points s, t in S be defined as the usual distance in the complex plane if s, t both belong to the same S_α. Otherwise let it be $|T|$. It is readily seen that (S, Σ, μ) is a regular measure space. For each s in S let $f(s) = s$ if s is in S_α. Clearly f is μ-measurable, and since the norm of the restriction of T to $\mathfrak{H}_\alpha$ is at most $|T|$, it follows from Lemma 3.2 that $|f(s)| \leq |T|$. Let $\mathfrak{D}_1$ consist of all finite sums of the form $x = \sum x_{\alpha_i}$ with $x_{\alpha_i} \in \mathfrak{H}_{\alpha_i}$, and let U_1 be the map from $\mathfrak{D}_1$ into $L_2(S, \Sigma, \mu)$ defined by the equation $U_1(\sum x_{\alpha_i}) = \sum U_{\alpha_i} x_{\alpha_i}$, where the functions $U_\alpha x_\alpha$ are defined as in Theorem 2 for s in S_α and are assumed to be defined on all of S by the requirement that $(U_\alpha x_\alpha)(s) = 0$ if s is not in S_α. Clearly U_1 is linear, isometric, and preserves inner products. Since the domain and range of U_1 are dense in $\mathfrak{H}$ and $L_2(S, \Sigma, \mu)$ respectively, it has a unique continuous extension U which is an isometric map between $\mathfrak{H}$ and $L_2(S, \Sigma, \mu)$ preserving inner products. Since, for $x = \sum x_{\alpha_i}$ in $\mathfrak{D}_1$, we have

$$\begin{aligned}(U_1 Tx)(s) &= \sum (U_{\alpha_i} T x_{\alpha_i})(s) \\ &= f(s)(\sum U_{\alpha_i} x_{\alpha_i})(s) \\ &= f(s)(U_1 x)(s);\end{aligned}$$

it follows that $UTx = f(\cdot)Ux$ for every x in $\mathfrak{H}$. Q.E.D.

5 COROLLARY. *If $\mathfrak{H}$ is separable then the measure space (S, Σ, μ) in Corollary 4 may be taken to be finite.*

PROOF. If $\mathfrak{H}$ is separable then there are a countable number of mutually orthogonal admissible spaces $\mathfrak{H}_n$, $n = 1, 2, \ldots$, which span $\mathfrak{H}$. It will suffice to choose x_n in $\mathfrak{H}_n$ with $|x_n|^2 = 1/2^n$ and such that linear combinations of $T^i T^{*j} x_n$, $i, j \geq 0$, are dense in $\mathfrak{H}_n$. For such a choice we have $\mu_n(S_n) = 1/2^n$ and $\mu(S) \leq 1$. Q.E.D.

The spectral representation discussed in the preceding theorems gives important information about the structure of a normal operator. It is clear, however, that the admissible measures of Definition 1 may be chosen in many ways. It will now be shown that in case the Hilbert space $\mathfrak{H}$ is separable, there is a certain "best" way to make the choice and that when this "best" choice is made, the resulting family of measures characterizes the operator T to within unitary equiv-

alence. This result comprises what is known as *spectral multiplicity* theory.

6 DEFINITION. Let T be a bounded normal operator in the Hilbert space $\mathfrak{H}$ and let E be its spectral resolution. A vector x in $\mathfrak{H}$ is called *maximal* relative to T if every measure of the form $(E(\cdot)y, y)$, $y \in \mathfrak{H}$, is continuous with respect to $(E(\cdot)x, x)$.

7 LEMMA. *Let T be a bounded normal operator in the separable Hilbert space $\mathfrak{H}$. Let y_0 be a given vector in $\mathfrak{H}$. Then there is an x in $\mathfrak{H}$, maximal relative to T, such that y_0 lies in the subspace*

$$\mathfrak{H}(x) = \overline{\text{sp}}\,\{T^n(T^*)^m x \mid n, m \geqq 0\}$$

of $\mathfrak{H}$.

PROOF. We can clearly suppose that $|y_0| = 1$. Let $y_0, y_1, y_2, \ldots$ be an orthonormal basis for $\mathfrak{H}$, whose initial element is y_0. Let E be the spectral resolution for T and let $v_n(e) = (E(e)y_n, y_n)$ for each Borel set e. Using the Lebesgue decomposition theorem (III.4.14), let $\{e_n\}$ be a sequence of Borel sets such that $\sum_{j=0}^{n-1} v_j(e_n) = 0$, and such that if e is a Borel subset of the complement e_n' of e_n and $\sum_{j=0}^{n-1} v_j(e) = 0$, then $v_n(e) = 0$. Let σ_0 be the entire complex plane, and put $\sigma_n = \bigcup_{j=n}^{\infty} e_j$ for $n \geq 1$. Then $\{\sigma_n\}$ is a decreasing sequence of Borel sets such that $\sum_{j=0}^{n-1} v_j(\sigma_n) = 0$, and such that if e is a Borel subset of σ_n' and $\sum_{j=0}^{n-1} v_j(e) = 0$, then $v_n(e) = 0$.

Put $x = \sum_{j=0}^{\infty} 2^{-j} E(\sigma_j)y_j$. To see that x is maximal let e be a Borel set for which

$$(E(e)x, x) = |E(e)x|^2 = 0.$$

Since $v_j(\sigma_n) = 0$ if $j < n$, it follows that $E(\sigma_n)y_j = 0$ if $j < n$. Thus

$$\begin{aligned} E\big(e(\sigma_{n-1}-\sigma_n)\big)x &= \sum_{j=1}^{\infty} 2^{-j} E\big(e(\sigma_{n-1}-\sigma_n)\big)E(\sigma_j)y_j \\ &= 2^{-(n-1)}E\big(e(\sigma_{n-1}-\sigma_n)\big)y_{n-1} \end{aligned}$$

so that

$$|E\big(e(\sigma_{n-1}-\sigma_n)\big)x|^2 = 2^{-2(n-1)}v_{n-1}\big(e(\sigma_{n-1}-\sigma_n)\big).$$

We then have

$$\begin{aligned} \sum_{j=0}^{n-1} 2^{-2j} v_j\big(e(\sigma_{n-1}-\sigma_n)\big) &= |E\big(e(\sigma_{n-1}-\sigma_n)\big)x|^2 \\ &= |E(\sigma_{n-1}-\sigma_n)E(e)x|^2 = 0, \end{aligned}$$

all terms but the final term in the sum on the left of the equation vanishing. Hence

$$\sum_{j=0}^{n-1} v_j(e(\sigma_{n-1}-\sigma_n)) = 0.$$

Since $\sigma_{n-1}-\sigma_n \subseteq \sigma_n'$, it follows that

$$\sum_{j=0}^{n} v_j\,(e(\sigma_{n-1}-\sigma_n)) = 0.$$

Since $\sigma_{n-1}-\sigma_n \subset \sigma_{n+1}'$, it follows that

$$\sum_{j=0}^{n+1} v_j(e(\sigma_{n-1}-\sigma_n)) = 0.$$

Continuing inductively in this way, we have $v_j(e(\sigma_{n-1}-\sigma_n)) = 0$ for all j. By summing over n it is seen that $v_j(e) = 0$ for all j. Thus, since $v_j(e) = |E(e)y_j|^2$, we have $E(e)y_j = 0$ for all j. Since $\{y_j\}$ is an orthonormal basis for $\mathfrak{H}$, it follows that $E(e)y = 0$ for all y in $\mathfrak{H}$. Thus $(E(e)y, y) = 0$ for all y in $\mathfrak{H}$, and x is maximal.

Clearly we have $y_0 = E(\sigma_0-\sigma_1)x$. Hence the theorem will be proved if it is shown that

$$E(\sigma_0-\sigma_1)x \in \overline{\mathrm{sp}}\,\{T^n(T^*)^m x | n, m \geqq 0\}.$$

By the Weierstrass approximation theorem and by Corollary 2.8,

$$\overline{\mathrm{sp}}\,\{T^n(T^*)^m x | n, m \geqq 0\} = \overline{\mathrm{sp}}\,\{f(T)x | f \in C(\sigma(T))\}.$$

Let $\varepsilon > 0$ be given. By the remarks preceding Corollary 2.4, we can find an open set $U \supseteq \sigma_0 - \sigma_1$ and a closed set $C \subseteq \sigma_0-\sigma_1$ such that $|E(U-C)x| < \varepsilon$. It is then clear that if f is a continuous function which is identically zero on U', identically one on C, and which lies between zero and one, and if χ is the characteristic function of $\sigma_0-\sigma_1$, we have

$$|E(\sigma_0-\sigma_1)x - f(T)x|^2 = \int_{U-C} |f(\lambda)-\chi(\lambda)|^2 (E(d\lambda)x, x) \leqq \varepsilon.$$

Thus

$$E(\sigma_0-\sigma_1)x \in \overline{\mathrm{sp}}\,\{f(T)x | f \in C(\sigma(T))\}. \qquad \text{Q.E.D.}$$

8 LEMMA. *Let T be a bounded normal operator in a separable*

Hilbert space $\mathfrak{H}$ *and let* E *denote its resolution of the identity. Then there exists a sequence* $\{x_n\} \subseteq \mathfrak{H}$ *such that* $\mathfrak{H} = \sum_{i=1}^{\infty} \mathfrak{H}(x_i)$, *where*

$$\mathfrak{H}(x_i) = \overline{\mathrm{sp}}\,\{f(T)x_i | f \in C(\sigma(T))\},$$

and a decreasing sequence $\{e_n\}$ *of Borel sets such that* $(E(e)x_n, x_n) = (E(ee_n)x_1, x_1)$, $n \geqq 1$.

PROOF. Let $\{y_n\}$ be a complete orthonormal system in $\mathfrak{H}$. Using Lemma 7, select z_1 in such a way that y_1 is in $\mathfrak{H}(z_1)$ and z_1 is maximal relative to T. Let T_2 denote the restriction of T to the orthogonal complement $\mathfrak{K}(z_1)$ of $\mathfrak{H}(z_1)$. Since T and T^* map $\mathfrak{H}(z_1)$ into itself, T and T^* map $\mathfrak{K}(z_1)$ into itself. Thus it follows immediately that T_2 is normal.

Now select z_2 in $\mathfrak{K}(z_1)$ in such a way that z_2 is maximal relative to T_2, and $\mathfrak{H}(z_2) = \overline{\mathrm{sp}}\,\{T^n(T^*)^m z_2 | n, m \geqq 0\}$ contains the perpendicular projection of y_2 on $\mathfrak{K}(z_1)$. Then y_1 and y_2 are in $\mathfrak{H}(z_1) \oplus \mathfrak{H}(z_2)$. Moreover, from the maximality of z_1 it follows that the measure $\mu_2 = (E(\cdot)z_2, z_2)$ is continuous relative to $\mu_1 = (E(\cdot)z_1, z_1)$.

Let T_3 denote the restriction of T_2 to the orthogonal complement $\mathfrak{K}(z_2)$ of $\mathfrak{H}(z_2)$ in $\mathfrak{K}(z_1)$, etc. It is clear that, proceeding inductively, we obtain a sequence $\{z_n\} \subseteq \mathfrak{H}$ and measures $\mu_n = (E(\cdot)z_n, z_n)$ such that the manifolds $\mathfrak{H}(z_n)$ are orthogonal, the vectors $y_1, \ldots, y_n$ are in $\mathfrak{H}(z_1) \oplus \ldots \oplus \mathfrak{H}(z_n)$, and such that μ_{n+1} is continuous with respect to μ_n for $n \geq 1$. Since y_n is in $\sum_{i=1}^{\infty} \mathfrak{H}(z_i)$ for each $n \geq 1$, it is clear that

$$\mathfrak{H} = \sum_{i=1}^{\infty} \mathfrak{H}(z_i).$$

Using the Radon-Nikodým theorem, let ρ_n be a Borel measurable density of μ_n relative to μ_1, so that

$$\mu_n(e) = \int_e \rho_n(\lambda)\mu_1(d\lambda),$$

for each Borel set e. Since μ_1 and μ_n are both non-negative measures, we may take ρ_n to be non-negative. Let $e_n = \{\lambda | \rho_n(\lambda) > 0\}$. Since μ_{n+1} is μ_n-continuous, $\mu_1(e_{n+1} - e_n) = 0$. Thus, modifying ρ_{n+1} if necessary on a set of μ_1-measure zero, we may suppose that the sequence of sets e_n is decreasing and that e_1 is the whole plane.

For $n \geqq 2$, let $x_n = \lim_{k\to\infty} x_{nk}$ where

$$x_{nk} = \int_{S_{nk}} (\rho_n(\lambda))^{-1/2} E(d\lambda) z_n, \qquad S_{nk} = \left\{\lambda \,\middle|\, \rho_n(\lambda) > \frac{1}{k}\right\}.$$

This limit exists, since if $k > j$, we have

$$\begin{aligned}|x_{nk}-x_{nj}|^2 &= \int_{S_{nk}-S_{nj}} \rho_n(\lambda)^{-1}|E(d\lambda)z_n|^2 \\ &= \int_{S_{nk}-S_{nj}} \rho_n(\lambda)^{-1}\rho_n(\lambda)\mu_1(d\lambda) \\ &= \mu_1(S_{nk}-S_{nj}) \leq \mu_1(e_n-S_{nj})\end{aligned}$$

so that $|x_{nk}-x_{nj}| \to 0$ as $j, k \to \infty$. Putting $x_1 = z_1$, we have, for $n > 1$,

$$\begin{aligned}(E(e)x_n, x_n) = |E(e)x_n|^2 &= \lim_{k\to\infty}\int_{eS_{nk}} \rho_n(\lambda)^{-1}|E(d\lambda)z_n|^2 \\ &= \lim_{k\to\infty}\int_{eS_{nk}} |E(d\lambda)x_1|^2 \\ &= (E(ee_n)x_1, x_1). \qquad \text{Q.E.D.}\end{aligned}$$

Throughout the rest of this section, the notation $\mu_1 \lessapprox \mu_2$ will be used to indicate that the measure μ_1 is μ_2-continuous. If $\mu_1 \lessapprox \mu_2$ and $\mu_2 \lessapprox \mu_1$ the notation $\mu_1 \cong \mu_2$ will be employed.

9 Definition. Let μ be a positive measure defined on the Borel sets of the complex plane and let $\{e_n\}$ be a decreasing sequence of Borel sets whose first element e_1 is the entire plane. A spectral representation of a separable Hilbert space $\mathfrak{H}$ relative to a bounded normal operator T onto $\sum_{n=1}^{\infty} L_2(e_n, \mu)$ is said to be an *ordered representation* of $\mathfrak{H}$ relative to T. The measure μ is called the *measure of the ordered representation*. The sets e_n will be called the *multiplicity sets of the ordered representation*. If $\mu(e_k) > 0$ and $\mu(e_{k+1}) = 0$ then the ordered representation is said to have *multiplicity k*. If $\mu(e_k) > 0$ for all k, the representation is said to have *infinite multiplicity*. Two ordered representations U and $\tilde{U}$ of $\mathfrak{H}$ relative to T and $\tilde{T}$ respectively, with measures μ and $\tilde{\mu}$, and multiplicity sets $\{e_n\}$ and $\{\tilde{e}_n\}$ will be called *equivalent* if $\mu \simeq \tilde{\mu}$ and $\mu(e_n \Delta \tilde{e}_n) = 0 = \tilde{\mu}(e_n \Delta \tilde{e}_n)$ for $n = 1, 2, \ldots$.

Remark. We note that two equivalent ordered representations have the same multiplicity. For if $\mu(e_k) = 0$ for some k, then $\tilde{\mu}(e_k) = 0$ since $\mu \simeq \tilde{\mu}$. Since $\tilde{\mu}(e_k \Delta \tilde{e}_k) = 0$ we have $\tilde{\mu}(\tilde{e}_k) = 0$. By symmetry it is seen that $\tilde{\mu}(\tilde{e}_k) = 0$ if and only if $\mu(e_k) = 0$.

→ 10 Theorem. *A separable Hilbert space $\mathfrak{H}$ has an ordered representation relative to any bounded normal operator T in $\mathfrak{H}$ and any two ordered representations of $\mathfrak{H}$ relative to T are equivalent.*

Proof. The first statement follows from Lemma 8 and Theorem 2 (cf. the proof of Theorem 2). Consequently only the uniqueness assertion need be proved here.

Let U and $\tilde{U}$ be ordered representations of $\mathfrak{H}$ relative to T onto the Hilbert spaces $\mathfrak{R} = \sum_{n=1}^{\infty} L_2(e_n, \mu)$ and $\tilde{\mathfrak{R}} = \sum_{n=1}^{\infty} L_2(\tilde{e}_n, \tilde{\mu})$ respectively. The notation $f = [f_1, f_2, \ldots]$, where f_i is in $L_2(e_i, \mu)$, will be used for the elements of $\mathfrak{R}$ and a similar one for the elements of $\tilde{\mathfrak{R}}$. Let $V = \tilde{U}U^{-1}$, so that V is an isometric map of $\mathfrak{R}$ onto $\tilde{\mathfrak{R}}$. We recall that if F is a bounded Borel function and y is in $\mathfrak{H}$ then

$$(UF(T)y)_n(\lambda) = F(\lambda)(Uy)_n(\lambda), \qquad \mu\text{-almost everywhere on } e_n,$$
$$(\tilde{U}F(T)y)_n(\lambda) = F(\lambda)(\tilde{U}y)_n(\lambda), \qquad \tilde{\mu}\text{-almost everywhere on } \tilde{e}_n.$$

It follows that

$$(*) \qquad V[F(\cdot)f_n(\cdot)] = [F(\cdot)(Vf)_n(\cdot)], \qquad n = 1, 2, \ldots.$$

It will now be shown that $\mu \cong \tilde{\mu}$. Because of symmetry it will suffice to prove that $\mu \lessapprox \tilde{\mu}$. To make an indirect proof of this, suppose that for some bounded Borel set e we have $\tilde{\mu}(e) = 0$ and $0 < \mu(e) < \infty$ and consider the vector

$$f = [\chi_e, 0, 0, \ldots] \in \mathfrak{R}.$$

By taking χ_e for the function F in equation (*) it is seen that

$$Vf = V[\chi_e^2, 0, 0, \ldots] = [\chi_e(Vf)_1, \chi_e(Vf)_2, \ldots].$$

Now $|f|^2 = \mu(e) > 0$, but

$$|Vf|^2 = \sum_{n=1}^{\infty} \int_{\tilde{e}_n} |\chi_e(\lambda)(Vf)_n(\lambda)|^2 \tilde{\mu}(d\lambda)$$
$$= \sum_{n=1}^{\infty} \int_{\tilde{e}_n \cap e} |(Vf)_n(\lambda)|^2 \tilde{\mu}(d\lambda) = 0,$$

which contradicts the fact that V is an isometry and proves that $\mu \cong \tilde{\mu}$.

It will next be shown by induction that $\mu(e_n \Delta \tilde{e}_n) = 0 = \tilde{\mu}(e_n \Delta \tilde{e}_n)$ for $n = 1, 2, \ldots$. The case $n = 1$ follows from the facts that e_1 and $\tilde{e}_1$ are the whole complex plane and that $\mu \cong \tilde{\mu}$. Suppose it is known that

$$\mu(e_j \Delta \tilde{e}_j) = 0 = \tilde{\mu}(e_j \Delta \tilde{e}_j), \qquad 1 \leq j \leq n.$$

To see that this equation holds also for $j = n+1$ it is sufficient to prove that $\mu(e_{n+1} - \tilde{e}_{n+1}) = 0$; for then by symmetry, $\tilde{\mu}(\tilde{e}_{n+1} - e_{n+1}) = 0$

and since $\mu \cong \tilde{\mu}$, it follows that $\tilde{\mu}(e_{n+1}-\tilde{e}_{n+1}) = 0$ and $\mu(\tilde{e}_{n+1}-e_{n+1}) = 0$.

Suppose that $\mu(e_{n+1}-\tilde{e}_{n+1}) > 0$ and let σ be an arbitrary fixed Borel subset of $e_{n+1}-\tilde{e}_{n+1}$ such that $0 < \mu(\sigma) < \infty$. Let the vectors $f^1, \ldots, f^{n+1}$ in $\mathfrak{H}$ be defined by the equations

$$\begin{aligned} f^1 &= [\chi_\sigma, 0, 0, \ldots], \\ f^2 &= [0, \chi_\sigma, 0, \ldots], \\ &\vdots \\ f^{n+1} &= [0, \ldots, 0, \chi_\sigma, 0, \ldots], \end{aligned}$$

the function χ_σ appearing in the $(n+1)$st place. These vectors have positive length and are orthogonal in $\mathfrak{H}$. Thus the vectors $Vf^1, \ldots, Vf^{n+1}$ have positive length and are orthogonal in $\tilde{\mathfrak{H}}$. Let

$$Vf^i = [\tilde{f}^i_j], \qquad j = 1, 2, \ldots, \qquad i = 1, \ldots, n+1.$$

Since $\sigma \subseteq \tilde{e}'_{n+1}$ and the sequence $\{\tilde{e}_m\}$ is decreasing, it follows that $\tilde{\mu}(\sigma \cap \tilde{e}_m) = 0$ for $m > n$ and thus for $m > n$, $\tilde{f}^i_m(\lambda) = 0$, μ-almost everywhere on σ. Now let δ be an arbitrary Borel subset of σ with $\mu(\delta) > 0$ and define functions $g^1, \ldots, g^{n+1}$ by replacing χ_σ by χ_δ in the formulas for the functions f^i. Then the vectors Vg^i are non-zero and orthogonal in $\tilde{\mathfrak{H}}$ and by equation $(*)$

$$Vg^i = [\chi_\delta \tilde{f}^i_j].$$

Thus

$$(Vg^i, Vg^k) = \int_\delta \left[\sum_{j=1}^{n} \tilde{f}^i_j(\lambda)\overline{\tilde{f}^k_j(\lambda)}\right] \tilde{\mu}(d\lambda) = \delta_{ik}\mu(\delta), \qquad i, k = 1, \ldots, n+1.$$

Since δ is arbitrary it follows that we have for $\tilde{\mu}$-almost all λ in σ,

$$\sum_{j=1}^{n} \tilde{f}^i_j(\lambda)\overline{\tilde{f}^k_j(\lambda)} = 0, \qquad 1 \leq i \neq k \leq n+1,$$

$$\sum_{j=1}^{n} |\tilde{f}^i_j(\lambda)|^2 > 0, \qquad i = 1, \ldots, n+1.$$

If λ_0 is a point at which all of these equations hold, then the n-tuples $[\tilde{f}^i_1(\lambda_0), \ldots, \tilde{f}^i_n(\lambda_0)]$ form, for $i = 1, \ldots, n+1$, a set of $n+1$ non-zero orthogonal vectors in the n-dimensional unitary space E^n. This contradiction completes the proof. Q.E.D.

11 Lemma. *Let* (S, Σ, μ) *and* $(S, \Sigma, \tilde{\mu})$ *be measure spaces with* $\mu \cong \tilde{\mu}$. *Then there exists a linear isometric mapping* U *of* $L_2(S, \Sigma, \mu)$ *onto* $L_2(S, \Sigma, \tilde{\mu})$ *with the property that if* χ_e *is the characteristic function of a set* e *in* Σ, *then for* f *in* $L_2(S, \Sigma, \mu)$,

$$U^{-1}\chi_e U f = \chi_e f.$$

Proof. By the Radon-Nikodým theorem there are positive functions δ, $\tilde{\delta}$ such that

$$\tilde{\mu}(e) = \int_e \delta(\lambda)\mu(d\lambda); \qquad \mu(e) = \int_e \tilde{\delta}(\lambda)\tilde{\mu}(d\lambda), \qquad e \in \Sigma.$$

For f in $L_2(S, \Sigma, \mu)$ define the linear mapping

$$Uf = [\tilde{\delta}(\cdot)]^{1/2} f.$$

Then U is unitary, because

$$\begin{aligned}\int_S |f(\lambda)|^2 \mu(d\lambda) &= \int_S |f(\lambda)|^2 \tilde{\delta}(\lambda)\tilde{\mu}(d\lambda)\\ &= \int_S |(Uf)(\lambda)|^2 \tilde{\mu}(d\lambda).\end{aligned}$$

To see that U maps $L_2(S, \Sigma, \mu)$ onto $L_2(S, \Sigma, \tilde{\mu})$, define a second linear mapping V from $L_2(S, \Sigma, \tilde{\mu})$ into $L_2(S, \Sigma, \mu)$ by $Vf = [\delta(\cdot)]^{1/2} f$. It is clear that V is unitary, and because for all e in Σ

$$\mu(e) = \int_e \delta(\lambda)\tilde{\delta}(\lambda)\mu(d\lambda)$$

it follows that $\delta(\lambda)\tilde{\delta}(\lambda) = 1$ almost everywhere, and consequently that U and V are inverses of each other. Therefore U maps $L_2(S, \Sigma, \mu)$ onto $L_2(S, \Sigma, \tilde{\mu})$. The second assertion is evident from the definition of U. Q.E.D.

In connection with the following theorem it should be recalled that two operators T_1 and T_2 in Hilbert space $\mathfrak{H}$ are said to be *unitarily equivalent* if they are related by an equation $T_2 = VT_1V^{-1}$ where V is a unitary operator in $\mathfrak{H}$. It has been observed (cf. Section X.4) that unitarily equivalent operators have identical properties in $\mathfrak{H}$.

12 Theorem. *Two bounded normal operators in a separable Hilbert space* $\mathfrak{H}$ *are unitarily equivalent if and only if the corresponding ordered representations of* $\mathfrak{H}$ *relative to the operators are equivalent.*

Proof. Let T, $\tilde{T}$ be bounded normal operators in $\mathfrak{H}$ and let U, $\tilde{U}$ be ordered representations of $\mathfrak{H}$ relative to T, $\tilde{T}$ on $\sum_{n=1}^{\infty} L_2(e_n, \mu)$, $\sum_{n=1}^{\infty} L_2(\tilde{e}_n, \tilde{\mu})$ respectively. We shall first suppose that T and $\tilde{T}$ are

unitarily equivalent, i.e., that $\tilde{T} = VTV^{-1}$ where V is unitary. Under this assumption it will be shown that there is an ordered representation of $\mathfrak{H}$ onto $\sum_{n=1}^{\infty} L_2(\tilde{e}_n, \tilde{\mu})$ relative to T. It will follow from Theorem 10 that U and $\tilde{U}$ are equivalent. Let E and $\tilde{E}$ be the resolutions of the identity for T and $\tilde{T}$ respectively. From Corollary 2.7 it is seen that $\tilde{E} = VEV^{-1}$ and hence that

$$F(\tilde{T}) = VF(T)V^{-1}$$

for every bounded Borel function F. The mapping $W = \tilde{U}V$ of $\mathfrak{H}$ onto $\sum_{n=1}^{\infty} L_2(\tilde{e}_n, \tilde{\mu})$ is clearly an isometry and, furthermore, it satisfies the identity

$$\begin{aligned} WF(T)x &= \tilde{U}VF(T)x = \tilde{U}F(\tilde{T})Vx \\ &= F(\lambda)\tilde{U}Vx = F(\lambda)Wx. \end{aligned}$$

This shows that W is an ordered representation of $\mathfrak{H}$ onto $\sum_{n=1}^{\infty} L_2(\tilde{e}_n, \tilde{\mu})$ relative to T. This proves that U and $\tilde{U}$ are equivalent.

To prove the converse it is assumed that U and $\tilde{U}$ are equivalent. By Lemma 11 there is an isometry V_n of $L_2(e_n, \mu)$ onto $L_2(\tilde{e}_n, \tilde{\mu})$. An isometry V of $\sum_{n=1}^{\infty} L_2(e_n, \mu)$ onto $\sum_{n=1}^{\infty} L_2(\tilde{e}_n, \tilde{\mu})$ will be determined by the equation $VL_2(e_n, \mu) = V_n L_2(e_n, \mu)$. The linear transformation $Y = \tilde{U}^{-1}VU$ is then a unitary map in $\mathfrak{H}$ and for every x in $\mathfrak{H}$ we have, using Lemma 11,

$$\begin{aligned} YTx &= \tilde{U}^{-1}VUTx = \tilde{U}^{-1}V(\lambda Ux) \\ &= \tilde{U}^{-1}\lambda VUx = \tilde{T}\tilde{U}^{-1}VUx = \tilde{T}Yx, \end{aligned}$$

which proves that T and $\tilde{T}$ are unitarily equivalent. Q.E.D.

6. A Formula for the Spectral Resolution

In working with specific operators it is important to have a method for calculating the resolution of the identity. The following theorem gives a method for calculating the resolution of the identity for a self adjoint operator T in terms of its resolvent $R(\alpha; T) = (\alpha I - T)^{-1}$. It should be recalled (Theorem 4.2) that the spectrum of the self adjoint operator T is real and hence $R(\alpha; T)$ is defined for all non-real α.

1 THEOREM *If E is the resolution of the identity for the bounded*

self adjoint operator T and if (a, b) is the open interval $a < \lambda < b$ then, in the strong operator topology,

$$E((a, b)) = \lim_{\delta\to 0+} \lim_{\varepsilon\to 0+} \frac{1}{2\pi i}\int_{a+\delta}^{b-\delta} [R(\mu-\varepsilon i; T)-R(\mu+\varepsilon i; T)]d\mu.$$

PROOF. For $\varepsilon > 0$, $0 < \delta < (b-a)/2$, and λ real let

$$f(\delta, \varepsilon, \lambda) = \frac{1}{2\pi i}\int_{a+\delta}^{b-\delta} \left[\frac{1}{\mu-\varepsilon i-\lambda} - \frac{1}{\mu+\varepsilon i-\lambda}\right] d\mu.$$

An elementary integration shows that

$$f(\delta, \varepsilon, \lambda) = \frac{1}{\pi}\left[\arctan\frac{b-\delta-\lambda}{\varepsilon} - \arctan\frac{a+\delta-\lambda}{\varepsilon}\right],$$

and hence that $|f(\delta, \varepsilon, \lambda)| \leq 1$. It is clear that

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} f(\delta, \varepsilon, \lambda) = \chi_{(a,b)}(\lambda), \qquad -\infty < \lambda < \infty,$$

where $\chi_{(a,b)}$ is the characteristic function of the open interval $a < \lambda < b$. Thus, by Corollary 2.8(iii),

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} f(\delta, \varepsilon, T)x = E((a, b))x, \qquad x \in \mathfrak{H}.$$

To complete the proof it will therefore be sufficient to show that

$$[*] \qquad f(\delta, \varepsilon, T) = \frac{1}{2\pi i}\int_{a+\delta}^{b-\delta} [R(\mu-\varepsilon i; T)-R(\mu+\varepsilon i; T)]d\mu.$$

To see this we observe that the integrand defining $f(\delta, \varepsilon, \lambda)$ is continuous in (λ, μ) for λ in $\sigma(T)$ and μ in the interval $a+\delta \leq \mu \leq b-\delta$. Thus the Riemann sums which define the integral converge uniformly for λ in $\sigma(T)$ and [*] follows from Corollary IX.3.15. Q.E.D.

7. Perturbation Theory

The general perturbation theory discussed in Section VII.6 may be extended in case the operators considered are normal operators on Hilbert space. In this section we shall be concerned with perturbation in the strong operator topology and shall find it convenient to use the notation $T_n \to_s T$ to mean that $T_n \to T$ in the strong operator

topology, i.e., $T_n x \to Tx$ for every x in the space upon which the operators $T, T_1, T_2, \ldots$, are defined.

1 LEMMA. *Let $S, T, S_n, T_n, n \geq 1$ be bounded linear operators in Hilbert space with $S_n \to S$, $T_n \to T$ in the strong operator topology. Then $S_n+T_n \to S+T$, $\alpha S_n \to \alpha S$, and $S_n T_n \to ST$ in the strong operator topology. If each S_n is normal and S is normal then $S_n^* \to S^*$ in the strong operator topology.*

PROOF. The first two statements are obvious. The uniform boundedness theorem (II.3.21) yields a constant K with $|S_n| < K$, and thus

$$
\begin{aligned}
|S_n T_n x - STx| &< |S_n T_n x - S_n Tx| + |S_n Tx - STx| \\
&\leq K|T_n x - Tx| + |(S_n - S)Tx| \to 0,
\end{aligned}
$$

which proves that $S_n T_n \to ST$ in the strong operator topology. Now suppose that each S_n is normal and that S is normal. Then

$$
\begin{aligned}
|S_n^* x|^2 &= (S_n^* x, S_n^* x) = (S_n S_n^* x, x) \\
&= (S_n^* S_n x, x) = (S_n x, S_n x) \\
&= |S_n x|^2 \to |Sx|^2 = (Sx, Sx) \\
&= (S^* S x, x) = (SS^* x, x) = (S^* x, S^* x) = |S^* x|^2,
\end{aligned}
$$

and thus

$$
|S_n^* x - S^* x|^2 = |S_n^* x|^2 - (S^* x, S_n^* x) - (S_n^* x, S^* x) + |S^* x|^2 \to 0,
$$

which proves that $S_n^* \to S^*$ strongly. Q.E.D.

2 THEOREM. *Let the sequence $\{T_n\}$ of bounded normal operators in Hilbert space converge in the strong operator topology to the normal operator T. Then, for every complex bounded Borel function defined on the complex plane we have $f(T_n) \to f(T)$ in the strong operator topology, provided that the resolution of the identity for T vanishes on a closed set containing the discontinuities of f.*

PROOF. By the principle of uniform boundedness (II.3.21) there is a constant K with $|T_n|, |T| \leq K$. Therefore, by Lemma 3.2, the spectra $\sigma(T_n)$, $\sigma(T)$ are contained in the compact disk $D = \{\lambda \,|\, |\lambda| \leq K\}$ and thus (2.4) we may restrict our attention to bounded Borel functions on D. Let $\mathfrak{B}$ be the B^*-algebra, with norm $|f| =$

$\sup_{\lambda \in D} |f(\lambda)|$, of all complex bounded Borel functions on D, and let $\mathfrak{C}$ be the subset of $\mathfrak{B}$ consisting of those f for which $f(T_n) \to f(T)$ in the strong operator topology. The set $\mathfrak{C}$ is closed in $\mathfrak{B}$, for if $f_m \in \mathfrak{C}$, $|f_m - f| \to 0$, then

$$\begin{aligned} &|f(T_n)x - f(T)x| \\ &\qquad \leq |(f-f_m)(T_n)x-(f-f_m)(T)x| + |f_m(T_n)x-f_m(T)x| \\ &\qquad \leq 2|f-f_m||x|+|f_m(T_n)x-f_m(T)x|, \end{aligned}$$

and

$$\limsup_{n\to\infty} |f(T_n)x - f(T)x| \leq 2|f-f_m|\,|x|, \qquad m \geq 1,$$

which shows, since $|f-f_m| \to 0$, that $f(T_n) \to_s f(T)$. It follows from Lemma 1 that $\mathfrak{C}$ is a B^*-subalgebra of $\mathfrak{B}$. Since the function $f(\lambda) = \lambda$ is in $\mathfrak{C}$ by hypothesis, it follows that every polynomial in λ and $\bar{\lambda}$ is in $\mathfrak{C}$. Thus, by the Weierstrass theorem, every continuous function is in $\mathfrak{C}$. Now let $E(\sigma) = 0$ where σ is a closed set containing all points of discontinuity of f. By Urysohn's lemma (I.5.2) there is, for each $m \geq 1$, a continuous function g_m with $0 \leq g_m(\lambda) \leq 1$ and with

$$\begin{aligned} g_m(\lambda) &= 1, &\quad &\text{distance } (\lambda, \sigma) \geq 1/m, \\ &= 0, & &\lambda \in \sigma. \end{aligned}$$

Thus $g_m(T) \to I$ strongly by Corollary 2.8(iii). Since the sequence $\{f(T_n)\}$ is uniformly bounded, to see that $f(T_n) \to f(T)$ strongly it will suffice to show that for each $m \geq 1$, we have $f(T_n)g_m(T_n) \to f(T)g_m(T)$ strongly. But this follows since $f \cdot g_m$ is a continuous function. Q.E.D.

3 Corollary. *Let E_n, E be the resolutions of the identity for the normal operators T_n, T respectively, and let $T_n \to T$ in the strong operator topology. If E vanishes on the boundary of the Borel set σ, then $E_n(\sigma) \to E(\sigma)$ in the strong operator topology.*

8. Exercises

1 Show that an operator in $\mathfrak{H}$ is positive if and only if it can be written in the form TT^*.

2 If a commutative B^*-algebra of operators on $\mathfrak{H}$ admits no non-

trivial invariant subspaces then $\mathfrak{H}$ is one dimensional.

3 If a weakly closed B^*-algebra $\mathfrak{A}$ of operators on $\mathfrak{H}$ admits no non-trivial invariant subspaces then $\mathfrak{A} = B(\mathfrak{H})$. (Hint: Use the second theorem of Section IX.5.)

4 Let $\mathfrak{A}$ be a B^*-algebra of operators in a finite dimensional Hilbert space $\mathfrak{H}$. Show that $\mathfrak{H}$ may be written uniquely as the orthogonal direct sum $\mathfrak{H} = \mathfrak{H}_1 \oplus \ldots \oplus \mathfrak{H}_n$ of invariant subspaces $\mathfrak{H}_k$ of $\mathfrak{H}$ which contain no smaller non-trivial invariant subspaces of $\mathfrak{H}$.

5 Let T be a normal operator in Hilbert space. Show that T is compact if and only if

(a) $\sigma(T)$ is a denumerable set $\{\lambda_n\}$;

(b) $\sigma(T)$ has no limit point except possibly 0;

(c) if $\lambda_n \neq 0$ then $E(\lambda_n)$ is finite dimensional.

6 A positive operator has a unique positive square root.

7 The product of two positive commuting operators is positive.

8 Show that if N is normal, the closure of the set of values assumed by (Nx, x) for $|x| = 1$ is the smallest closed convex set containing $\sigma(T)$.

9 Let $\mathfrak{A}$ be a commutative B^*-algebra of operators in a finite dimensional Hilbert space $\mathfrak{H}$. Show that there is an orthonormal basis for $\mathfrak{H}$, each of whose elements is an eigenvector for every $T \in \mathfrak{A}$.

10 A positive transformation has a bounded inverse if and only if for some $\varepsilon > 0$, $T - \varepsilon I$ is positive definite.

11 A bounded operator T has a bounded inverse if and only if for some $\varepsilon > 0$, $TT^* - \varepsilon I$ and $T^*T - \varepsilon I$ are both positive definite.

12 A normal projection is self adjoint. If E_1 and E_2 are self adjoint projections, then $E_1 \geq E_2$ in the sense of Section 4 if and only if $E_1 \geq E_2$ in the sense of Definition VI.3.4.

13 If N is a normal operator in $\mathfrak{H}$ and E is its spectral resolution, then E is in the weak closure of the B^*-algebra generated by N and N^*.

14 A weakly closed B^*-algebra $\mathfrak{A}$ is generated by the self adjoint projections in $\mathfrak{A}$.

15 Let P denote the product space of a countable collection of replicas of the complex plane, and let I denote the unit interval. Show that there is a one to one map h of P onto I such that $h(e)$ is a Borel set if and only if e is a Borel set. (Hint: Use the decimal representation of real numbers.)

16 Let $N_1, N_2, \ldots$ be a countable sequence of normal operators in $\mathfrak{H}$, all commuting with each other. Show that there exists a single Hermitian operator T such that each N_k is a Borel function of T. (Hint: Use Theorem 2.1 and Exercise 15).

17 For operators A, B, and C in Hilbert space show that

(a) $A \leq B$ and $B \leq A$ imply $A = B$;

(b) $A \leq B$ and $B \leq C$ imply $A \leq C$;

(c) $A_1 \leq A_2$ and $B_1 \leq B_2$ imply $A_1+B_1 \leq A_2+B_2$;

(d) $A \leq B$ and $\alpha \geqq 0$ imply $\alpha A \leq \alpha B$;

(e) $A \leq B$ implies $-B \leq -A$;

(f) if A is Hermitian, there are numbers m and M such that $mI \leq A \leq MI$.

18 Show that $0 \leq A \leq B$ implies that $A^2 \leq B^2$ if A and B commute, but not if A and B fail to commute.

19 If T is a positive operator in $\mathfrak{H}$ then $|(Tx, y)|^2 \leq (Tx, x)(Ty, y)$.

20 Let T_α be a uniformly bounded generalized sequence of positive operators in $\mathfrak{H}$. Then $T_\alpha \to 0$ in the strong operator topology if and only if $T_\alpha \to 0$ in the weak operator topology, or equivalently, if and only if $(T_\alpha x, x) \to 0$ for each $x \in \mathfrak{H}$. (Hint: Use Exercise 19).

21 If T_α is a uniformly bounded generalized sequence of operators in $\mathfrak{H}$, and if $\alpha \leq \beta$ implies $T_\alpha \leq T_\beta$, then the strong limit $\lim_\alpha T_\alpha$ exists. (Hint: Use Exercise 20).

22 Let T_1 and T_2 be compact Hermitian operators. Let $\lambda_1, \lambda_2, \ldots$ be the sequence of positive eigenvalues of T_1 arranged in decreasing order and repeated according to multiplicity. Let $\mu_1, \mu_2, \ldots$ be the corresponding sequence formed for T_2, and $\nu_1, \nu_2, \ldots$ that formed for T_1+T_2. Then

(a) $\nu_{k+i-1} \leq \mu_k+\lambda_i$;

(b) $|\nu_k-\lambda_k| \leq |T_2|$.

23 Show that the sequence $\lambda_1, \lambda_2, \ldots$ of positive eigenvalues of the compact Hermitian operator T, arranged in decreasing order and counted according to multiplicity, may be obtained from the formula

$$\lambda_n = \max_{\mathfrak{S}_n} \min_{x \in \mathfrak{S}_n} (Tx, x)/(x, x),$$

where $\mathfrak{S}_n$ denotes an arbitrary n-dimensional subspace of $\mathfrak{H}$.

24 In the Hilbert space of all measurable functions on $[0, 1]$

whose squares are Lebesgue integrable, consider the operators defined by the integral kernels

$$K_1(x, y) = i \operatorname{sgn}(x-y), \qquad K_2(x, y) = |x-y|,$$

and compute their eigenvalues and eigenfunctions.

25 Show that the bounded operator T in Hilbert space is compact if and only if either

(a) $Tx_n \to 0$ strongly whenever $x_n \to 0$ weakly, or

(b) $(Tx_n, x_n) \to 0$ whenever $x_n \to 0$ weakly.

(Hint: For (b), show that the hypothesis implies that $(Tx_n, y_n) \to 0$ whenever the sequences $\{x_n\}$ and $\{y_n\}$ are both weakly convergent to zero.)

9. Notes and Remarks

The basic result of this chapter concerning the spectral theory of Hermitian operators goes back to the work of Hilbert [1, IV]. Proofs of this result were also given by F. Riesz [20, 6], and contributions were made by many others. The reader is referred to the encyclopedic account of Hellinger and Toeplitz [3] concerning the early development of Hilbert space theory, and to the treatise of Wintner [1], which is phrased in terms of infinite matrices, where many references are given. After a period of relative quiet, around 1930 the fundamental work of von Neumann [7, 8, 2] and Stone [10, 3] ushered in the modern form of spectral theory in Hilbert space. Stone's book [3] gives a thorough discussion of spectral theory in a separable Hilbert space, and is a very valuable reference; it is primarily concerned with unbounded operators. The book of Sz.-Nagy [3] gives a concise account of both the bounded and unbounded cases, as does the more recent work of Riesz and Sz.-Nagy [1]. Halmos [6] discusses the bounded case only, but develops a theory of multiplicity. Reference is also made to the books of Ahiezer and Glazman [1] and Cooke [1], as well as to the lucid treatment given by C. T. Ionescu Tulcea in his book (in Rumanian) *Spatii Hilberti*, Editura Acad. Rep. Populare Romane, 1956.

Since there are a number of references available which document the historical aspect of the development, we shall give only very brief remarks concerning the results presented in the text. This will enable

us to comment on various developments which are of importance and interest, but which we have not formally discussed.

The spectral theorem. The spectral theorem for bounded self adjoint operators in Hilbert space is due to Hilbert [1: IV]. The reader should also see the proofs of F. Riesz [20, 6] which are quite modern in spirit. Many other proofs of the spectral theorem for self adjoint, unitary, or normal operators have been given, both in the bounded and unbounded cases. We refer the reader to the treatises of Ahiezer and Glazman [1], Halmos [6], Loomis [1], Riesz and Sz.-Nagy [1], Stone [3], Sz.-Nagy [8], and Wintner [1]. Additional proofs of these results, both for bounded and unbounded operators, may be found in the following: Carleman [1], Christian [1], Cooper [1, 2], Eberlein [5], Esser [1], Friedrichs [4], Hellinger [1], Kodaira [2], Koopman and Doob [1], Lengyel [1], Lengyel and Stone [1], Lorch [5], McShane [3], Nakano [7, 12, 13], von Neumann [2, 7, 16, 22], Ogasawara [7], Rellich [8], F. Riesz [14], Riesz and Lorch [1], Smith [1], Stone [7], Teichmüller [2], Tsuji [1], Wecken [1], Wintner [2], Yosida [11, 12; I], and Yosida and Nakayama [1]. The volume of Cooke [1] reproduces some of these proofs. See also Fell, J. M. G. and Kelley, J. L., An Algebra of Unbounded Operators. *Proc. Nat. Acad. Sci. U.S.A* 38, 592–598 (1952).

Von Neumann [2; p. 401] proved that a commuting family of normal operators on a separable Hilbert space has a simultaneous spectral reduction. The same fact follows for a countable system in a general Hilbert space (see also Haar [2; pp. 781 790], Sz.-Nagy [3; pp. 66–69], and Riesz and Sz.-Nagy [1; Secs. 130–131]). The first treatment of a B^*-algebra of operators as in Theorem 2.1 without countability restrictions was by Dunford [13], although weakly closed algebras had been considered earlier by Yosida [12; I], and similar results had been obtained for Abelian rings of operators by Stone [7].

A systematic use of the operational calculus of the form given in Theorem 1.1 was introduced by von Neumann [18] and Stone [8; Chap. VI]; see also Lorch [2, 11].

The following theorem was essentially proved by von Neumann [2; p. 393], [18; p. 213].

THEOREM. *Let A be a bounded self adjoint operator on a separable Hilbert space, and let T be a bounded operator which commutes with every operator which commutes with A. Then there exists a bounded measurable function f such that $T = f(A)$.*

This theorem was stated explicitly by F. Riesz [21]. Mimura [1] simplified Riesz's proof, and extended the result to unbounded operators. A more elementary proof was given by Sz.-Nagy [3; pp. 63–65] (see also Riesz and Sz.-Nagy [1; Sec. 129], Nakano [8, 9] and Wecken [2]). The theorem, as stated, fails in the non-separable case, but an appropriate extension has been obtained by Segal [5; II, p. 88]. A generalization of this result to B-spaces has been obtained by Bade [3, 4] and will be discussed in Chapter XVII.

The spectrum of a self adjoint operator. Theorem 4.3 is one of many procedures for the calculation of eigenvalues of a compact self adjoint operator. For some additional discussion, see Riesz and Sz.-Nagy [1; Secs. 95–96]. For more extended accounts of the Rayleigh-Ritz, the Weinstein, and the many other methods that have been studied, the reader should consult the treatise of Collatz [1]. Much recent work on this problem has been done by Aronszajn and his collaborators; see for example Aronszajn [3, 4], where other references may be found. Lengyel and Stone [1] show that if T is self adjoint, then the upper or lower bound of the set $\{(Tx, x) \mid |x| = 1\}$ is attained if and only if it is an eigenvalue of T.

It is sometimes convenient to have the relation that if T is a normal operator, then λ is in $\sigma(T)$ if and only if for every $\varepsilon > 0$ there is an $x \neq 0$ such that $|Tx - \lambda x| < \varepsilon |x|$. For a proof, see Halmos [3; p. 51]. For some results and references pertaining to the spectrum, particularly of compact operators, we refer to Riesz and Sz.-Nagy [1; Secs. 133 134], and Chiang [1].

For references to the classical literature on this subject, see Hellinger and Toeplitz [3; Secs. 32–35].

Spectral representation and multiplicity theory. The results of Section 5 are closely related to the theory of multiplicity which was studied by Hellinger [1] and Hahn [5]. For a more recent treatment see also Stone [3; Chap. 7] and von Neumann [15] for the case of a separable Hilbert space and Halmos [6] for a general space. Halmos is concerned with the multiplicity of a spectral measure, and his results

extend the work of Nakano [10. 11], Plessner and Rohlin [1] and Wecken [2]. Ahiezer and Glazman [1; Secs. 69–73] also has a treatment for unbounded operators.

If T is a self adjoint operator, let $\mathfrak{A}(T)$ be the ring of bounded operators which commute with T. Yosida [3] showed that T_1 and T_2 are unitarily equivalent if and only if $\mathfrak{A}(T_1)$ and $\mathfrak{A}(T_2)$ are *-isomorphic under a mapping sending T_1 into T_2.

Other formulations of multiplicity theory concerned with algebras of operators in Hilbert space have been given by Segal [5; II] and Kelley [6]. We state a theorem of Segal which can frequently be used in place of the spectral theorem.

THEOREM. *A maximal Abelian self adjoint algebra of bounded operators on a Hilbert space is unitarily equivalent to the algebra of all multiplications by bounded measurable functions in L_2 over some measure space.*

Calculation of the resolution. The formula in Theorem 6.1 is essentially due to Stieltjes [1, pp. 72–75], and was a basic tool in the work of Hellinger [1]. See also Stone [8; p. 183] and the papers of Lengyel [1] and Koopman and Doob [1].

Perturbation theory. References for perturbation theory have already been given in Section VII.11. The results in Section 7 are essentially due to Rellich [2; II]. See also Riesz and Sz.-Nagy [1; Secs. 134 136].

Invariant subspaces. If T is an operator in a B-space $\mathfrak{X}$, and if $\mathfrak{M}$ is a closed linear subspace which is neither $\{0\}$ nor $\mathfrak{X}$ for which we have $T\mathfrak{M} \subseteq \mathfrak{M}$, then $\mathfrak{M}$ is called a (non-trivial) *invariant subspace* of $\mathfrak{X}$ with respect to T. If $\mathfrak{X}$ is a Hilbert space and if both $\mathfrak{M}$ and its orthocomplement $\mathfrak{X} \ominus \mathfrak{M}$ are invariant subspaces of $\mathfrak{X}$ with respect to T, then $\mathfrak{M}$ is said to *reduce* T. It is not difficult to see that a non-trivial subspace of Hilbert space may be an invariant subspace for an operator but not reduce the operator. In fact, an operator may have many non-trivial invariant subspaces and no non-trivial reducing subspaces (cf. Halmos [6; p. 40] for an elementary example).

The spectral theorem assures that if T is a normal operator (and is neither the zero nor the identity operator), then T is reduced by at least one non-trivial subspace. For operators which are not normal

this is far from clear, and it is of considerable interest to find non-trivial invariant subspaces for a given operator. It is not known whether every operator, distinct from the zero and identity operators, has a non-trivial invariant subspace. It is readily seen from Theorem VII.3.10 that if T is a bounded linear operator in a B-space $\mathfrak{X}$ and if $\sigma(T)$ contains at least two components, then T has an invariant subspace. Aronszajn and Smith [1] have shown that every compact operator has an invariant subspace even when $\sigma(T) = \{0\}$. It follows from a theorem of Godement [1; p. 136] that if T is an isometric linear operator mapping a B-space $\mathfrak{X}$ onto itself, then T has an invariant manifold. If $T \in B(\mathfrak{X})$ and $|T^n| \leq K$, $n = 0, \pm 1, \pm 2, \ldots$, then $\mathfrak{X}$ can be renormed to make T an isometry, so the same conclusion follows under this assumption. Wermer [1] has shown that if $|T^n| = O(e^{|n|^\alpha})$ for some α with $0 < \alpha < 1$, and if $\sigma(T)$ contains at least two points, then T has an invariant subspace. He also proved that if $|T^n| = O(|n|^k)$ for some k, the same conclusion is obtained. (See also Wermer [2].)

Wermer [4; p. 275] proved that if T is a normal operator and if $\sigma(T)$ has no interior and does not separate the plane, then every closed invariant subspace of T is also an invariant subspace for T^* and therefore reduces T. His paper also considers the problem of when an operator T in $B(\mathfrak{X})$, whose eigenvectors are fundamental in $\mathfrak{X}$, has the property that if $\mathfrak{M}$ is a closed invariant manifold of T, then $\mathfrak{M}$ is spanned by the eigenvectors it contains. This problem is related to the problem of spectral synthesis to be discussed in Section XI.4. A similar problem was discussed by Beurling [4], who was able to find *all* of the closed invariant subspaces for the operator of multiplication by z on a Hilbert space of functions analytic in $|z| < 1$.

Restrictions of operators. Let $\mathfrak{M}$ be a closed subspace of a Hilbert space $\mathfrak{H}$. It is easily seen that if $\mathfrak{M}$ is an invariant subspace of a self adjoint operator T, then the restriction of T to $\mathfrak{M}$ is a self adjoint operator in Hilbert space $\mathfrak{M}$. The situation is radically different for normal operators. For example, if $\mathfrak{H} = L_2$ on the disk $|z| < 1$ and $\mathfrak{M}$ is the subspace of analytic functions, then the operator T of multiplication by z is normal on $\mathfrak{H}$ but not on $\mathfrak{M}$, since the adjoint of T in $\mathfrak{M}$ is given by

$$(T^*f)(z) = z^{-1}(f(z) - f(0)), \qquad f \in \mathfrak{M}.$$

The theorem of Wermer [4] cited in the preceding paragraph gives a condition under which the restriction of a normal operator to every invariant subspace is again normal. Wermer [5] studied the restriction of an operator T to a subspace $\mathfrak{M}$ which is not invariant under T^{-1} and such that for some $x \in \mathfrak{M}$, the vectors $T^n x$, $n = 0, 1, 2, \ldots$, are fundamental in $\mathfrak{M}$. The restriction of T to $\mathfrak{M}$ is then represented as multiplication by z on a space of analytic functions. Applications are made to the restrictions of normal and unitary operators.

Halmos, Lumer, and Schäffer [1] proved that the restriction of a normal operator may possess an inverse but not a square root (see also Halmos and Lumer [1]).

Dilations of operators. The preceding paragraph indicates that it is of interest to inquire when a given operator A in a Hilbert space $\mathfrak{H}$ can be extended (in some sense) to an operator B in a Hilbert space $\mathfrak{K}$, containing $\mathfrak{H}$, in such a way that B has properties not possessed by A.

One possible type of extension is the following: if $\mathfrak{K}$ is a Hilbert space containing $\mathfrak{H}$, and if P is the self adjoint projection of $\mathfrak{K}$ onto $\mathfrak{H}$, then B is called a *dilation* of A to $\mathfrak{K}$ in case $AP = PBP$. Halmos [1] has proved that every bounded operator A on a Hilbert space $\mathfrak{H}$ has a dilation B to $\mathfrak{K} = \mathfrak{H} \oplus \mathfrak{H}$, and that if A is a *contraction* (i.e., $|A| \leq 1$), then B may be taken to be unitary. Further, as observed by E. A. Michael, if A is a positive operator on $\mathfrak{H}$ with $|A| \leq 1$, then there is a self adjoint projection on $\mathfrak{K} = \mathfrak{H} \oplus \mathfrak{H}$ which is a dilation of A. A number of interesting consequences follow from these results; for example, if $\mathfrak{H}$ is an infinite dimensional Hilbert space, then the closure in the weak operator topology of the set of all unitary operators is the set of all contractions and the closure of all projections is the set of all positive contractions. An important extension of one of the results cited above is the following theorem due to Sz.-Nagy [8, 10].

THEOREM. *If A is a contraction defined on a Hilbert space $\mathfrak{H}$, then there exists a Hilbert space $\mathfrak{K} \supseteq \mathfrak{H}$ and a unitary operator U on $\mathfrak{K}$ such that if P is the self adjoint projection of $\mathfrak{K}$ onto $\mathfrak{H}$, then $A^n P = PU^n P$, $n = 1, 2, \ldots$. In addition, $\mathfrak{K}$ may be chosen to be minimal in the sense that it is spanned by the elements $\{U^k x | x \in \mathfrak{H}, k = 0, \pm 1, \pm 2, \ldots\}$ in which case $\mathfrak{K}$ and U are unique to within unitary equivalence.*

A very elementary proof of this result, excluding the minimality condition, has been given by Schäffer [1]. An extension to B-spaces was given by Sz.-Nagy [10; p. 114], and similar results for semi-groups of contractions are found in Sz.-Nagy [8, 10] (see also Cooper [3]). Sz.-Nagy's original proof depended on the following theorem due to Neumark [8].

THEOREM. *Let S be an abstract set and Σ a field (resp. σ-field) of subsets of S. Let F be an additive (resp. weakly countably additive) function on Σ to the set of positive operators on a Hilbert space $\mathfrak{H}$ satisfying $F(\phi) = 0$ and $F(S) = I$. Then there exists a Hilbert space $\mathfrak{K} \supseteq \mathfrak{H}$ and a self adjoint projection valued additive (resp. weakly countable additive) function E on Σ to $B(\mathfrak{K})$ such that if P is the self adjoint projection of $\mathfrak{K}$ onto $\mathfrak{H}$, then $F(e)P = PE(e)P$, for all $e \in \Sigma$.*

For an extended discussion of these important results and a general theorem unifying them, see Sz.-Nagy [11].

It follows from the work of Sz.-Nagy that if A is a contraction on a Hilbert space $\mathfrak{H}$, then there exists a strongly countably additive positive operator valued set function F on $|z| = 1$ such that

$$A^{(n)} = \int_0^{2\pi} e^{in\theta} F(d\theta),$$

where $A^{(n)} = A^n$ if $n \geqq 0$ and $A^{(n)} = A^{*|n|}$ if $n < 0$. Schreiber [1] has used this representation to obtain an operational calculus for functions which are boundary values of bounded analytic functions on $|z| < 1$.

Extension of operators. Let A be an operator in a Hilbert space $\mathfrak{H}$. It is of interest to obtain conditions under which A has a normal *extension* to some Hilbert space $\mathfrak{K} \supseteq \mathfrak{H}$, i.e., when there exists a normal operator B on $\mathfrak{K}$ such that if P is the self adjoint projection of $\mathfrak{K}$ onto $\mathfrak{H}$, then $AP = BP$. Such operators are called *subnormal* by Bram [1], who has made an extensive study of them, starting from the work of Halmos [1]. As opposed to dilations, not every operator has a normal extension (cf. Halmos [1; p. 133]). The following result is an improvement, due to Bram, of a theorem of Halmos (see also Sz.-Nagy [11; p. 18]).

THEOREM. *A necessary and sufficient condition that a bounded linear operator A on a Hilbert space $\mathfrak{H}$ have a normal extension is that*

$$\sum_{m,n=0}^{r} (A^n x_m, A^m x_n) \geq 0,$$

for every finite set $x_0, x_1, \ldots, x_r$ in $\mathfrak{H}$. If this condition is satisfied, the minimal normal extension is unique up to unitary equivalence.

Another criterion has been given by Bram [1; p. 79], where the relation of the spectra of A and its minimal normal extension and other questions are investigated. Halmos [9] also considers the relation of the spectra.

The spectral sets of von Neumann. If T is a bounded linear operator in a Hilbert space, then von Neumann [8] defines a closed set S of the complex sphere to be a *spectral set* of T if $f(T)$ exists and $|f(T)| \leq 1$ whenever f is a rational function such that $|f(z)| \leq 1$ for all $z \in S$. It is seen that if S is a spectral set, then $\sigma(T) \subseteq S$, and that if T is a normal operator, then $\sigma(T)$ is a spectral set in this sense. The following theorem is of interest.

THEOREM. *If $|T| \leq 1$, if f is analytic on a domain containing $|z| \leq 1$, and if $|f(z)| \leq 1$ for $|z| \leq 1$, then $|f(T)| \leq 1$.*

This implies that the disk $|z| \leq 1$ is a spectral set of T if and only if $|T| \leq 1$. This theorem was first proved by von Neumann [3; p. 269]; another proof has been given by Heinz [2] and a particularly elementary one is due to Sz.-Nagy [8]; [11; p. 15]. It is proved that T is normal if and only if $\sigma(T)$ is a spectral set (cf. von Neumann [3; p. 280]), and that T is self adjoint (resp. unitary) if and only if a bounded subset of the real axis (resp. the unit circle) is a spectral set. For proofs and additional results, see von Neumann [3] or Riesz and Sz.-Nagy [1; Secs. 152—155].

Square roots of operators. If A is an operator, then an operator B is said to be a *square root* of A in case $B^2 = A$. It is an elementary consequence of the spectral theorem that every positive self adjoint operator in Hilbert space has a positive self adjoint square root. For proofs which do not use the spectral theorem, see Visser [2], Wecken [1], or Riesz and Sz.-Nagy [1; p. 262]. Halmos, Lumer, and Schäffer [1] have shown that there exist invertible operators in Hilbert space which do not have square roots; in fact, Halmos and Lumer [1] proved that the operators with square roots are not even dense in those with inverses (relative to the uniform topology).

See Schäffer [2] for a similar result. Julia [1–4] has made a systematic study of the determination of all self adjoint square roots of a given positive operator, both in the bounded and unbounded cases. Extensions to nth roots and nonselfadjoint square roots are considered in Julia [5, 6].

Commutativity of operators. It is trivial to prove that if A and B are commuting self adjoint operators in a Hilbert space, then AB is self adjoint. It has been seen (cf. Exercise X.8.7) that if A and B are commuting positive operators, then AB is positive. If A is a normal operator and if B is an operator which commutes with A, then B commutes with A^*. This fact was conjectured by von Neumann, and proofs have been given by Fuglede [1] and Halmos [3], [6; p. 68]; another proof follows from Corollary XV. 3. 7. It follows that if A and B are commuting normal operators, then AB is a normal operator.

Brown [1] showed that if both T and T^* commute with TT^*, then T is normal, and Putnam [3] showed that the same conclusion follows from the commutativity of T and $TT^* - T^*T$.

Generalizing a result of Fuglede [1], Putnam [1] showed that if A and B are normal operators and if T is an invertible operator such that $A = TBT^{-1}$, then there exists a unitary operator U such that $A = UBU^*$.

Surprisingly enough, if A, B, and AB are normal operators, then it does not follow that BA is a normal operator (cf. Kaplansky [6]). However, extending some work of Wiegmann [1], Kaplansky has obtained conditions under which this is true (e.g., if in addition either A or B is compact, then the normality of BA follows). Also if A and AB are normal, then BA is normal if and only if B commutes with A^*A.

Given any two operators A and B, the magnitude of the *commutator* $AB - BA$ of A and B is a measure of the extent to which they fail to commute. Wintner [3] showed that if A and B are bounded and self adjoint, then their commutator cannot be a nonzero multiple of the identity, and Putnam [2] observed that Wintner's method does not require the self adjointness. Wielandt [2] obtained a more general result, applicable to B-algebras. Halmos [10; II] (see also Schäffer [2]) proved that if A commutes with $AB - BA$,

then $AB - BA$ is a two-sided topological divisor of zero. In analogy with a theorem of N. Jacobson, Kaplansky conjectured that if A commutes with $AB - BA$, then the latter is quasi-nilpotent. Putnam [8] demonstrated that if A and B both commute with $AB - BA$, then $AB - BA$ is a quasi-nilpotent, thereby almost settling the conjecture. Vidav [1] has given a different proof of Putnam's result valid in any B-algebra. A somewhat stronger result has been proved by Singer and Wermer [1] who obtained a theorem implying that if A and B are bounded operators on a B-space and if $AB - BA$ lies in the uniformly closed algebra generated by A and I, then $AB - BA$ is a quasi-nilpotent.

Polar decomposition. The polar decomposition, or canonical factorization, is particularly important for unbounded operators and will be studied in Section XII.7. However, it is of sufficient interest in the case of bounded operators to be stated here. An operator U in a Hilbert space $\mathfrak{H}$ is called a *partial isometry* with initial domain $\mathfrak{M}$ if $|Ux| = |x|$ for $x \in \mathfrak{M}$ and $Ux = 0$ for $x \in \mathfrak{H} \ominus \mathfrak{M}$.

THEOREM. *Every bounded linear transformation T in a Hilbert space can be written in the form UP where U is a partial isometry and P is a positive operator. If T is normal, then U may be taken to be unitary and such that U and P commute with each other and with all operators that commute with T and T^*.*

For an elementary proof, see Riesz and Sz.-Nagy [1; Sec. 110]. In general, U and P do not commute, but Brown [1] has made a study of operators T for which they do commute, and proved that such is the case if and only if T commutes with T^*T. It is seen that such an operator T may be decomposed into the direct sum of a normal operator and a generalized shift operator.

An inequality of Heinz. If A and B are positive bounded operators and Q is a bounded linear operator in a Hilbert space $\mathfrak{H}$ such that $|Qx| \leq |Ax|$ and $|Q^*x| \leq |Bx|$ for all x in $\mathfrak{H}$, then for any real number α with $0 \leq \alpha \leq 1$ and all x, y in $\mathfrak{H}$, we have $|(Qx, y)| \leq |A^\alpha x|\, |B^{1-\alpha} y|$. This is a sharpening, due to Kato [5], of an inequality proved by Heinz [1]. They showed that it remains valid for unbounded operators. Another proof, based on the convexity theorem of Riesz, and valid for multilinear forms, was given by Dixmier [6]. Recently, Heinz [4] has given a short elegant proof of this result.

Symmetrizable and normalizable operators. It may happen that an operator T is not symmetric (or normal) but for some non-trivial operator A, either the operator AT or TA is symmetric (or normal). In such a case, it is sometimes possible to obtain significant information concerning the spectral nature of T. Such operators arise naturally in the study of integral equations (cf. Hellinger and Toeplitz [8; Sec. 88]), and both the integral operators and the abstract operators have been studied to a considerable extent, particularly in the compact case. For this theory, the reader is referred to the book of Zaanen [5; Chap. 12] and the papers of Reid [1, 2], Wilkins [1], Zaanen [3, 7, 8], and Zimmerberg [1, 2]. Particular interest is attached to the possibility of eigenfunction expansions.

CHAPTER XI

Miscellaneous Applications

This chapter is devoted to applications of the spectral theory of normal operators to problems in a variety of fields of mathematics. Since these topics are not in the main stream of our interest we shall not give anything approaching an exhaustive treatment of the subjects considered. However, due to the comprehensiveness and power of the general spectral theory, it is possible not only to give a satisfactory foundation for these subjects, but also to develop their principal results, and this we shall attempt to do. The topics discussed are group theory and the Peter-Weyl theorem; Harald Bohr's theory of almost periodic functions; the Fourier transform, convolutions and the Plancherel theory in $L_2(G)$, where G is a locally compact Abelian group; Wiener's closure theorem in L_1 and the associated classical Tauberian theorems; Hilbert-Schmidt operators and their Fredholm theory; the Hilbert transform and the Calderón-Zygmund inequality.

1. Compact Groups

This first section is concerned with the study of compact topological groups. The existence of Haar measure is established for such a group (Theorem 1) by using one of the fixed point theorems in Section V.10. The Haar measure is used to obtain the basic result of Peter and Weyl which, in turn, is used to establish the fundamental property of the continuous characters on a compact group.

1 THEOREM. *For every compact topological group G there is a unique non-negative regular countably additive measure μ defined on the Borel sets Σ of G such that $\mu(G) = 1$ and $\mu(sE) = \mu(E)$ for each s in G and E in Σ. In addition, $\mu(Es) = \mu(E)$ and $\mu(E^{-1}) = \mu(E)$ for each s in G and E in Σ.*

PROOF. Let $C(G)$ denote the B-space of all continuous real functions on G, and let e be the function identically equal to 1. Let K

be the subset of $C^*(G)$ consisting of all x^* such that $x^*(e) = 1$ and $x^*(f) \geqq 0$ whenever $f(s) > 0$ for every s in G. It is clear that K is convex and closed in the $C(G)$ topology of $C^*(G)$. If the function f is in the unit sphere of $C(G)$ then $-e(s) \leqq f(s) \leqq e(s)$ and so, for x^* in K, $-1 \leqq x^*f \leqq 1$, which shows that $|x^*| \leqq 1$. Thus K is a convex subset of the unit sphere in $C^*(G)$ which is closed in the $C(G)$ topology of $C^*(G)$. It follows from Corollary V.4.3 that K is compact in this topology.

For each s in G let the linear map L_s in $C(G)$ be defined by the equation $(L_s f)(t) = f(st)$. Clearly $L_s L_t = L_{st}$ and $L_1 = I$ where 1 is the unit in G. Since $L_t^* L_s^* = (L_s L_t)^* = L_{st}^*$ the family $\{L_s^* | s \in G\}$ is a group of operators in $C^*(G)$. It is also evident that $L_s^* K \subseteq K$ for every s in G.

It will first be shown that the group $\{L_s^* | s \in G\}$ of operators in $C^*(G)$ with its $C(G)$ topology is an equicontinuous family (V.10.7) on the set K. To do this let V be the neighborhood of the origin in $C^*(G)$ determined by the positive number ε and the functions $f_1, \ldots, f_n$ in $C(G)$. We shall define a neighborhood U of the origin in $C^*(G)$ such that for every pair x^*, y^* in K with $x^* - y^*$ in U the vector $L_s^*(x^* - y^*)$ is in V for every s in G.

The finite set $f_1, \ldots, f_n$ is compact in $C(G)$ and hence, by Corollary IV.6.9, to every positive number δ corresponds a neighborhood N of the identity in G such that for every pair t, u in G with $u^{-1}t$ in N we have

$$|f_j(u) - f_j(t)| < \delta, \qquad j = 1, \ldots, n.$$

Suppose that $u^{-1}t$ is in N. Then for every s in G the element $(su)^{-1}(st) = u^{-1}t$ is in N and so

$$|(L_s f_j)(u) - (L_s f_j)(t)| = |f_j(su) - f_j(st)| < \delta, \qquad j = 1, \ldots, n.$$

It follows from Corollary IV.6.9 that the closure of the set $\{L_s f_j | s \in G, j = 1, \ldots, n\}$ is compact in $C(G)$ and thus, by Theorem I.6.15, it is totally bounded. Hence there are functions $g_1, \ldots, g_m$ in $C(G)$ such that each element $L_s f_j$ has a distance of less than $\varepsilon/4$ from one of the elements $g_1, \ldots, g_m$. The neighborhood U of the origin in $C^*(G)$ is defined to be that determined by the elements $g_1, \ldots, g_m$ and the positive number $\varepsilon/2$. Now if x^*, y^* are in K with

$x^* - y^*$ in U then, by choosing i properly and recalling that $|x^*|$, $|y^*| \leqq 1$, it is seen that

$$
\begin{aligned}
|L_s^*(x^*-y^*)f_j| &= |(x^*-y^*)L_sf_j| \\
&\leq |(x^*-y^*)(L_sf_j-g_i)|+|(x^*-y^*)g_i| \\
&\leqq 2\,|L_sf_j-g_i|+|(x^*-y^*)g_i| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon,
\end{aligned}
$$

for every s in G. Thus the family $\{L_s^*\}$ is equicontinuous on K as asserted.

Theorem V.10.8 may now be applied to yield the existence of a point x^* in K such that $L_s^*x^* = x^*$ for every s in G. It follows from the Riesz representation theorem (IV.6.3) that there is a uniquely determined countably additive regular measure μ on the family Σ of Borel sets in G for which

$$x^*f = \int_G f(s)\mu(ds), \qquad f \in C(G).$$

Since x^* is a positive functional it follows from the Riesz theorem that the measure μ is non-negative and since $x^*(e) = 1$ it follows that $\mu(G) = 1$. Now, if μ_s is defined by the equation $\mu_s(E) = \mu(s^{-1}E)$ then, since $x^* = x^*L_s$, we have

$$\int_G f(t)\mu(dt) = \int_G f(st)\mu(dt) = \int_G f(t)\mu_s(dt), \qquad f \in C(G).$$

It may be readily verified that the measure μ_s is regular and hence the Riesz theorem and the above identity show that $\mu = \mu_s$ which proves the first part of the theorem.

By an argument similar to the preceding, there is a non-negative regular countably additive set function ν on Σ with $\nu(G) = 1$ and $\nu(Es) = \nu(E)$ for E in Σ and s in G. Let ν_1 be any such function and let μ_1 be any non-negative regular measure on Σ with $\mu_1(G) = 1$ and $\mu_1(sE) = \mu_1(E)$. It will be shown that $\mu_1 = \nu_1$ which will establish the uniqueness of μ and show that $\mu(Es) = \mu(E)$. To see this let f be a continuous scalar function on G so that the function f_1 defined on the Cartesian product $G\times G$ by the equation $f_1(s, t) = f(st)$ is continuous and hence $\nu_1\times\nu_1$ measurable. Since $\mu_1(G) = \nu_1(G) = 1$ it follows from the Fubini theorem (III.11.9) that

$$\begin{aligned}\int_G f(t)\mu_1(dt) = \int_G f(st)\mu_1(dt) &= \int_G \left\{\int_G f(st)\mu_1(dt)\right\} \nu_1(ds) \\ &= \int_G \left\{\int_G f(st)\nu_1(ds)\right\} \mu(dt) \\ &= \int_G f(st)\nu_1(ds) = \int_G f(s)\nu_1(ds).\end{aligned}$$

Since this holds for every continuous function f it follows from the Riesz theorem (IV.6.3) that $\mu_1 = \nu_1$. Finally, to see that $\mu(E) = \mu(E^{-1})$ let the measure λ on Σ be defined by the equation $\lambda(E) = \mu(E^{-1})$. Then $\lambda(G) = 1$ and $\lambda(sE) = \mu(E^{-1}s^{-1}) = \mu(E^{-1}) = \lambda(E)$. Furthermore since the map $s \to s^{-1}$ is a homeomorphism in G the measure λ is regular. It follows therefore, from the uniqueness of μ, that $\lambda = \mu$. Q.E.D.

2 Definition. The unique measure whose existence is established in Theorem 1 is called the *Haar measure* on the compact group G.

By using Haar measure it will be shown that the class of continuous functions on G which are finite dimensional in the sense of the following definition form a fundamental set in $C(G)$ and also in $L_2(G, \Sigma, \mu)$.

3 Definition. Let f be a complex valued function defined on a group G. For s in G let f^s be defined on G by the equation $f^s(t) = f(ts)$. The function f^s is the *translate* of f by s. The function f is said to be *finite dimensional* if its set $\{f^s | s \in G\}$ of translates is a finite dimensional vector space of functions.

The spectral theorem will be used in the proof of the following theorem and so the field of scalars is taken to be the field of complex numbers.

4 Theorem. (*Peter-Weyl*) *Let G be a compact topological group, with Σ its Borel field and μ its Haar measure. Then the set of complex continuous finite dimensional functions is fundamental both in $C(G)$ and in $L_2(G, \Sigma, \mu)$.*

Proof. Let $L_2 = L_2(G, \Sigma, \mu)$, and let $\mathfrak{N}$ be the ortho-complement in L_2 of the set of continuous finite dimensional functions. We shall prove density in L_2 by showing that $\mathfrak{N} = \{0\}$. To show this it will be shown that an arbitrary element in $\mathfrak{N}$ is orthogonal to

every element in $C(G)$, a dense set in L_2. Now an arbitrary element f in $C(G)$ may be written as $f = h+ig$ where

$$h(s) = \tfrac{1}{2}[f(s)+\overline{f(s^{-1})}], \quad g(s) = \frac{i}{2}[\overline{f(s^{-1})}-f(s)].$$

Since the functions h and g satisfy the relations $\overline{h(s)} = h(s^{-1})$, $\overline{g(s)} = g(s^{-1})$ it will suffice to show that every continuous function g with $\overline{g(s)} = g(s^{-1})$ is orthogonal to $\mathfrak{N}$.

The proof of this will depend upon some properties of the linear operator T_g in L_2 defined by the equation

$$(T_g f)(s) = \int_G g(su^{-1})f(u)\mu(du), \qquad f \in L_2.$$

Since g is in $C(G)$ it follows from Corollary IV.6.9 that for every $\varepsilon > 0$ there is a neighborhood U of the identity in G such that if st^{-1} is in U then $|g(s) - g(t)| < \varepsilon$. Now let f be in L_2 and have norm $|f| < 1$ and let st^{-1} be in U. Then, since $(su^{-1})(tu^{-1})^{-1} = st^{-1}$ is in U,

$$\begin{aligned}|(T_g f)(s) - (T_g f)(t)| &= \left|\int_G \{g(su^{-1}) - g(tu^{-1})\}f(u)\mu(du)\right| \\ &\leq \varepsilon\left\{\int_G |f(u)|\mu(du)\right\} \leq \varepsilon\left\{\int_G |f(u)|^2\mu(du)\right\}^{\frac{1}{2}} \leq \varepsilon.\end{aligned}$$

In view of Corollary IV.6.9 this inequality shows that T_g maps the unit sphere in L_2 into a conditionally compact set in $C(G)$. Thus T_g is a compact operator (VI.5.1) as a map from L_2 into $C(G)$ and, *a fortiori*, as a map from L_2 into L_2.

Next, by using the property that $\overline{g(s)} = g(s^{-1})$, it may be seen that T_g is a self adjoint operator. For it follows from the Fubini theorem that every pair f, h of elements in L_2 satisfies the equations

$$\begin{aligned}(f, T_g h) &= \int_G f(s)\overline{\left\{\int_G g(st^{-1})h(t)\mu(dt)\right\}}\mu(ds) \\ &= \int_G \overline{h(t)}\left\{\int_G \overline{g(st^{-1})}f(s)\mu(ds)\right\}\mu(dt) \\ &= \int_G \overline{h(t)}\left\{\int_G g(ts^{-1})f(s)\mu(ds)\right\}\mu(dt) = (T_g f, h).\end{aligned}$$

By Theorem X.3.5 there is a complete orthonormal set $\{\varphi_\alpha\}$ of eigenfunctions of T_g and by Theorem VII.4.5 the eigenfunctions correspond-

ing to an eigenvalue $\lambda \neq 0$ form a finite dimensional space. Now if $T_g\varphi = \lambda\varphi$ then

$$\int_G g(su^{-1})\varphi(u)\mu(du) = \lambda\varphi(s), \qquad s \in G,$$

is a continuous function. By replacing s by st and u by ut and using the fact that $\mu(Et) = \mu(E)$ it is seen that

$$\int_G g(su^{-1})\varphi(ut)\mu(du) = \lambda\varphi(st),$$

i.e., every translate φ^t of an eigenfunction φ corresponding to λ is also an eigenfunction corresponding to λ. Thus every eigenfunction of T_g which corresponds to a non-zero eigenvalue is a finite dimensional continuous function. Hence $\mathfrak{N}$ is orthogonal to every eigenfunction of T_g except to those corresponding to $\lambda = 0$. It follows from Theorem X.3.4 that for h in $\mathfrak{N}$, $h = \sum(h, \varphi_\alpha)\varphi_\alpha$ where the sum is taken over those α for which $T_g\varphi_\alpha = 0$. Thus $T_g h = 0$, i.e.,

$$0 = \int_G g(su^{-1})h(u)\mu(du) = \int_G \overline{g(us^{-1})}h(u)\mu(du).$$

If in this equation s is replaced by the unit in G, it reduces to the equation $(h, g) = 0$, which shows that g is orthogonal to $\mathfrak{N}$ and completes the proof of the fact that the finite dimensional continuous functions form a fundamental set in L_2.

It remains to be shown that the finite dimensional continuous functions are dense in $C(G)$. To do this let k in $C(G)$ and $\varepsilon > 0$ be given. As before it is seen from Corollary IV.6.9 that there is a neighborhood V of the identity in G such that $|k(s)-k(ts)| < \varepsilon/2$ for every s in G and every t in V. Since the map $s \to s^{-1}$ is a homeomorphism there is a neighborhood W of the identity in G such that W and W^{-1} are both contained in V and thus the set $U = W \cup W^{-1}$ is a neighborhood of the identity in G with the properties that $U \subseteq V$ and $U = U^{-1}$. Since a compact space is normal (I.5.9) it follows from the Urysohn theorem (I.5.2) that there is a non-negative continuous function $h \neq 0$ on G which vanishes on the complement of U. These properties of h will be preserved if we normalize it by requiring that its $L_1(G, \Sigma, \mu)$ norm $|h|_1 = 1$. The function h' defined by the equation $h'(s) = [h(s)+h(s^{-1})]/2$ has all the properties listed

for h and in addition the property that $h'(s) = h'(s^{-1})$. Thus we may and shall assume that $h(s) = h(s^{-1})$. Now the $C(G)$ norm of $k - T_h k$ is

$$|k - T_h k| = \sup_{s \in G} |k(s) - \int_G h(st^{-1})k(t)\mu(dt)|,$$

and if t is replaced by ts it is seen that

$$\begin{aligned}
|k - T_h k| &= \sup_{s \in G} \left| \int_G h(t^{-1})\{k(s) - k(ts)\}\mu(dt) \right| \\
&= \sup_{s \in G} \int_U h(t^{-1})|k(s) - k(ts)|\mu(dt) \\
&\leq \frac{\varepsilon}{2} \int_U h(t^{-1})\mu(dt) = \frac{\varepsilon}{2} \int_U h(t)\mu(dt) = \frac{\varepsilon}{2} |h|_1 = \frac{\varepsilon}{2}.
\end{aligned}$$

According to the first part of the theorem there is a linear combination g of continuous finite dimensional functions with

$$|k - g|_2 \leq \frac{\varepsilon}{2|h|_2}.$$

Since

$$\begin{aligned}
|T_h k - T_h g| &= \sup_{s \in G} \left| \int_G h(st^{-1})\{k(t) - g(t)\}\mu(dt) \right| \\
&\leq \sup_{s \in G} \left\{ \int_G h(st^{-1})^2 \mu(dt) \right\}^{\frac{1}{2}} |k - g|_2 \\
&\leq |h|_2 |k - g|_2 \leq \frac{\varepsilon}{2},
\end{aligned}$$

it follows that $|k - T_h g| < \varepsilon$. Thus, to complete the proof, it will suffice to show that $T_h g$ is itself a linear combination of finite dimensional continuous functions and hence is finite dimensional. Since it has already been observed that $T_h g$ is continuous, to finish the proof it suffices to show that $T_h g$ is a finite dimensional function if g is a finite dimensional function.

Since

$$(T_h g)(s) = \int_G h(st^{-1})g(t)\mu(dt)$$

it is seen, by replacing s by su and t by tu, that

$$(T_h g)(su) = \int_G h(st^{-1}) g(tu) \mu(dt),$$

which means that $(T_h g)^u = T_h(g^u)$. Now if $g^{u_1}, \ldots, g^{u_n}$ form a basis for the space of translates of g then for any u in G the translate g^u has the form $\sum_{i=1}^n \alpha_i g^{u_i}$ and so

$$(T_h g)^u = \sum_{i=1}^n \alpha_i T_h(g^{u_i}) = \sum_{i=1}^n \alpha_i (T_h g)^{u_i},$$

which shows that the translates of $T_h g$ form a finite dimensional space and thus that $T_h g$ is a finite dimensional function. Q.E.D.

In the case of compact Abelian groups, Theorem 4 may be sharpened to some extent.

5 Definition. If G is an Abelian group, then a *character* of G is a complex valued function χ defined on G which is such that $\chi(e) = 1$ and $\chi(st) = \chi(s)\chi(t)$ for all s, t in G.

6 Theorem. *Let G be a compact Abelian group, with Σ its Borel field and μ its Haar measure. Then the set of continuous characters is fundamental both in $C(G)$ and in $L_2(G, \Sigma, \mu)$.*

Proof. In view of Theorem 4, it suffices to prove that any continuous finite dimensional function f is a linear combination of continuous characters. For s in G let R_s be defined on L_2 by the equation $R_s g = g^s$. Since G is Abelian, it is readily seen that the family $\{R_s | s \in g\}$ is a commutative family of unitary operators on L_2. By hypothesis, the subspace $\mathfrak{F}$ generated by the translates of f is a finite dimensional manifold which is invariant under each of the operators R_s.

Let $\mathfrak{F}_1 \neq \{0\}$ be a subspace of $\mathfrak{F}$ which is invariant under each of the operators R_s, and which has no non-trivial subspaces with this property. To see that $\mathfrak{F}_1$ is one-dimensional, let s be fixed and let $\lambda(s)$ be an eigenvalue of the operator R_s considered in the space $\mathfrak{F}_1$. Let $\mathfrak{H} = \{h \in \mathfrak{F}_1 | R_s h = \lambda(s) h\}$; then since $R_s(R_t h) = \lambda(s)(R_t h)$, it is seen that $\mathfrak{H}$ is invariant under the set of operators $\{R_t\}$, and since $\mathfrak{H} \neq \{0\}$ it follows that $\mathfrak{H} = \mathfrak{F}_1$. This implies that in $\mathfrak{F}_1$ the arbitrary operator R_s is $\lambda(s)$ times the identity operator, and hence the dimension of $\mathfrak{F}_1$ must be one, for otherwise it would have non-trivial invariant subspaces. Since the operators R_s are unitary, they also

leave $\mathfrak{F} \ominus \mathfrak{F}_1$ invariant and the above argument may be repeated to decompose $\mathfrak{F}$ into a direct sum $\mathfrak{F}_1 \oplus \ldots \oplus \mathfrak{F}_n$ of non-trivial orthogonal one-dimensional manifolds. Furthermore, on each subspace $\mathfrak{F}_i$ each operator R_s acts as multiplication by some scalar $\lambda_i(s)$, i.e.,

$$f_i(ts) = \lambda_i(s) f_i(t), \qquad f_i \in \mathfrak{F}_i.$$

It follows that if $f_i \neq 0$ then f vanishes nowhere and thus, by placing $t = e$, the identity in G, in the above equation, it is seen that λ_i is in $\mathfrak{F}_i$ and that $\lambda_i(st) = \lambda_i(s)\lambda_i(t)$. Since $\mathfrak{F}_1$ is one-dimensional it consists of scalar multiples of the character λ_i. Thus every function f in $\mathfrak{F}$ is a finite linear combination of continuous characters of G. Q.E.D.

2. Almost Periodic Functions

The theory of compact groups developed in the preceding section may be applied in an interesting and significant manner to the analysis of the space AP of almost periodic functions on the real line $R = (-\infty, +\infty)$. In fact, the theory will be applied in this section to prove the principal result in H. Bohr's theory of almost periodic functions. It has been observed (IV.7.6) that there is a compact Hausdorff space S such that AP is isometrically and *-isomorphic to the space $C(S)$. In fact, R may be embedded as a dense subset of S in such a manner that every function f in AP has a unique extension to a function f_1 in $C(S)$. This follows from Corollary IV.6.19 and the observation that the periodic (and hence the almost periodic) functions distinguish between the points of R. Moreover, the correspondence $f \leftrightarrow f_1$ is an isometric and algebraic isomorphism between AP and $C(S)$. In this way one may regard AP as the family of all the restrictions of functions in $C(S)$ to the dense set R.

The next step will be that of showing that the group structure of R may be extended by continuity to S in such a manner that S becomes a compact Abelian topological group. In doing this we make use of a lemma.

1 LEMMA. *Let f be in AP and let $\varepsilon > 0$. Then there is a finite set $h_1, \ldots, h_m$ in AP and a number $\delta > 0$ with the property that if x_1, x_2, y_1, y_2 are any four numbers such that*

[*] $$|h_i(x_1)-h_i(x_2)| < \delta, \quad |h_i(y_1)-h_i(y_2)| < \delta,$$

for $i = 1, \ldots, m$, *then it follows that*

$$|f(x_1+y_1)-f(x_2+y_2)| < \varepsilon.$$

PROOF. Let B be the set of all functions g of the form $g(x) = f(x+u)$, where $-\infty < u < +\infty$. Then B is contained in a compact set of AP (IV.7.2). Consequently (I.6.15), there is a finite set $h_1, \ldots, h_m$ in AP such that for each g in B there is an h_i with $|g-h_i| < \varepsilon/8$. Suppose that [*] is satisfied with $\delta = \varepsilon/4$. Then for any g in B we have

$$\begin{aligned}|g(x_1)-g(x_2)| &\leqq |g(x_1)-h_i(x_1)| \\ &\quad +|h_i(x_1)-h_i(x_2)|+|h_i(x_2)-g(x_2)| \\ &< \frac{\varepsilon}{8}+\frac{\varepsilon}{4}+\frac{\varepsilon}{8} = \frac{\varepsilon}{2}.\end{aligned}$$

Similarly $|g'(y_1)-g'(y_2)| < \varepsilon/2$, for any g' in B. Now for fixed i, take g and g' to be the functions in B defined by the equations

$$g(x) = f(x+y_2), \quad g'(x) = f(x+x_1), \qquad x \in R.$$

Since $g(x_1) = f(x_1+y_2) = g'(y_2)$, on combining, we have

$$\begin{aligned}|f(x_1+y_1)-f(x_2+y_2)| &\leqq |g'(y_1)-g'(y_2)| \\ &\quad +|g(x_1)-g(x_2)| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon,\end{aligned}$$

which proves the lemma. Q.E.D.

2 THEOREM. *The line R may be embedded as a dense subgroup of a compact Abelian topological group S in such a way as to make AP the family of all restrictions $f|R$ to R of functions f in $C(S)$. The operation $f \to f|R$ is an isometric *-isomorphism of $C(S)$ onto AP. The group S is called the Bohr compactification of the real numbers.*

PROOF. Everything in this statement has already been established except the fact that S is an Abelian topological group under an operation which coincides on R with addition. Let s_1 and s_2 be points of S; we will define $s_1 \oplus s_2$. Let $\mathscr{U}_1$ denote the collection of neigh-

borhoods of s_1 and $\mathscr{U}_2$ the neighborhoods of s_2. For $U_i \in \mathscr{U}_i$, $i = 1, 2$, let

$$W(U_1, U_2) = \overline{U_1 \cap R + U_2 \cap R},$$

where the sum is taken in R and the closure in S. If $V_i \in \mathscr{U}_i$, then $W(U_1, U_2) \cap W(V_1, V_2) \supseteq W(U_1 \cap V_1, U_2 \cap V_2)$ and so the family $W = \{W(U_1, U_2) | U_i \in \mathscr{U}_i\}$ has the finite intersection property Since S is compact, $T(s_1, s_2) = \bigcap W$ is not void; we will show that $T(s_1, s_2)$ contains only one point, which will be called $s_1 \oplus s_2$.

Suppose that $T(s_1, s_2)$ contains two distinct points t_1, t_2. By the Urysohn theorem (I.5.2), there is a positive function f in $C(S)$ with $f(t_1) = 1$, $f(t_2) = 0$. Applying Lemma 1 to the restriction of f to R, we conclude that there is a set $h_1, \ldots, h_m$ in AP and a number $\delta > 0$ such that if x_1, x_2, y_1, y_2 are any four real numbers such that $|h_i(x_1) - h_i(x_2)| < \delta$ and $|h_i(y_1) - h_i(y_2)| < \delta$ for $i = 1, \ldots, m$, then $|f(x_1 + y_1) - f(x_2 + y_2)| < 1/2$. Each h_i has a unique extension to a continuous function on S and we denote this extension by the same letter. Hence there are neighborhoods U_1 of s_1 and U_2 of s_2 such that for s in U_1, $|h_i(s_1) \quad h_i(s)| < \delta/2$ and for s in U_2, $|h_i(s_2) - h_i(s)| < \delta/2$, for $i = 1, \ldots, m$. Choose x_1, x_2 in $U_1 \cap R$ and y_1, y_2 in $U_2 \cap R$. Then $|h_i(x_1) - h_i(x_2)| < \delta$ and $|h_i(y_1) - h_i(y_2)| < \delta$ for $i = 1, \ldots, m$ which implies that $|f(x_1 + y_1) - f(x_2 + y_2)| < 1/2$. This shows that if u_1, u_2 are in the set $U_1 \cap R + U_2 \cap R$ then $|f(u_1) - f(u_2)| < 1/2$. By continuity we may conclude that if u_1, u_2 are in $W(U_1, U_2)$, then $|f(u_1) - f(u_2)| \leq 1/2$. Since t_1, t_2 are in $W(U_1, U_2)$ for any choice of U_1, U_2 we have

$$1 = |f(t_1) - f(t_2)| \leq \tfrac{1}{2},$$

which is a contradiction. This proves that $T(s_1, s_2)$ consists of precisely one point, which we denote by $s_1 \oplus s_2$.

It is clear that if s_1, s_2 are in R then $s_1 \oplus s_2 = s_1 + s_2$, so that the operation $\oplus$ is an extension of the addition in R. To show that $\oplus$ is a continuous function on $S \times S$, let V be a neighborhood of $s_1 \oplus s_2$. If, for every U_1 in $\mathscr{U}_1$ and U_2 in $\mathscr{U}_2$, $W(U_1, U_2)$ intersects the complement V' of V, then the family $\{V' \cap W(U_1, U_2) | U_i \in \mathscr{U}_i\}$ has the finite intersection property and therefore a non-void intersection, i.e., $V' \cap T(s_1, s_2) = V' \cap \bigcap W \neq \phi$. We have seen above that $T(s_1, s_2)$ contains only the point $s_1 \oplus s_2$, and hence the supposi-

tion just made leads to a contradiction. Thus there are neighborhoods $U_i \in \mathscr{U}_i$ such that $W(U_1, U_2) \subset V$. Now if $u_i \in U_i$, $i = 1, 2$, then

$$\{u_1 \oplus u_2\} = T(u_1, u_2) \subset W(U_1, U_2)$$

and hence $u_1 \oplus u_2 \in V$. This proves that $\oplus$ is continuous. From the continuity of $\oplus$, the fact that R is dense in S, and that $\oplus$ coincides with $+$ on R, we conclude immediately that $s \oplus 0 = s$, $s_1 \oplus s_2 = s_2 \oplus s_1$ and

$$s_1 \oplus (s_2 \oplus s_3) = (s_1 \oplus s_2) \oplus s_3.$$

It remains to be shown that inverse elements under $\oplus$ exist in S. Consider the mapping $H : AP \to AP$ defined by $(Hg)(x) = g(-x)$, $x \in R$. Clearly H is an algebraic isomorphism of AP and preserves conjugation, i.e., $H(\bar{g}) = \overline{Hg}$, $g \in AP$. Since AP is equivalent to $C(S)$ the mapping H may be regarded as operating on $C(S)$. By Theorem IV.6.26 there is a homeomorphism h of S onto itself such that $(Hf)(s) = f(h(s))$, $f \in C(S)$, $s \in S$. In particular, if s is in R then $h(s) = -s$, and so $s \oplus h(s) = 0$ for s in a dense set R and thus identically on S. Consequently $h(s)$ is the inverse of s, and S is a topological group. Q.E.D.

3 Lemma. *The continuous characters of the compact Abelian group S are the extensions to S of functions of the form $e^{i\lambda x}$, $x \in R$, where λ is an arbitrary real number.*

Proof. We first observe that if χ is a function of the form $\chi(x) = e^{i\lambda x}$, $x \in R$, then since χ is periodic it is, *a fortiori*, almost periodic and has a continuous extension χ_1 to S. Furthermore, since $\chi(x+y) = \chi(x)\chi(y)$ and $|\chi(x)| = 1$ for all $x, y \in R$, these identities hold for the extension χ_1 and so χ_1 is a character of S.

Conversely, if χ_1 is a continuous character of S, let χ be the restriction of χ_1 to R. Then $\chi(0) = 1$, $\chi(x+y) = \chi(x)\chi(y)$, $|\chi(x)| = 1$, $x, y \in R$ and χ is continuous on R. Let $\alpha > 0$ be such that if $|x| \leqq \alpha$, then $|\chi(x) - 1| < 1$, and let θ be such that $\chi(\alpha) = e^{i\theta}$. Since $\chi(\alpha) = [\chi(\alpha/2)]^2$ we must have $\chi(\alpha/2) = e^{i\theta/2}$, and by induction $\chi(\alpha/2^n) = e^{i\theta/2^n}$. This implies that if $r = 2^{-n}m$, where m and n are integers then $\chi(r\alpha) = e^{i\theta r}$. By continuity, this equation holds for all real

numbers r. By placing $\lambda = \theta/\alpha$ we have $\chi(x) = e^{i\lambda x}$, $x \in R$, as asserted. Q.E.D

With these preliminaries at our disposal we are now able to prove the main theorem concerning almost periodic functions on the line.

4 THEOREM. (*Bohr*) *A continuous function on* $R = (-\infty, \infty)$ *is almost periodic if and only if it may be uniformly approximated by finite linear combinations of functions in the set* $\{e^{i\lambda x} | \lambda \in R\}$.

PROOF. Since AP is a B-space (IV.7.5) it is clear that the family of functions in the closed linear manifold spanned by the periodic functions is contained in AP. On the other hand, it has been seen (Theorem 2) that AP is isometric and isomorphic with $C(S)$, where S is a compact Abelian group, and also (Lemma 3) that the continuous characters of S are of the form $e^{i\lambda x}$. By Theorem 1.6, the set of continuous characters is fundamental in $C(S)$; consequently, their restrictions to R are a fundamental set in the space AP. Q.E.D.

From the isometric isomorphism of the spaces AP and $C(S)$ and the Riesz representation theorem (IV.6.3.) we may state the following result.

5 THEOREM. *The space* AP^* *is isometrically isomorphic with the space* $rca(S)$ *of all regular countably additive measures defined on the Borel subsets of the Bohr compactification* S *of the real numbers. The isomorphism* $x^* \to \mu_1 \in rca(S)$ *is given by the formula*

$$x^* f = \int_S f_1(s)\mu_1(ds), \qquad f \in AP,$$

where f_1 *is the unique extension to* S *of the function* f *in* AP.

3. Convolution Algebras

In Chapter IX it was seen that a number of spaces of functions are B-algebras under pointwise multiplication, and so the theory of such algebras can be applied to them immediately. However, if L_1 denotes the space of functions on the real line $(-\infty, +\infty)$ which are Lebesgue integrable, then it is readily seen that L_1 does not form a B-algebra under pointwise multiplication. It is an important

and useful fact that L_1 can be given a multiplication with respect to which it becomes, after the adjunction of a unit, a B-algebra to which the results of Chapter IX are applicable. As the "product" of two functions f and g in L_1 we shall take the convolution $f*g$, defined by the equation

$$(f*g)(x) = \int_{-\infty}^{+\infty} f(x-y)g(y)dy, \qquad \infty < x < \infty.$$

It will be seen that the analysis of the algebra L_1 under convolution as "product" is closely connected with the theory of the Fourier transform.

Instead of restricting our consideration to the case of the additive group of real numbers, we shall discuss the case of a locally compact Abelian group which we denote by R. We assume throughout that R is σ-compact, i.e., the union of countably many compact sets. Every such group has a non-negative countably additive measure which is defined on the Borel sets Σ, finite on compact sets, positive or infinite on open sets, invariant under translation, i.e., $\lambda(x+E) = \lambda(E)$ for E in Σ and x in R, and which enjoys the regularity property $\sup \lambda(F) = \lambda(E) = \inf \lambda(G)$ for E in Σ, where F varies over the closed subsets of E and G over the open sets containing E. Such a measure is unique up to multiplication by positive numbers, and is called *Haar measure*. In the case $R = (-\infty, +\infty)$, the Haar measure may be taken to be Lebesgue measure; in the case of a compact group, its existence and uniqueness was proved in Theorem 1.1. The reader who is unfamiliar with Haar measure may wish to consult the remarks under the heading *Convolution Algebras* in Section 11. He may also wish to suppose, on first reading, that R is the real number system. However, the reader who is familiar with the theory of Haar measure will naturally wish to note that the proofs given below apply without change to locally compact, σ-compact Abelian groups in general. For this reader we call attention to the fact that in the proof of Lemma 3 we require R to be *non-discrete*, and so, throughout the section, we shall suppose that R is non-discrete. To make the special considerations required for the discrete case is quite easy, and some remarks on this score will also be found in the notes.

Thus we treat a non-discrete locally compact, σ-compact Abelian group R, making use of the elementary properties of its Haar measure

which are well-known in the case of Lebesgue measure on the line. When integration is with respect to Haar measure, as is generally the case, we write dx instead of $\lambda(dx)$. Throughout, we denote $L_p(R, \Sigma, \lambda)$ by $L_p(R)$.

To begin the study, some basic properties of convolutions will be obtained.

1 Lemma. (a) *If f is λ-measurable, then the function $f(x-y)$ is a $\lambda \times \lambda$-measurable function.*

(b) *For $f, g \in L_1(R)$ the function $f(x-y)g(y)$ is integrable in y for almost all x and the convolution $f*g$ of f and g, which is defined by the equation*

$$(f*g)(x) = \int_R f(x-y)g(y)dy,$$

is in $L_1(R)$ and satisfies the inequality

$$|f*g|_1 \leq |f|_1 |g|_1.$$

With convolution as multiplication the linear space $L_1(R)$ is a commutative and associative algebra.

(c) *For f in $L_1(R)$ and g in $L_2(R)$ the convolution $(f*g)(x) = \int_R f(x-y)g(y)dy$ exists for almost all x, belongs to $L_2(R)$, and $|f*g|_2 \leq |f|_1 |g|_2$. The product $f*g$ is linear in each variable, and satisfies the equation $h*(f*g) = (h*f)*g$ for h, f in $L_1(R)$ and g in $L_2(R)$. For f in $L_1(R)$, the Hilbert space adjoint of the bounded linear transformation $g \to f*g$ in $L_2(R)$ is the transformation $g \to \tilde{f}*g$, where $\tilde{f}(x) = \overline{f(-x)}$.*

(d) *If f is in $L_1(R)$ and g is in $L_\infty(R)$, then the convolution integral of f and g exists for all x and defines a function in $C(R)$ of norm at most $|f|_1 |g|_\infty$.*

(e) *If f and g are in $L_2(R)$, the convolution integral of f and g exists for all x and defines a function in $C(R)$ of norm at most $|f|_2 |g|_2$.*

(f) *If f is in L_p, $1 \leq p < \infty$, and if f_y is defined by the equation $f_y(x) = f(x-y)$, $x \in R$, then the mapping $y \to f_y$ is a continuous function on R to $L_p(R)$.*

Proof. Statement (a) was proved in Lemma VIII.1.24 for the case where R is the additive group of real numbers. The general case will be discussed in the notes at the end of the chapter.

Since the integrand in the convolution integral is measurable, we see, from Tonelli's theorem (III.11.14), that

$$\int_{R\times R}|f(x-y)g(y)|d(x\times y) = \int_R |g(y)|\left\{\int_R |f(x-y)|dx\right\}dy = |f|_1|g|_1,$$

and so $f(x-y)g(y)$ is $\lambda\times\lambda$-integrable. The first conclusion in (b) then follows from Fubini's theorem (III.11.9). The remainder of (b) is proved by making elementary changes of variable which, since they have been given before (see Lemma VIII.1.25), we omit here.

Next we prove (f). If $1 \leqq p < \infty$, it is readily seen from Corollary III.3.8 and the regularity of λ that the collection of functions which are continuous and vanish outside of compact sets is dense in $L_p(R)$. Hence for f in $L_p(R)$ let k be such a continuous function with $|f-k|_p < \varepsilon$. Since k is uniformly continuous, we see, for z sufficiently close to y, that $|k_z - k_y|_p < \varepsilon$. Therefore

$$|f_z-f_y|_p < |f_z - k_z|_p + |k_z-k_y|_p + |k_y - f_y|_p < 3\varepsilon,$$

as desired.

Statement (d) now follows readily from (f), for since the map $y \to f_y$ is a continuous map of R into $L_1(R)$, the integral $\int f(x-y)g(y)dy$ is, for g in $L_\infty(R)$, continuous in x and

$$\left|\int f(x-y)g(y)dy\right| \leqq |f|_1|g|_\infty .$$

The proof of (e) is similar.

To prove (c) let f be in $L_1(R)$ and g, h in $L_2(R)$. From (e) it is seen that

$$\int_R |g(x-y)\overline{h(x)}|dx$$

is a continuous function of y and is bounded by $|g|_2|h|_2$. Hence, by Tonelli's theorem we have

$$\int_{R\times R}|f(y)g(x-y)\overline{h(x)}|d(x\times y) = \int_R |f(y)|\left\{\int_R |g(x-y)\overline{h(x)}|dx\right\}dy$$
$$\leqq |f|_1|g|_2|h|_2.$$

This shows that the double integral on the left is finite and thus from Fubini's theorem we conclude that the inner product

$$
\begin{aligned}
(f*g, h) &= \int_R \left\{\int_R f(x-y)g(y)dy\right\} \overline{h(x)}dx \\
&= \int_R \left\{\int_R f(y)g(x-y)dy\right\} \overline{h(x)}dx \leq |f|_1|g|_2|h|_2,
\end{aligned}
$$

for all h in $L_2(R)$. This proves that $f*g$ is in $L_2(R)$ and that $|f*g|_2 \leq |f|_1|g|_2$. The second sentence in (c) follows from the corresponding assertions in (b) and the fact that $L_1(R) \cap L_2(R)$ is dense in $L_2(R)$. The final clause in (c) amounts to showing that $(f*g, h) = (g, \tilde{f}*h)$; i.e.,

$$
\int_R \left\{\int_R f(x-y)g(y)\,dy\right\} \overline{h(x)}dx = \int_R g(y) \left\{\overline{\int_R \tilde{f}(y-x)h(x)dx}\right\} dy.
$$

But this equality follows from Fubini's theorem which asserts that both of the above integrals equal the double integral

$$
\int_{R\times R} f(x-y)g(y)\overline{h(x)}d(x\times y),
$$

which we have already seen to exist. This completes the proof. Q.E.D.

As a consequence of parts (b) and (c) of Lemma 1, we have the following corollary.

2 Corollary. *With convolution as multiplication the linear space $L_1(R)$ satisfies all the hypotheses of a commutative B-algebra except perhaps for the one asserting the existence of a unit. Furthermore, the unary map $f \to \tilde{f}$ in $L_1(R)$ is an involution, i.e.,*

$$
\begin{aligned}
\widetilde{(f+g)} &= \tilde{f}+\tilde{g}, & \widetilde{(fg)} &= \tilde{g}\tilde{f}, \\
\widetilde{(\alpha f)} &= \bar{\alpha}\tilde{f}, & \tilde{\tilde{f}} &= f.
\end{aligned}
$$

Later we shall adjoin a unit to $L_1(R)$ making it a commutative B-algebra. The preceding corollary shows that the extended algebra is a commutative algebra with an involution. It is not a B^*-algebra however, since the identity $|\tilde{f}*f| = |f|^2$ is not satisfied.

3 Lemma. *If, for f in $L_1(R)$, the operator $T(f)$ in $L_2(R)$ is defined by the equation*

$$
T(f)g = f*g, \qquad g \in L_2(R),
$$

then the map $f \to T(f)$ is a continuous isomorphism of the algebra

$L_1(R)$ onto a commutative algebra $\mathfrak{A}_0$ of operators in the Hilbert space $L_2(R)$. It also has the properties

$$T(f)^* = T(\bar{f}), \qquad |T(f)| \leq |f|_1.$$

Moreover the closure of $\mathfrak{A}_0$ in the uniform topology of operators does not contain the identity operator.

PROOF. A comparison of this statement with that of Lemma 1(c) shows that it is sufficient to prove the final assertion and the fact that $T(f) = 0$ implies $f = 0$.

It is in the following proof that the standing implicit assumption of the non-discreteness of R is first used. Suppose, first, that the closure of $\mathfrak{A}_0$ in the uniform topology of operators does contain the identity operator, so that an f in $L_1(R)$ exists with $|f*g - g|_2 < (1/2)|g|_2$ for all g in $L_2(R)$. Since $L_1(R) \cap L_\infty(R)$ is dense in $L_1(R)$, we may assume without loss of generality that f is in $L_1(R) \cap L_\infty(R)$. Since the measure of the single point $x = 0$ is zero, it follows from the regularity of the measure λ, and the fact that every open set has positive measure, that there are neighborhoods U_n, $n = 1, 2, \ldots$, of $\{0\}$ such that $0 \neq \lambda(U_n) < (1/n^2)$. Let $g_n(x) = (\lambda(U_n))^{-\frac{1}{2}}$ for x in U_n and $g_n(x) = 0$ elsewhere. Then g_n is in $L_1(R) \cap L_2(R)$, $|g_n|_2 = 1$, $|g_n|_1 \leq 1/n$. Consequently

$$\begin{aligned}
\tfrac{1}{4} > |g_n - f*g_n|_2^2 &= \int_R |g_n(x) - (f*g_n)(x)|^2 \, dx \\
&\geq \int_{U_n} |g_n(x) - (f*g_n)(x)|^2 \, dx \\
&\geq \inf_{x \in U_n} \{|g_n(x) - (f*g_n)(x)|^2\} \lambda(U_n).
\end{aligned}$$

By Lemma 1(d), $|(f*g_n)(x)| \leq |f|_\infty / n$ for every x in R and we therefore conclude, from the preceding chain of inequalities, that

$$\tfrac{1}{4} > \{(\lambda(U_n))^{-\frac{1}{2}} - |f|_\infty n^{-1}\}^2 \lambda(U_n).$$

Since the right side approaches 1, we have obtained a contradiction. Thus the closure of $\mathfrak{A}_0$ in the uniform operator topology does not contain the identity operator.

Suppose that $T(f) = 0$ for some f in $L_1(R)$ so that for each g in $L_2(R)$ we have $(f*g)(x) = 0$ for almost all x. Let h be any function in $L_\infty(R)$ which vanishes outside of a compact set C; then h is in

$L_1(R) \cap L_2(R)$. Since h is in $L_2(R)$ it follows that $(f*h)(x) = 0$ almost everywhere, and since h is $L_\infty(R)$ it follows from Lemma 1(d) that $f*h$ is continuous. Consequently

$$0 = (f*h)(0) = \int_C f(-y)h(y)dy.$$

Now the function h defined by the equations $h(y) = |f(-y)|(f(-y))^{-1}$, $y \in C$, and $h(y) = 0$, $y \notin C$ is clearly in $L_\infty(R)$. The above argument allows us to conclude that

$$\int_C |f(x)|dx = 0$$

for any compact set C, and thus $|f|_1 = 0$. This proves that the map $f \to T(f)$ is one-to-one. Q.E.D.

We now introduce a number of definitions and notations which will be of fundamental importance throughout the rest of this section. Since much of the subsequent analysis will be couched in terms introduced in the next paragraph, the reader should study it with particular care.

For the remainder of this section $\mathfrak{A}$ will denote the B^*-algebra of operators $\overline{\mathfrak{A}}_0 \oplus \{\alpha I\}$, where I is the identity operator in the Hilbert space $L_2(R)$, and where $\overline{\mathfrak{A}}_0$ is the closure of $\mathfrak{A}_0$ in the uniform operator topology. The letter $\mathscr{M}$ will denote the space of maximal ideals of $\mathfrak{A}$, and $\tau: \mathfrak{A} \to C(\mathscr{M})$ will be the isometric isomorphism of $\mathfrak{A}$ onto $C(\mathscr{M})$ whose existence is asserted by Corollary IX.3.8. For f in $L_1(R)$, we usually write $\tau(Tf)$ simply as τf. The letter E will denote the projection-valued measure defined for the Borel subsets of $\mathscr{M}$, whose existence is asserted by Theorem X.2.1. Thus, for f in $L_1(R)$ and g in $L_2(R)$, we have

$$f*g = \int_{\mathscr{M}} (\tau f)(m)E(dm)g.$$

Since $h(I) = 1$ for every (complex-valued) non-trivial multiplicative linear functional h, and since, by IX.2.3, every such function is continuous, it follows that a multiplicative linear functional on $\mathfrak{A}$ is entirely determined by its restriction to $\mathfrak{A}_0$. Thus, there is a unique point p_∞ in $\mathscr{M}$ such that $(\tau f)(p_\infty) = 0$ for every f in $L_1(R)$; this point p_∞ corresponds to the multiplicative linear functional h defined by the equations $h(f) = 0$ for f in $\overline{\mathfrak{A}}_0$, $h(I) = 1$. For every other point p in $\mathscr{M}$ we have $(\tau f)(p) \neq 0$ for some f in $L_1(R)$. The point p_∞ in

$\mathscr{M}$ will henceforth be referred to as the point at infinity in $\mathscr{M}$. For convenience, let $\mathscr{M}_0 = \mathscr{M} - \{p_\infty\}$.

4 Lemma. *In the notation introduced above, we have $E(\{p_\infty\}) = 0$.*

Proof. Let $g = E(\{p_\infty\})g$ so that for a Borel set δ not containing p_∞ we have $E(\delta)g = 0$ and thus for f in $L_1(R)$

$$f * g = \int_{\mathscr{M}} (\tau f)(m)E(dm)g = \int_{\{p_\infty\}} (\tau f)(m)E(dm)g.$$

Since $(\tau f)(p_\infty) = 0$ for every f in $L_1(R)$, this equation shows that $f * g = 0$ for every f in $L_1(R)$. If f is in $L_1(R) \cap L_2(R)$, then $f * g$ is continuous (cf. Lemma 1(e)) and so

$$0 = (f * g)(0) = \int_R f(-y)g(y)dy, \qquad f \in L_1(R) \cap L_2(R).$$

Since $L_1(R) \cap L_2(R)$ is dense in $L_2(R)$, it follows that $(g, h) = 0$ for every h in $L_2(R)$. Thus $g = 0$. Q.E.D.

5 Lemma. *If e is a Borel subset of $\mathscr{M}$ such that $\bar{e}$ excludes p_∞, then $E(e)$ is bounded, not only as a mapping of $L_2(R)$ into $L_2(R)$, but also as a mapping of $L_2(R)$ into $C(R)$.*

Proof. Since $L_1(R) \cap L_2(R)$ is a dense linear subspace of $L_1(R)$ we may, for each p in $\bar{e}$, choose an f in $L_1(R) \cap L_2(R)$ such that $(\tau f)(p) = 2$. Since by Lemma 1, $f * \tilde{f}$ is in $L_1(R) \cap L_2(R)$ and

$$\begin{aligned}\big(\tau(f * \tilde{f})\big)(m) = \big(\tau(T(f * \tilde{f}))\big)(m) &= \big(\tau(T(f)T(f)^*)\big)(m) \\ &= |(\tau f)(m)|^2,\end{aligned}$$

it may be assumed that τf is a non-negative on $\mathscr{M}$. Since $(\tau f)(p) = 2$ there is a neighborhood $N(p)$ of p with $(\tau f)(m) \geq 1$ for every m in $N(p)$. Since $\bar{e}$ is compact, a finite collection $N(p_1), \ldots, N(p_n)$ of these neighborhoods cover $\bar{e}$ and the corresponding functions $f_1, \ldots, f_n$ have the property that their sum $g = f_1 + \ldots + f_n$ has $(\tau g)(m) \geq 1$ for every m in $\mathscr{M}$. Thus $(\tau g)(m)^{-1} \leq 1$ on $\mathscr{M}$ and it follows from Theorem X.1.1 that the operator

$$P = \int_e (\tau g)(m)^{-1}E(dm)$$

is a bounded operator in $L_2(R)$ with $T(g)P = E(e)$. By Lemma 1(e)

the operator $T(g)$ is bounded as a map of $L_2(R)$ onto $C(R)$ and so $E(e)$ is likewise. Q.E.D.

In view of Lemma 5 it is seen that for every functional x^* in $C(R)^*$ and every Borel set e in $\mathscr{M}$ whose closure excludes p_∞ the scalar $x^*E(e)g$ is continuous and linear for g in $L_2(R)$. According to Theorem IV.4.5 then, every x^* in $C(R)^*$ determines a unique point h in $L_2(R)$ such that $x^*E(e)g = (g, h)$ for every g in $L_2(R)$. This observation applied to the functional δ in $C(R)^*$ which is defined by the identity

$$\delta f = f(0), \qquad f \in C(R),$$

will be used in the following lemma.

6 Lemma. *There is a unique non-negative regular measure μ defined on the family $\mathscr{B}$ of all Borel subsets in $\mathscr{M}_0$ and having the properties:*

(i) *μ is finite on compact sets;*

(ii) *if $\mu(e)$ is finite, e is contained in a countable union of compact sets;*

(iii) *μ is positive on non-void open sets; and*

(iv) *for every Borel set e with $p_\infty \notin \bar{e}$, $\mu(e) = |\psi(e)|^2$, where $\psi(e)$ is the point in $L_2(R)$ uniquely determined by the identity*

$$\delta E(e)g = (g, \psi(e)), \qquad g \in L_2(R),$$

and where the symbol δ is used for the vector in $C(R)^*$ defined by the equation $\delta f = f(0)$.

Proof. Let $\mathscr{B}_0$ consist of those sets e in $\mathscr{B}$ with $p_\infty \notin \bar{e}$. It was observed in the discussion preceding the statement of Lemma 6 that for e in $\mathscr{B}_0$ there is a uniquely determined point $\psi(e)$ in $L_2(R)$ which satisfies the identity stated in (iv). Let the set function μ_0 be defined on $\mathscr{B}_0$ by the equation $\mu_0(e) = |\psi(e)|^2$. It will first be shown that μ_0 is countably additive on $\mathscr{B}_0$ and then that it has an extension to $\mathscr{B}$ with the desired properties.

If e_1, e_2 are in $\mathscr{B}_0$ then, since $E(e_2)$ is self adjoint (X.2.1), it follows that for all g in $L_2(R)$,

$$\begin{aligned}(g, E(e_2)\psi(e_1)) &= (E(e_2)g, \psi(e_1)) = \delta E(e_1)E(e_2)g \\ &= \delta E(e_1 \cap e_2)g = (g, \psi(e_1 \cap e_2)).\end{aligned}$$

It follows that $E(e_2)\psi(e_1) = \psi(e_1 \cap e_2)$ and thus that $E(e_2)\psi(e_2) =$

$\psi(e_2)$ and that $E(e_2)\psi(e_1) = 0$ if e_1 and e_2 are disjoint. Thus $\psi(e_1)$ and $\psi(e_2)$ are orthogonal whenever e_1 and e_2 are disjoint. Hence if e_1 and e_2 are disjoint then

$$\begin{aligned}\psi(e_1 \cup e_2) &= E(e_1 \cup e_2)\psi(e_1 \cup e_2) \\ &= [E(e_1)+E(e_2)]\psi(e_1 \cup e_2) \\ &= E(e_1)\psi(e_1 \cup e_2) + E(e_2)\psi(e_1 \cup e_2) \\ &= \psi(e_1) + \psi(e_2),\end{aligned}$$

so that the vector valued set function ψ is additive on $\mathscr{B}_0$. Therefore, if $e_1 \cap e_2 = \phi$, the set function μ_0 satisfies the equation

$$\begin{aligned}\mu_0(e_1 \cup e_2) &= (\psi(e_1 \cup e_2),\ \psi(e_1 \cup e_2)) \\ &= (\psi(e_1) + \psi(e_2),\ \psi(e_1) + \psi(e_2)) \\ &= (\psi(e_1),\ \psi(e_1)) + (\psi(e_2),\ \psi(e_2)) \\ &= \mu_0(e_1) + \mu_0(e_2),\end{aligned}$$

so that μ_0 is additive on $\mathscr{B}_0$.

To see that μ_0 is countably additive on $\mathscr{B}_0$ let e_n, $n \geqq 1$, be disjoint sets in $\mathscr{B}_0$ whose union e is also in $\mathscr{B}_0$. Let $r_n = e_n \cup e_{n+1} \cup \ldots$, so that $E(r_n)g \to 0$ for every g in $L_2(R)$ and, by Lemma 5,

$$(g, \psi(r_n)) = \delta E(e)E(r_n)g \to 0.$$

This argument shows that the vector valued additive set function ψ is weakly countably additive on the σ-field consisting of all Borel subsets of e. By a theorem of Pettis (IV.10.1) it is countably additive in the strong topology, i.e., $|\psi(r_n)|^2 = \mu_0(r_n) \to 0$, and this proves that μ_0 is countably additive on $\mathscr{B}_0$.

The set function μ_0 will now be extended to a countably additive function μ on $\mathscr{B}$. Let e be a set in $\mathscr{B}$. If e is contained in the union of an increasing sequence $\{e_n\}$ of sets in $\mathscr{B}_0$ let

$$(*) \qquad \mu(e) = \lim_n \mu_0(ee_n),$$

and otherwise let $\mu(e) = \infty$. Since μ_0 is non-negative, the sequence $\{\mu(ee_n)\}$ is non-decreasing, and so the limit defining $\mu(e)$ exists. To see that it depends only upon e and not on the sequence $\{e_n\}$ let $\{a_n\}$ be another increasing sequence in $\mathscr{B}_0$ whose union contains e. Let $b_n = e_n \cup a_n$. It is evidently sufficient to show that $\lim \mu_0(eb_n) = \lim \mu_0(ee_n)$. Since $eb_n \supseteq ee_n$ we have $\mu_0(eb_n) \geqq \mu_0(ee_n)$ so that to

prove the uniqueness of the limit it will suffice to show that if $\mu_0(eb_n) \geqq k$ for some n, then, for every $\varepsilon > 0$, $\mu_0(ee_m) > k - \varepsilon$ for some m. Since $\bigcup ee_m = e$, the sequence $\{ee_m b_n, m \geqq 1\}$ is an increasing sequence of sets whose union is eb_n. Since μ_0 is countably additive on $\mathscr{B}_0$, $\mu_0(eb_n) = \lim_m \mu_0(ee_m b_n) \geqq k$, and so for some m, $\mu_0(ee_m) \geqq \mu_0(ee_m b_n) > k - \varepsilon$. This shows that the set function μ is uniquely defined on $\mathscr{B}$.

Next we show that μ is countably additive. For this purpose, we let $\{a_n\}$ be a disjoint sequence in $\mathscr{B}$. It is clear that $\mu(\bigcup a_n) \geqq \mu(a_n)$, so that, if $\mu(a_n) = \infty$ for any n, the equation $\mu(\bigcup a_n) = \sum \mu(a_n)$ is trivially true. Hence we may and shall assume that $\mu(a_n) < \infty$ for each n. Consequently, there are sequences of increasing sequences $\{e_{nm}\}$ in $\mathscr{B}_0$ with $a_n \subseteqq \bigcup_{m=1}^{\infty} e_{nm}$. By letting $e_k = \bigcup_{m+n \leqq k} e_{mn}$, a single increasing sequence $\{e_k\}$ in $\mathscr{B}_0$ is obtained with $a = \bigcup a_n \subseteqq \bigcup e_k$. Since $\mu_0(a_n e_k)$ is a set of positive numbers increasing with k to the limit $\mu(a_n)$ it follows (III.6.17) that $\lim_k \sum_{n=1}^{\infty} \mu_0(a_n e_k) = \sum_{n=1}^{\infty} \mu(a_n)$ and thus, since μ_0 is countably additive on $\mathscr{B}_0$, that

$$\mu(a) = \lim_k \mu_0(ae_k) = \lim_k \sum_{n=1}^{\infty} \mu_0(a_n e_k) = \sum_{n=1}^{\infty} \mu(a_n),$$

which establishes the countable additivity of μ.

We shall now prove that μ is regular. Let e be in $\mathscr{B}$ with $\mu(e) < \infty$, and let $\{e_n\}$ be a sequence of disjoint sets in $\mathscr{B}_0$ with $e = \bigcup e_n$. Let $\varepsilon > 0$ be given, and let N be so large that $\sum_{N+1}^{\infty} \mu(e_n) < \varepsilon/2$. Put $\tilde{e} = \bigcup_1^N e_n$. Then $\tilde{e}$ is in $\mathscr{B}_0$ and thus $\psi(\tilde{e})$ is defined. Since, by X.2.1, for $\tilde{e}$ fixed the measure $|E(\cdot)\psi(\tilde{e})|^2 = \big(E(\cdot)\psi(\tilde{e}), \psi(\tilde{e})\big)$ is regular, we can find a closed, and hence compact, subset d of $\tilde{e}$ such that

$$|E(d)\psi(\tilde{e})|^2 + \frac{\varepsilon}{2} \geqq |E(\tilde{e})\psi(\tilde{e})|^2.$$

Now it has already been observed that $E(e_2)\psi(e_1) = \psi(e_1 \cap e_2)$ and so $\mu_0(d) = |\psi(d)|^2 = |E(d)\psi(\tilde{e})|^2$ and $\mu_0(\tilde{e}) = |\psi(\tilde{e})|^2 = |E(\tilde{e})\psi(\tilde{e})|^2$. Thus

$$\mu(d) + \varepsilon \geqq \mu(\tilde{e}) + \frac{\varepsilon}{2} \geqq \mu(e).$$

On the other hand, since $\mathscr{M}$ is compact and hence normal, there is a

sequence $\{v_n\}$ of open sets in $\mathscr{B}_0$ with $e_n \subseteq v_n$. Since the measure $|E(\cdot)\psi(v_n)|^2$ is regular, for each n there is an open set w_n with $v_n \supseteq w_n \supseteq e_n$ and such that

$$|E(w_n)\psi(v_n)|^2 = |\psi(w_n)|^2 = \mu_0(w_n) - \mu(w_n) \leq \mu(e_n) + \frac{\varepsilon}{2^n}.$$

Thus, the set $w = \bigcup_{n=1}^{\infty} w_n$ is an open set containing e and $\mu(w) \leq \mu(e) + \varepsilon$. Hence μ is regular.

If K is a compact subset of $\mathscr{M}_0$, then by definition $\mu(K) = \mu_0(K) < \infty$. To complete the proof, let u be a non-void open subset of $\mathscr{M}_0$. Since u contains a non-void open subset v of $\mathscr{M}_0$ such that $p_\infty \notin \bar{v}$, we may as well suppose that $p_\infty \notin \bar{u}$. It suffices to show that there exists a g in $L_2(R)$ such that $(E(u)g)(0) \neq 0$. If this is false, then for every h in $L_2(R)$ such that $E(u)h = h$, it follows that $h(0) = 0$. Thus, since all of the operators S in $\mathfrak{A}$ commute with $E(u)$, we have $(Sh)(0) = 0$ if $h = E(u)h$. In particular if f is in $L_1(R) \cap L_2(R)$, and if $h = E(u)h$, then

$$\begin{aligned} 0 &= [T(\tilde{f})h](0) = (\tilde{f} * h)(0) \\ &= \int_R \overline{f(y)}h(y)dy = (h, f). \end{aligned}$$

Since $L_1(R) \cap L_2(R)$ is dense in $L_2(R)$ we infer that $h = 0$ and hence that $E(u) = 0$. Now let F be a continuous function on $\mathscr{M}$ which vanishes on the complement of u, but does not vanish identically, and let $S_0 = \tau^{-1}(F)$ be the operator in $\mathfrak{A}$ which corresponds to F under the isometric isomorphism $\tau : \mathfrak{A} \to C(\mathfrak{M})$. By Theorem X.2.1(iii), we have

$$S_0 = \int_{\mathscr{M}} F(m)E(dm) = \int_u F(m)E(dm).$$

If $E(u) = 0$, then $E(a) = 0$ for $a \subseteq u$ and thus $S_0 = 0$. Since τ is one-to-one, this is a contradiction. Q.E.D.

Note. We observe that in the course of showing the regularity of μ it was demonstrated that if e is in B and $\mu(e) < \infty$, then for any $\varepsilon > 0$ there is an open set u and a *compact* set c such that $c \subseteq e \subseteq u$ and $\mu(u - c) < \varepsilon$.

For the remainder of this and the next section, the letter μ denotes the measure whose existence has just been established.

7 Lemma. *If f is in $L_1(R) \cap L_2(R)$ and if e is a Borel set in $\mathscr{M}$ whose closure does not contain p_∞, then*

$$\int_e (\tau f)(m)\mu(dm) = \delta[E(e)f] = (f, \psi(e))$$

where $\psi(e)$ is the vector in $L_2(R)$ defined in Lemma 6.

Proof. If we write ψ for $\psi(e)$, then

$$\begin{aligned}(f, \psi) &= \int_R f(x)\overline{\psi(x)}dx = \int_R f(x-0)\overline{\psi(x)}dx \\ &= \int_R \overline{\tilde{f}(0-x)\psi(x)}dx = \overline{(\tilde{f} * \psi)(0)} = \overline{\delta(\tilde{f} * \psi)}.\end{aligned}$$

Since the operation $T(\tilde{f})$ of convolution by $\tilde{f}$ commutes with $E(e)$ and since, as we have seen in the proof of Lemma 6, $E(e)\psi = \psi$, it follows from the definition of ψ that

$$\delta(\tilde{f} * \psi) = \delta(E(e)(\tilde{f} * \psi)) = (\tilde{f} * \psi, \psi).$$

Since $T(\tilde{f}) = T(f)^*$ the preceding calculations show that

$$\begin{aligned}\text{(i)} \qquad \delta E(e)f &= (f, \psi) = \overline{\delta(f * \psi)} = (\psi, \tilde{f} * \psi) \\ &= (f * \psi, \psi) = \int_{\mathscr{M}} (\tau f)(m)(E(dm)\psi, \psi).\end{aligned}$$

On the other hand, if a is a Borel set in $\mathscr{M}$, then, as shown in the proof of Lemma 6, $E(a)\psi(e) = \psi(ae)$ and hence

$$\begin{aligned}(\psi(ae), \psi(e)) &= (E(a)\psi, \psi) = (E(a)^2\psi, \psi) \\ &= (E(a)\psi, E(a)\psi) = (\psi(ae), \psi(ae)) = \mu(ae).\end{aligned}$$

Consequently,

$$\int_{\mathscr{M}} (\tau f)(m)(E(dm)\psi, \psi) = \int_e (\tau f)(m)\mu(dm)$$

which when combined with (i) completes the proof. Q.E.D.

8 Lemma. *If f is in $L_1(R) \cap L_2(R)$, then*

$$\int_{\mathscr{M}_0} |(\tau f)(m)|^2 \mu(dm) = \int_R |f(x)|^2 dx.$$

Proof. Let e be a Borel subset of $\mathscr{M}_0$ with compact closure, then $E(e)(f * \tilde{f}) = E(e)T(f)\tilde{f} = f * (E(e)\tilde{f})$. Applying the preceding lemma, we obtain

$$\delta[f * E(e)\tilde{f}] = \delta[E(e)(f * \tilde{f})]$$
$$= \int_e (\tau(f * \tilde{f}))(m)\mu(dm) = \int_e |(\tau f)(m)|^2 \mu(dm).$$

Since τf is continuous on $\mathscr{M}$ and vanishes at p_∞ the set $a_n = \{m \mid |(\tau f)(m)| > 1/n\}$ is a Borel set of $\mathscr{M}_0$ with compact closure; further, τf vanishes on the complement of $\bigcup a_n$. By Lemma 4, $E(p_\infty)\tilde{f} = 0$ and thus, since $|E(e)\tilde{f}|^2$ is a regular measure, there is an increasing sequence $\{b_n\}$ of compact subsets of $\mathscr{M}_0$ such that $E(b_n)\tilde{f} \to \tilde{f}$ in the norm of $L_2(R)$. Setting $e_n = a_n \cup b_n$, we have

$$\int_{\mathscr{M}_0} |(\tau f)(m)|^2 \mu(dm) = \lim_{n\to\infty} \int_{e_n} |(\tau f)(m)|^2 \mu(dm)$$
$$= \lim_{n\to\infty} \delta[f * E(e_n)\tilde{f}] = \lim_{n\to\infty} \delta[T(f)E(e_n)\tilde{f}].$$

Since, by Lemma 1(e), $T(f)$ is a bounded map of $L_2(R)$ into $C(R)$, the last expression equals

$$\delta[T(f) * \tilde{f}] = \delta(f * \tilde{f})$$
$$= \int_R f(0-y)\overline{f(-y)}dy = \int_R |f(y)|^2 dy.$$

This proves the lemma. Q.E.D.

At this point we pause to review what has been done. We have seen that each function in $L_1(R)$ gives rise under convolution to a bounded linear operator on the Hilbert space $L_2(R)$. Letting $\mathfrak{A}$ be the smallest B^*-algebra of operators which contains these convolution operators, we introduced the compact topological space $\mathscr{M}$ of maximal ideals of $\mathfrak{A}$ so that each element of $\mathfrak{A}$ corresponds to a continuous function on $\mathscr{M}$. By deleting the point at infinity of $\mathscr{M}$, we obtained a locally compact space $\mathscr{M}_0$. The resolution of the identity for the algebra $\mathfrak{A}$ was used to introduce a countably additive regular measure μ on the Borel subsets $\mathscr{B}$ of the space $\mathscr{M}_0$. Now each f in $L_1(R)$ corresponds to some continuous function τf on $\mathscr{M}$ which vanishes at infinity, and we always have $|\tau f|_\infty = |T(f)| \leq |f|_1$. However if f is in $L_1(R) \cap L_2(R)$ it was seen that the mapping τ is an isometry into the space $L_2(\mathscr{M}_0)$. It will now be demonstrated that the mapping τ can be uniquely extended to be an isometric isomorphism between $L_2(R)$ and $L_2(\mathscr{M}_0)$, and that if e is in $\mathscr{B}$, then operating on a function f in $L_2(R)$ by the projection $E(e)$ corresponds

to multiplying the corresponding function τf in $L_2(\mathscr{M}_0)$ by the characteristic function of the set e. Similarly, convolution in $L_2(R)$ corresponds to pointwise multiplication in $L_2(\mathscr{M}_0)$.

9 Theorem. (a) (*Plancherel*) *The mapping* $f \to \tau f$ *sending* $L_1(R, \Sigma, \lambda) \cap L_2(R, \Sigma, \lambda)$ *into* $L_2(\mathscr{M}_0, \mathscr{B}, \mu)$ *has a unique extension to an isometry of* $L_2(R, \Sigma, \lambda)$ *onto* $L_2(\mathscr{M}_0, \mathscr{B}, \mu)$.

(b) *If this extension is denoted by the same symbol* τ, *we have* $\tau E(e)\tau^{-1} = \varphi(e)$ *for each Borel subset* e *of* $\mathscr{M}_0$, *where* $[\varphi(e)f](m) = f(m)$ *if* $m \in e$ *and* $[\varphi(e)f](m) = 0$ *if* $m \notin e$. *Thus* $E(e) = \tau^{-1}\varphi(e)\tau$.

Proof. It follows immediately from the preceding lemma, from the fact that $L_1(R) \cap L_2(R)$ is dense in $L_2(R)$, and from I.6.17 that τ has a unique extension to an isometry of $L_2(R)$ into $L_2(\mathscr{M}_0)$. Thus (a) is established except for the statement that the extension, which will also be denoted by τ, maps $L_2(R)$ onto all of $L_2(\mathscr{M}_0)$.

We now turn to the proof of (b). First recall that if f is in $L_1(R)$ and g is in $L_1(R) \cap L_2(R)$, then

$$\tau(f * g) = \tau f \cdot \tau g.$$

Since τ is continuous as a mapping of $\mathfrak{A}$ into $C(\mathscr{M})$, we conclude that $\tau(Sg) = \tau S \cdot \tau g$ for S in $\mathfrak{A}_0$ and g in $L_1(R) \cap L_2(R)$. But $\tau(Ig) = \tau g = \tau I \cdot \tau g$ and hence $\tau(Sg) = \tau S \cdot \tau g$ for S in $\mathfrak{A}$ and g in $L_1(R) \cap L_2(R)$. Since $L_1(R) \cap L_2(R)$ is dense in $L_2(R)$ and τ is continuous as a map of $L_2(R)$ into $L_2(\mathscr{M}_0)$, we conclude that

$$\tau(Tg) = \tau T \cdot \tau g, \qquad T \in \mathfrak{A}, \qquad g \in L_2(R).$$

If g_1 is another element of $L_2(R)$, then

$$\begin{aligned}\int_{\mathscr{M}} (\tau T)(m)(E(dm)g, g_1) = (Tg, g_1) = (\tau(Tg), \tau g_1) &= \int_{\mathscr{M}_0} (\tau(Tg))(m)\overline{(\tau g_1)(m)}\mu(dm)\\ &= \int_{\mathscr{M}_0} (\tau T)(m)(\tau g)(m)\overline{(\tau g_1)(m)}\mu(dm)\\ &= \int_{\mathscr{M}_0} (\tau T)(m)\nu(dm), \qquad T \in \mathfrak{A},\end{aligned}$$

where we have put $\nu(e) = \int_{e-p_\infty} (\tau g)(m)\overline{(\tau g_1)(m)}\mu(dm)$. Since the measure ν is μ-continuous, its regularity follows from that of μ.

Now from the regularity of $(E(\cdot)g, g_1)$, the above equality, and the uniqueness part of the Riesz representation theorem (IV.6.2), it follows that

$$(E(e)g, g_1) = \nu(e)$$

for any Borel set $e \subseteq \mathscr{M}_0$. Using the portion of (a) already proved, we obtain

$$(\tau E(e)g, \tau g_1) = (E(e)g, g_1) = \int_e (\tau g)(m)\overline{(\tau g_1)(m)}\mu(dm).$$

Since this identity is valid for any g_1 in $L_2(R)$, we conclude that $\tau(E(e)g)(m) = \tau g(m)$ for μ-almost all m in e and $\tau(E(e)g)(m) = 0$ for μ-almost all m in $\mathscr{M}_0 - e$. Once it has been shown that τ^{-1} is everywhere defined on $\mathscr{M}_0$, this will prove part (b).

All that remains for us to prove is that τ maps $L_2(R)$ onto all of $L_2(\mathscr{M}_0)$. For this it is sufficient to show that $\tau L_2(R)$ contains the characteristic function of every set of finite μ-measure and it follows from Lemma 6 that it is sufficient to show that $\tau L_2(R)$ contains the characteristic function of every set e in $\mathscr{B}_0$. As in the proof of Lemma 5, there is a function g in $L_1(R) \cap L_2(R)$ with $\tau g(p) > 1 > 0$ for p in $\bar{e}$. Let Q be a bounded operator in $\mathfrak{A}$ such that τQ coincides with the reciprocal of τg on $\bar{e}$. It follows from what has already been proved that $\tau(Qg) = \tau Q \cdot \tau g$, and so $\tau(Qg)(p) = 1$ for p in $\bar{e}$. From the part of (b) already demonstrated we see that $\tau[E(e)Qg] = \varphi(e)\tau(Qg)$, so that $\tau[E(e)Qg]$ is the characteristic function of e. Q.E.D.

10 Corollary. *For each f in $L_1(R)$ and each Borel subset a of the complex plane, let $(\tau f)^{-1}(a) = \{m \in \mathfrak{M}_0 | \tau f(m) \in a\}$, and let $M(a)$ be the operation in $L_2(\mathfrak{M}_0)$ of multiplication by the characteristic function of $(\tau f)^{-1}(a)$. Then the spectral resolution of the convolution operator $T(f)$ in $L_2(R)$ may be expressed as $\tau^{-1}M(\cdot)\tau$. The spectrum of $T(f)$ is the range of τf, and a complex number α is in the point spectrum of $T(f)$ if and only if $\mu((\tau f)^{-1}(\alpha)) \neq 0$.*

Proof. The first two parts follow from the preceding theorem and from X.2.10 and X.2.9(iii). To prove the statement concerning the point spectrum, recall that if g is in $L_2(R)$ then

$$\tau(f * g) = \tau f \cdot \tau g.$$

If $(\tau f)(m) = \alpha$ for m in a Borel set e of $\mathscr{M}_0$, then $g = \tau^{-1}\chi_e$ is an eigenfunction of $T(f)$ corresponding to α. Conversely, if $f * g = \alpha g$, then we have

$$(\tau f - \alpha)\cdot \tau g = 0.$$

If $g \neq 0$, then since τ is an isometry by Theorem 9(a), it follows that $\tau g(m) \neq 0$ for m in a set of positive μ-measure, and hence $\tau f(m) = \alpha$ for m in this set. Q.E.D.

In the next few paragraphs it will be shown that there is a one-to-one correspondence between the points of $\mathscr{M}_0$ and the continuous homomorphisms of R into the multiplicative group of complex numbers of unit modulus. The set of all such homomorphisms forms an Abelian group. Thus in a natural manner the space $\mathscr{M}_0$ can be endowed with the structure of a locally compact Abelian group.

The next proof uses the representation of $L_\infty(R)$ as the adjoint space of $L_1(R)$. This follows from Theorem IV.8.5 if R is σ-finite. The general case is discussed in the notes at the end of the present chapter.

11 Theorem. *There is a one-to-one correspondence between points m in $\mathscr{M}_0$ and continuous complex valued functions h_m on R satisfying the identities $|h_m(x)| = 1$, $h_m(x+y) = h_m(x)h_m(y)$, $x, y \in R$. This correspondence is given by the formula*

$$(\tau f)(m) = \int_R h_m(x)f(x)dx, \qquad f \in L_1(R).$$

Proof. If m is in $\mathscr{M}_0$ then the function φ_m defined by the equation $\varphi_m(f) = (\tau f)(m)$, $f \in L_1(R)$, is a non-zero linear functional on $L_1(R)$ which satisfies the identity $\varphi_m(f * g) = \varphi_m(f)\varphi_m(g)$. Since $|\tau f|_\infty \leq |f|_1$ it follows that φ_m has norm at most 1, so by Theorem IV.8.5 there is an essentially unique function $h_m \in L_\infty(R)$ with $|h_m|_\infty \leq 1$ such that

$$\varphi_m(f) = \int_R h_m(x)f(x)dx.$$

We have, by the Fubini theorem,

$$\begin{aligned}\int_R \left\{g(y)\int_R h_m(x)f(x-y)dx\right\} dy &= \int_R \left\{h_m(x)\int_R f(x-y)g(y)dy\right\} dx \\ = \varphi_m(f * g) = \varphi_m(f)\varphi_m(g) &= \int_R \left\{g(y)\int_R h_m(x)h_m(y)f(x)dx\right\} dy,\end{aligned}$$

for every pair f, g in $L_1(R)$. From the equality of the extreme terms for all g in $L_1(R)$ we infer that

$$[*] \qquad \int_R h_m(x)f(x-y)dx = h_m(y)\int_R h_m(x)f(x)dx$$

for almost all y. For some choice of f the integral on the right of [*] is not zero and since, by Lemma 1(d), the integral on the left of [*] is continuous, we conclude that h_m agrees almost everywhere with a continuous function. By redefining h_m on a set of measure zero, we may take it to be continuous. A change of variables in [*] shows that for every f in $L_1(R)$,

$$\int_R h_m(x+y)f(x)dx = \int_R h_m(y)h_m(x)f(x)dx,$$

which implies that for each y, $h_m(x+y) = h_m(y)h_m(x)$ for almost all x. Since both sides of this last equation are continuous, $h_m(x+y) = h_m(x)h_m(y)$ for all x, y in R. Since h_m does not vanish identically, it follows from the identity $h_m(x)h_m(0) = h_m(x)$ that $h_m(0) = 1$. Since $|h_m|_\infty \leq 1$, we conclude that $|h_m(x)| \leq 1$ for x in R. But $|h_m(x)h_m(-x)| = |h_m(0)| = 1$ and so $|h_m(x)| = 1$ for x in R. This shows that each point m in $\mathscr{M}_0$ determines a function h_m as described. It is evident that the function h_m is unique.

Let us complete the proof of the theorem by showing that, conversely, if a continuous function H which satisfies the identities $|H(x)| = 1$ and $H(x+y) = H(x)H(y)$ is given, then there exists an (evidently unique) m in $\mathscr{M}_0$ such that $(\tau f)(m) = \int_R H(x)f(x)dx$ for f in $L_1(R)$. Let m_0 be a point in $\mathscr{M}_0$ and, using what has already been proved, let H_0 be a continuous function on R which satisfies the identities $|H_0(x)| = 1$, $H_0(x+y) = H_0(x)H_0(y)$, and

$$(\tau f)(m_0) = \int_R H_0(x)f(x)dx, \qquad f \in L_1(R).$$

Let $H_1(x) = H(x)\overline{H_0(x)}$, and define the map Φ by the equation

$$(\Phi f)(x) = H_1(x)f(x).$$

Then Φ may be regarded as a norm preserving linear operator both in $L_1(R)$ and in $L_2(R)$. Since $H_1^{-1}(x) = H_1(-x)$ we have for f in $L_1(R)$ and g in $L_2(R)$

$$\begin{aligned}\{\Phi T(f)\Phi^{-1}g\}(x) &= \int_R H_1(x)f(x-y)H_1(y)^{-1}g(y)dy \\ &= \int_R H_1(x-y)f(x-y)g(y)dy \\ &= \{T(\Phi f)g\}(x).\end{aligned}$$

Thus $\Phi T(f)\Phi^{-1} = T(\Phi f)$, and consequently $\Phi T(L_1(R))\Phi^{-1} \subseteqq T(L_1(R))$. Since Φ is a unitary mapping of $L_2 = L_2(R)$ into itself, the mapping $A \to \Phi A\Phi^{-1}$ is a norm-preserving mapping of the space of bounded linear operators from $L_2(R)$ into itself. It follows immediately by continuity that $\Phi\mathfrak{A}\Phi^{-1} \subseteqq \mathfrak{A}$. Thus $A \to \Phi A\Phi^{-1}$ is a norm-preserving isomorphism of $\mathfrak{A}$ onto itself. Consequently, the map ψ defined by $\psi f = \tau(\Phi(\tau^{-1}f)\Phi^{-1})$ is a norm-preserving isomorphism of $C(\mathcal{M})$ onto itself. Since $\Phi AA_1\Phi^{-1} = \Phi A\Phi^{-1}\Phi A_1\Phi^{-1}$, ψ is actually an algebraic isomorphism of $C(\mathcal{M})$ onto itself, so that, by Theorem IV.6.26, ψ has the form $(\psi f)(m) = f(\chi m)$, $m \in \mathcal{M}$, for some homeomorphism χ of $\mathcal{M}$ onto $\mathcal{M}$. Let $m = \chi(m_0)$.

Now, as was seen above, $\Phi T(f)\Phi^{-1} = T(\Phi f)$ for f in $L_1(R)$, and so for such f,

$$\begin{aligned}(\tau f)(m) &= (\psi\tau f)(m_0) = (\tau T(\Phi f))(m_0) \\ &= (\tau\Phi f)(m_0) = \int_R H_0(x)(\Phi f)(x)dx \\ &= \int_R H_0(x)H_1(x)f(x)dx = \int_R H(x)f(x)dx. \quad \text{Q.E.D.}\end{aligned}$$

12 Corollary. *The function h_m of the theorem which corresponds to the point m in $\mathcal{M}_0$ is given by the equation*

$$h_m(y) = \frac{(\tau f_y)(m)}{(\tau f)(m)},$$

where f is any function in $L_1(R)$ for which $(\tau f)(m) \neq 0$, and f_y is the function in $L_1(R)$ defined by the identity $f_y(x) = f(x-y)$, $x \in R$.

Proof. This is just the content of formula [*] of the first paragraph of the proof of the preceding theorem. Q.E.D.

The next few results represent the topology of the space $\mathcal{M}_0$ directly in terms of the functions of Theorem 11.

13 Definition. Let $\hat{R}$ be the set of all continuous functions h defined on R which satisfy the identities $h(x+y) = h(x)h(y)$,

$|h(x)| = 1$. It is evident that $\hat{R}$ forms an Abelian group with respect to the natural definition of multiplication for functions, the identity of this group being the function identically equal to unity. The group $\hat{R}$ is called the ***character group*** or the ***dual group*** of R. We topologize $\hat{R}$ by taking the sets

$$N(h, K, \varepsilon) = \{h_1 \in \hat{R} \mid |h_1(x) - h(x)| < \varepsilon, \quad x \in K\},$$

as a base for its topology, where $\varepsilon > 0$ is arbitrary and K is an arbitrary compact subset of R.

14 LEMMA. *The character group $\hat{R}$ is a topological group.*

PROOF. Verification that the neighborhoods $N(h, K, \varepsilon)$ are a base for a topology will be left to the reader. If $h_1 \in N(h, K, \varepsilon)$ and $h_2 \in N(h_0, K, \varepsilon)$ then $h_1 h_2 \in N(hh_0, K^2, \varepsilon)$ so that multiplication is continuous. If $h_1 \in N(h, K, \varepsilon)$ then $h_1^{-1} \in N(h^{-1}, K, \varepsilon)$, so the mapping $h \to h^{-1}$ is also continuous. Q.E.D.

15 THEOREM. *The one-to-one mapping $m \to h_m$, whose existence was established in Theorem* 11, *is a homeomorphism of $\mathscr{M}_0$ onto $\hat{R}$.*

PROOF. We first show that the mapping $m \to h_m$ is continuous. Let m_0 be an arbitrary point in $\mathscr{M}_0$, $0 < \varepsilon < 1$, and let $N(h_{m_0}, K, \varepsilon)$, be a neighborhood of h_{m_0}. By IV.8.19 the integrable continuous functions on R are dense in $L_1(R)$ so there is a continuous function f on R such that $|f|_1 < 1$ and $(\tau f)(m_0) \neq 0$. Let $\alpha = |(\tau f)(m_0)|$ so that $0 < \alpha < 1$ and let U be a neighborhood of m_0 such that if m is in U then $|(\tau f)(m) - (\tau f)(m_0)| < \alpha^2\varepsilon/4$. By Corollary 12, we have

$$[*] \qquad h_m(y) = \frac{(\tau f_y)(m)}{(\tau f)(m)}, \qquad m \in U.$$

By Lemma 1(f) the mapping $y \to f_y$ is continuous on R to $L_1(R)$. Similarly the map $y \to \tau f_y$ of R into $C(\mathscr{M})$ is continuous. From the compactness of the set $K \subseteq R$ and Theorem IV.6.7, we conclude that $\{\tau f_y \mid y \in K\}$ is an equi-continuous set in $C(\mathscr{M})$. Let $V \subseteq U$ be a neighborhood of m_0 such that $|(\tau f_y)(m) - (\tau f_y)(m_0)| < \alpha\varepsilon/4$ for every m in V and y in K. An elementary calculation using [*] and the two inequalities already established shows that $|h_m(y) - h_{m_0}(y)| < \varepsilon$ for all m in V and all y in K. Hence if m is in V then h_m is in $N(h_{m_0}, K, \varepsilon)$ and so the map $m \to h_m$ is continuous.

Conversely, let U be a neighborhood of m_0 such that $p_\infty \notin U$. There is a continuous function H on $\mathscr{M}$ such that $H(m_0) = 1$ and $H(m) = 0$ for $m \notin U$. Since H is in $\tau\mathfrak{A}_0$ and operators $T(f)$ corresponding to functions f in $L_1(R)$ are dense in $\mathfrak{A}_0$ there is a function f in $L_1(R)$ such that $(\tau f)(m_0) > 3/4$ and $|(\tau f)(m)| < 1/4$ for $m \notin U$. Let K be a compact subset of R such that

$$\int_{R-K} |f(x)| dx < \tfrac{1}{16},$$

and let $\varepsilon = (8 |f|_1)^{-1}$. Then if h_m is in $N(h_{m_0}, K, \varepsilon)$, it follows from Theorem 11 and Hölder's inequality that

$$\begin{aligned} |(\tau f)(m) - (\tau f)m_0)| &= \left|\int_R f(x)(h_m(x) - h_{m_0}(x)) dx\right| \\ &\leq 2 \int_{R-K} |f(x)| dx + \varepsilon \int_K |f(x)| dx \\ &< \tfrac{1}{8} + \tfrac{1}{8} = \tfrac{1}{4}. \end{aligned}$$

Consequently, $(\tau f)(m) > 1/2$ and so m is in U. This shows that the mapping $h_m \to m$ is continuous, and hence it is a homeomorphism. Q.E.D.

In view of the theorem just proved, it is customary to define a group operation on $\mathscr{M}_0$ by the equation

$$h_{m_1+m}(x) = h_{m_1}(x) h_{m_2}(x).$$

With this definition of addition, the set $\mathscr{M}_0$ becomes a locally compact Abelian group, and is topologically and algebraically isomorphic with the character group $\hat{R}$. It is desirable to have a more symmetric notation for $h_m(x)$ and, for the sake of simplicity in some of the formulas of the next section, we introduce the notation

$$[x, m] = \overline{h_m(x)}.$$

Since h_m is a homomorphism, so is $\overline{h_m}$. (It is more convenient for some purposes to use the complex conjugate $\bar{h}$ rather than h. This convention has the additional advantage that in case R is the multiplicative group of the unit circle it coincides with the notation customarily used in the theory of Fourier series.) It follows that $|[x, m]| = 1$, $[x_1 + x_2, m] = [x_1, m][x_2, m]$, and $[x, m_1 + m_2] = [x, m_1][x, m_2]$ for all x, x_1, x_2, in R and m, m_1, m_2 in $\hat{R}$. It follows

from these relations that $[-x, m] = [x, -m] = \overline{[x, m]}$. Moreover, it is evident from Definition 13 and Theorem 15 that $[x, m]$ is a continuous function of both variables. In this notation the formula in Theorem 11 becomes

$$(\tau f)(m) = \int_R \overline{[x, m]} f(x)dx, \qquad f \in L_1(R), \qquad m \in \mathscr{M}_0.$$

In Theorem 9 the domain of τ was extended to $L_2(R)$; we wish to obtain a similar integral representation for this extension. Let $\mathscr{E}$ be the family of all Borel subsets of R with finite measure and direct $\mathscr{E}$ by inclusion. If χ_e denotes the characteristic function of the set e in $\mathscr{E}$, and if f is in $L_2(R)$, then $\chi_e f$ is in $L_1(R) \cap L_2(R)$ and f is the limit in the norm of $L_2(R)$ of the generalized sequence $\{\chi_e f\}$. Hence, by Theorem 9, τf is the limit in the norm of $L_2(\mathscr{M}_0)$ of the generalized sequence $\{\tau(\chi_e f)\}$. Equivalently, we write

$$\tau f = \lim_e \int_e \overline{[x, \cdot]} f(x)dx, \qquad f \in L_2(R),$$

where the limit is taken in $L_2(\mathscr{M}_0)$.

We now show that the function f can be retrieved from τf by a similar limiting procedure.

16 Theorem. *Let $\hat{\mathscr{E}}$ denote the family of compact subsets of $\mathscr{M}_0$, directed by inclusion. Then if f is in $L_2(R)$, it is equal to the limit in the norm of this space of the generalized sequence of functions $\{f_e, e \in \hat{\mathscr{E}}\}$, defined by the equation*

$$f_e(x) = \int_e [x, m](\tau f)(m)\mu(dm), \qquad x \in R.$$

Proof. It follows from Corollary 12 that

$$[*] \qquad (\tau f_y)(m) = \overline{[y, m]}(\tau f)(m)$$

for every f in $L_1(R)$, y in R, and m in $\mathscr{M}_0$ with $(\tau f)(m) \neq 0$. If $(\tau f)(m) = 0$ then we must have $(\tau f_y)(m) = 0$, since otherwise it would follow from Corollary 12 that

$$(\tau f)(m) = (\tau(f_y)_{-y})(m) = \overline{[-y, m]}(\tau f_y)(m) \neq 0.$$

Thus [*] is true also in case $(\tau f)(m) = 0$. Replacing y by $-x$ in [*] gives

$$[**] \qquad [x, m](\tau f)(m) = (\tau f_{-x})(m).$$

If e is a Borel set of $\mathscr{M}_0$ and if f is in $L_1(R) \cap L_2(R)$ it follows from equation [**] and Theorem 9(b) that

$$\tau[E(e)f_y] = \chi_e \tau(f_y) = [-y, \cdot]\chi_e \tau f = \tau[(E(e)f)_y],$$

and thus that $E(e)f_y = (E(e)f)_y$. Using this, Lemma 7, and formula [**] it is seen that

$$\int_e [x, m](\tau f)(m)\mu(dm) = \delta[E(e)(f_{-x})] = \delta[(E(e)f)_{-x}] = (E(e)f)(x),$$

provided that f is in $L_1(R) \cap L_2(R)$. Now for e in $\hat{\mathscr{E}}$ and f in $L_2(R)$, the integral in the preceding equality exists. Since $L_1(R) \cap L_2(R)$ is dense in $L_2(R)$ we conclude that

$$\int_e [x, m](\tau f)(m)\mu(dm) = (E(e)f)(x), \qquad e \in \hat{\mathscr{E}}, \qquad f \in L_2(R).$$

Since the generalized sequence $\{E(e)f, e \in \hat{\mathscr{E}}\}$ converges to f in the norm of $L_2(R)$, the theorem is proved. Q.E.D.

17 Corollary. *If f is in $L_1(R)$ then $(\tau f_y)(m) = \overline{[y, m]}(\tau f)(m)$ for all m in $\mathscr{M}_0$. If f is in $L_2(R)$ the equation is valid for μ-almost all m in $\mathscr{M}_0$.*

Proof. If f is in $L_1(R)$, the stated equality was proved for all m in $\mathscr{M}_0$ in the formula [*] of the preceding proof. If f is in $L_2(R)$, the assertion follows from Plancherel's theorem, the fact that $L_1(R) \cap L_2(R)$ is dense in $L_2(R)$, and from III.3.6 and III.6.13(a). Q.E.D.

The next lemma gives a similar result for $L_2(\mathscr{M}_0)$.

18 Lemma. *If F is in $L_2(\mathscr{M}_0)$, and if F_p is defined by the equation $F_p(m) = F(m-p)$, $m \in \mathscr{M}_0$, then $(\tau^{-1}F_p)(x) = [x, p](\tau^{-1}F)(x)$ for λ-almost all x in R. If h is in $L_1(R)$ and if $h_1(x) = [x, p]h(x)$ for x in R, then $(\tau h_1)(m) = (\tau h)(m-p)$ for all m in $\mathscr{M}_0$.*

Proof. To prove the second assertion, note that since characters have unit modulus, h_1 is also in $L_1(R)$. It follows that

$$\begin{aligned}(\tau h_1)(m) &= \int_R \overline{[x, m]}h_1(x)dx \\ &= \int_R \overline{[x, m-p]}h(x)dx = (\tau h)(m-p), \qquad m \in \mathscr{M}_0.\end{aligned}$$

To prove the first statement observe that there is a sequence $\{F^n\}$ converging in the norm of $L_2(\mathscr{M}_0)$ to F and such that $\tau^{-1}F^n \in L_1(R) \cap L_2(R)$, $n = 1, 2, \ldots$. The functions h^n defined by the equation $h^n(x) = [x, p]\tau^{-1}F^n(x)$ for x in R are in $L_1(R) \cap L_2(R)$ and form a Cauchy sequence in $L_2(R)$. By what has been proved $(\tau h^n)(m) = F^n(m-p)$ for m in $\mathscr{M}_0$. By III.3.6 and III.6.13(a) we may suppose that $F^n(m) \to F(m)$ for μ-almost all m in $\mathscr{M}_0$, and that $\lim_{n\to\infty} \tau^{-1}F^n(x)$ exists for λ-almost all x in R. It follows that $F_p(m) = \lim_{n\to\infty} (\tau h^n)(m)$ for μ-almost all m in $\mathscr{M}_0$. Therefore $\tau^{-1}F_p(x) = \lim_{n\to\infty} h^n(x) = [x, p]\tau^{-1}F(x)$ for λ-almost all x in R. Q.E.D.

19 Theorem. *The measure μ on $\mathscr{M}_0 = \hat{R}$ is invariant under translation, that is,*

$$\mu(e+p) = \mu(e), \qquad e \in B, \qquad p \in \mathscr{M}_0.$$

Proof. First note that the desired equality is trivial if $\mu(e+p) = \mu(e) = \infty$, so that we need consider only the case where at least one of the numbers $\mu(e+p)$ and $\mu(e)$ is finite. Let $\mu(e) < \infty$ and for F in $L_2(\mathscr{M}_0)$ let F_p be defined by the equation $F_p(m) = F(m-p)$ for m, p in $\mathscr{M}_0$. If χ_e denotes the characteristic function of the set e, then it is readily seen that $(\chi_e)_p = \chi_{e+p}$. From the preceding lemma it is seen that

$$(\tau^{-1}\chi_{e+p})(x) = [x, p]\tau^{-1}\chi_e, \qquad x \in R.$$

Since characters have modulus equal to unity, it follows from Plancherel's theorem that

$$\{\mu(e+p)\}^2 = \{\mu(e)\}^2.$$

Hence if $\mu(e) < \infty$, we have proved that $\mu(e+p)$ is also finite and equals $\mu(e)$. If $\mu(e+p)$ were known to be finite we would replace e and p by $e+p$ and $-p$ in the argument just given to conclude that $\mu(e) = \mu(e+p-p) < \infty$ and equals $\mu(e+p)$. Q.E.D.

We now suppose that R denotes the group of real numbers and show that its character group $\hat{R}$ is algebraically and topologically isomorphic with R.

20 Theorem. *Let R be the additive group of real numbers and let $\hat{R}$ be its character group. Then there is a homeomorphic isomorphism*

t mapping $\hat{R}$ onto all of R with the properties

$$[x, m] = e^{ixt(m)}, \qquad x \in R, \qquad m \in \hat{R}$$

and

$$2\pi\mu(e) = \lambda(t(e)), \qquad e \in \mathscr{B},$$

where $\mathscr{B}$ is the family of Borel sets in $\hat{R}$ and where λ is Haar measure on R.

PROOF. For a fixed m in $\mathscr{M}_0 = \hat{R}$, the character $[x, m]$, $x \in R$, is a continuous function of x with $|[x, m]| = 1$ and $[x, m][y, m] = [x+y, m]$ for all x, y in R. It is a well-known and elementary fact (cf. VIII.1.2 for a generalized version) that this implies the existence of a real number $t(m)$ such that

$$[x, m] = e^{ixt(m)}, \qquad x \in R.$$

If $t(m_1) = t(m_2)$, then $[x, m_1] = [x, m_2]$ for x in R, so that by Theorem 11, $m \to t(m)$ is one-to-one. Since the function $h(x) = e^{ixt}$ satisfies the identities $|h(x)| = 1$ and $h(x+y) = h(x)h(y)$, the map $m \to t(m)$ maps $\hat{R}$ onto all of R. To see that the map t is a homeomorphism note that

$$\begin{aligned} |[x, 0] - [x, m]| &= |1 - e^{ixt(m)}| \\ &= \{(1-\cos xt(m))^2 + (\sin xt(m))^2\}^{\frac{1}{2}} \end{aligned}$$

and that this quantity is small for all x in the compact set $|x| \leqq K$ if and only if $|t(m)|$ is small; thus m is near the identity character if and only if $t(m)$ is near zero. Since

$$|[x, m_1] - [x, m]| = |[x, 0] \quad [x, m_2 - m_1]|,$$

the mapping $t : \hat{R} \to R$ is a homeomorphism. The identity

$$\begin{aligned} e^{ixt(m_1+m_2)} = [x, m_1+m_2] &= [x, m_1][x, m_2] \\ &= e^{ix\{t(m_1)+t(m_2)\}}, \end{aligned}$$

shows that $t(m_1+m_2) = t(m_1)+t(m_2)$, and so t is an algebraic homeomorphism.

To prove the final assertion, let λ_1 be defined, for each Borel subset e of R, by the equation $\lambda_1(e) = \mu(t^{-1}(e))$. In view of Theorem 19, the set function λ_1 is invariant under translations, i.e., $\lambda_1(e)$

$=\lambda_1(e+x)$ for x in R, and so, by the uniqueness of Haar measure, λ_1 is a constant multiple of the measure λ. Let $\lambda_1 = c\lambda$. We shall show that $c = 1/2\pi$, thereby completing the proof. To do this we shall first compute the transform of the function $f(x) = e^{-x^2/2}$, $x \in R$, which is in $L_1(R) \cap L_2(R)$. Denoting $t(m)$ briefly by t, we have

$$(\tau f)(m) = \int_{-\infty}^{+\infty} e^{-itx} e^{-x^2/2}\, dx = e^{-t^2/2} \int_{-\infty}^{+\infty} e^{-(x+it)^2/2}\, dx.$$

Now the expression

$$\int_{-\infty}^{+\infty} e^{-(x+z)^2/2}\, dx$$

defines an entire function of z which is equal to

$$\int_{-\infty}^{+\infty} e^{-x^2/2}\, dx = (2\pi)^{\frac{1}{2}}$$

for all real z and therefore for all z. Hence we conclude that

$$(\tau f)(m) = (2\pi)^{\frac{1}{2}} e^{-\{t(m)\}^2/2}.$$

By Plancherel's theorem

$$\int_{-\infty}^{+\infty} |e^{-x^2/2}|^2\, dx = \int_{-\infty}^{+\infty} |f(x)|^2\, dx = \int_{\mathscr{M}_0} |(\tau f)(m)|^2 \mu(dm).$$

By changing the variable in the last integral from m to $t = t(m)$, and recalling that $\lambda_1(e) = \mu(t^{-1}(e)) = c\lambda(e)$ for e in Σ, it is seen that

$$\int_{-\infty}^{+\infty} |e^{-x^2/2}|^2\, dx = 2\pi \int_{-\infty}^{+\infty} |e^{-t^2/2}|^2 \lambda_1(dt) = 2\pi c \int_{-\infty}^{+\infty} |e^{-t^2/2}|^2\, dt,$$

and so $2\pi c = 1$ and the proof is complete. Q.E.D.

Theorem 20 justifies the identification of the two spaces R and $\hat{R}$ provided that an adjustment is made in the definition of the Fourier transform to compensate for the factor 2π. This observation enables us to reformulate the Plancherel and inversion theorems (Theorems 9 and 16) in more conventional notation.

→ 21 THEOREM. *Let R be the additive group of real numbers. Then for each f in $L_2(R)$, the limit*

$$\text{(a)} \qquad g(t) = (Kf)(t) = \frac{1}{\sqrt{2\pi}} \lim_{N\to\infty} \int_{-N}^{N} f(x) e^{-ixt}\, dx$$

exists in the norm of the space $L_2(R)$. The mapping K is a unitary operator of $L_2(R)$ onto all of itself, and is called "the L_2-Fourier transform."

(b) *The inverse of K is given by the formula*

$$f(x) = (K^{-1}g)(x) = \frac{1}{\sqrt{2\pi}} \lim_{N\to\infty} \int_{-N}^{N} g(t)e^{+ixt}\,dt,$$

where the limit exists in the norm of $L_2(R)$.

(c) *If h is in $L_1(R)$, the spectrum of the convolution operator $T(h)$, defined for f in $L_2(R)$ by the formula $T(h)f = h * f$, is the set of values taken on by its "L_1-Fourier transform" which is given by the equation*

$$(\tau h)(t) = \int_{-\infty}^{\infty} h(x)e^{-ixt}\,dx.$$

The function τh is continuous and the point spectrum of $T(h)$ is the set of numbers α for which the set $\{t \in R \mid \alpha = (\tau h)(t)\}$ has positive measure.

(d) *If h is in $L_1(R)$ and f is in $L_2(R)$ then*

$$T(h)f = h * f = (K^{-1}M(\tau h)K)f,$$

where $M(\tau h)$ denotes the operation of multiplication by the continuous function τh.

The preceding discussion may be carried over to the locally compact additive group R^n of real n-dimensional vectors $x = [x_1, \ldots, x_n]$, and the reader can readily show by modifying the method used in the proof of Theorem 20 that the characters in this case have the representation given in the following theorem.

22 THEOREM. *Let R^n be the additive group of real n-dimensional vectors and let $\hat{R}^n$ be its character group. Then there is a homeomorphic isomorphism t mapping $\hat{R}^n$ onto all of R^n with the properties*

$$[x, m] = e^{i(x, t(m))} = \exp\{i(x_1 t_1(m) + \ldots + x_n t_n(m))\},$$

and

$$(2\pi)^n \mu(e) = \lambda(t(e)), \qquad e \in \mathscr{B}^n,$$

where $\mathscr{B}^n$ is the family of Borel sets in $\hat{R}^n$ and where λ is Lebesgue measure on R^n.

By using the form of the characters on R^n the Plancherel theorem may be given a more concrete formulation in the present case also. It is easily seen that this theorem asserts that if f is in $L_2(R^n)$, the limit

$$g(t_1, \ldots, t_n) = (2\pi)^{-n/2} \lim_{N\to\infty} \int_{-N}^{+N} \cdots \int_{-N}^{+N} f(x_1, \ldots, x_n) \cdot \exp\{-i(t_1x_1 + \ldots + t_nx_n)\} dx_1 \ldots dx_n$$

exists in the norm of $L_2(R^n)$, and defines a unitary mapping in this space whose inverse is given by the formula

$$f(x_1, \ldots, x_n) = (2\pi)^{-n/2} \lim_{N\to\infty} \int_{-N}^{+N} \cdots \int_{-N}^{+N} g(t_1, \ldots, t_n) \cdot \exp\{i(t_1x_1 + \ldots + t_nx_n)\} dt_1 \ldots dt_n,$$

where the limit exists in the norm of $L_2(R^n)$.

It is clear that the other parts of Theorem 21 may be generalized to several variables but such elaborations will be omitted here. Instead it will be indicated how Plancherel's theorem in two variables can be used to give some information concerning the Hankel transformation. We shall use the polar coordinates r, θ in the plane and be concerned with functions F in $L_2(R^2)$ having the special form $F(x, y) = f(r)e^{in\theta}$, where n is an integer. Since

$$\infty > \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F(x, y)|^2 dx dy = \int_0^{2\pi} \int_0^{\infty} |f(r)|^2 r dr d\theta = 2\pi \int_0^{\infty} r|f(r)|^2 dr,$$

the transformation U_n defined by the equation

$$(U_n f)(x, y) = \frac{1}{\sqrt{2\pi}} \frac{f(\sqrt{x^2+y^2})}{\sqrt[4]{x^2+y^2}} e^{in \arctan y/x}$$

is an isometric isomorphism of $L_2(0, \infty)$ onto a (necessarily closed) subspace $\mathfrak{X}$ of $L_2(R^2)$. The Fourier transform G of F is

$$G(u, v) = \lim_{r\to\infty} \frac{1}{2\pi} \iint_{x^2+y^2\leq r^2} f(r)e^{in\theta} e^{-i(xu+yv)} dx dy$$

where the limit is taken in the norm of $L_2(R^2)$. Upon introducing

polar coordinates $u = s\cos\varphi$, $v = s\sin\varphi$, we have $xu + yv = rs\cos(\theta - \varphi)$, and so

$$G(u, v) = \lim_{R\to\infty} \frac{1}{2\pi}\int_0^R f(r)r\,dr \int_0^{2\pi} e^{-i\{rs\cos(\theta-\varphi)-n\theta\}}\,d\theta.$$

By substituting θ' for $\theta-\varphi+(\pi/2)$ and simplifying, it is seen that

$$G(u, v) = \frac{1}{2\pi}(-ie^{i\varphi})^n \lim_{R\to\infty}\int_0^R f(r)r\,dr \int_0^{2\pi} e^{i(n\theta'-(rs\sin\theta'))}\,d\theta'.$$

Now the *Bessel function* J_n of order n is defined by the equation

$$J_n(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{i(n\theta - x\sin\theta)}\,d\theta;$$

hence we have

$$G(u, v) = \lim_{R\to\infty}(-ie^{i\varphi})^n \int_0^R rf(r)J_n(rs)\,dr,$$

the limit being taken in the mean of $L_2(R^2)$. By using the isometric isomorphism U_n introduced above, it is seen that

$$G(u, v) = \lim_{R\to\infty}\sqrt{2\pi}\, i^{-n}(U_n\tilde{g}_R)(u, v),$$

where

$$\tilde{g}_R(s) = s^{\frac{1}{2}}\int_0^R rf(r)J_n(rs)\,dr.$$

Thus, if we let $\tilde{f}(r) = r^{\frac{1}{2}}f(r)$, and

$$(H_n\tilde{f})(s) = \lim_{R\to\infty}\int_0^R (rs)^{\frac{1}{2}}J_n(rs)\tilde{f}(r)\,dr,$$

it is seen that the limit defining H_n exists in the norm of $L_2(0, \infty)$, and that $H_n = i^{-n}U_n^{-1}KU_n$, where K denotes the Fourier transform. Now, the inversion formula for the Fourier transform may evidently be interpreted as stating that $K^2 = M$, where $(Mf)(x, y) = f(-x, -y)$. Since

$$M\left(f(r)e^{in\theta}\right) = f(r)e^{in(\theta+\pi)} = (-1)^n f(r)e^{in\theta},$$

we have $U_n^{-1}MU_n = (-1)^n I$. Thus

$$H_n^2 = (i)^{-2n}U_n^{-1}KU_nU_n^{-1}KU_n = (i)^{-2n}U_n^{-1}MU_n^{-1} = (-1)^{-n}(-1)^n I = I,$$

This proves the following theorem:

23 Theorem. *Let f be in $L_2(0, \infty)$ with respect to Lebesgue measure. Then the limit*

$$(H_n f)(s) = \lim_{R\to\infty}\int_0^R (rs)^{\frac{1}{2}} J_n(rs) f(r) dr$$

exists in the norm of the space $L_2(0, \infty)$, and the Hankel transform defined by this formula is a self-inverse isometric isomorphism of $L_2(0, \infty)$ onto itself.

In Chapter XIII below we shall generalize this result to non-integral values of n, and show how more elaborate unitary transformations of this same type can be obtained.

4. Closure Theorems

As in the preceding section the letter R will stand for a non-discrete locally compact Abelian group and integration will always be performed with respect to a Haar measure on the group. It was observed in Corollary 5.2 that the complex B-space $L_1(R)$ is a commutative B-algebra with the operation of convolution as multiplication. In the present section this algebra is discussed with the purpose of presenting the closure theory centering around the famous result of Norbert Wiener (Theorem 7) which asserts that the non-vanishing of the Fourier transform of a function in $L_1(R)$ is a necessary and sufficient condition for linear combinations of its translates to be dense. This closure theorem of Wiener takes on a fuller meaning when it is interpreted as a Tauberian theorem and such applications are to be found in Section 5. A deep insight into the L_1-closure theory may be obtained by a study of A. Beurling's problem of spectral synthesis which is also discussed in this section. This is the problem of determining whether a bounded measurable function φ on R is in the L_1-closed linear manifold in $L_\infty(R)$ which is determined by the characters in the L_1-closed linear manifold spanned by the translates of φ. An analysis of this problem yields results, as in Theorem 21, more general than the original L_1-closure theorem of Wiener.

The following study of these closure properties in $L_1(R)$ will

be based upon two closely related commutative algebras of operators in the Hilbert space $L_2(R)$. One of these algebras, namely the algebra $\mathfrak{A}$ of the preceding section, we have met before. For convenience, its definition and some of its properties will be restated here. For every f in $L_1(R)$ the convolution

$$(f * g)(x) = \int_R f(x-y)g(y)dy, \qquad g \in L_2(R),$$

determines a bounded linear operator $T(f)$ in the Hilbert space $L_2(R)$ which is defined by the equation

$$T(f)g = f * g, \qquad g \in L_2(R).$$

The convolution $f * g$ is also defined if both f and g are in $L_1(R)$ but in this case it is a point of $L_1(R)$ and not necessarily in $L_2(R)$. The complex B-space $L_1(R)$ is a commutative normed algebra under convolution as multiplication and the mapping $f \to T(f)$ is a continuous isomorphism of the algebra $L_1(R)$ into the algebra $B(L_2(R))$ of bounded linear operators in $L_2(R)$. The algebra $T(L_1(R))$ does not contain the unit I in $B(L_2(R))$ nor does its closure $\overline{T(L_1(R))}$ in the uniform operator topology contain the unit. The algebra $\mathfrak{A}$ is, by definition, the B-algebra obtained by adjoining the unit I to $\overline{T(L_1(R))}$. Its elements have the form $\alpha I + A$ where A is in $\overline{T(L_1(R))}$. This algebra $\mathfrak{A}$ is also a B^*-algebra and for f in $L_1(R)$ the Hilbert space adjoint of $T(f)$ is given by the equations

$$T(f)^* = T(\tilde{f}), \qquad \tilde{f}(x) = \overline{f(-x)}.$$

If, for f in $L_1(R)$, the operator $T(f)$ in $L_2(R)$ is given the norm $|T(f)|_1 = |f|_1$, the $L_1(R)$ norm of f, then the algebra $T(L_1(R))$ is isometrically isomorphic to $L_1(R)$ and thus satisfies all the requirements for a B-algebra except for having a unit. The unit I may be adjoined by considering all pairs $(\alpha, T(f))$ with α in the complex field Φ and f in $L_1(R)$ as described in Section IX.1.1. The algebra $\mathfrak{A}_1$ is this algebra of pairs $(\alpha, T(f))$ under the norm

$$|(\alpha, T(f))|_1 = |\alpha| + |f|_1.$$

Since the subalgebras of $\mathfrak{A}_1$ consisting of elements of the form $(\alpha, 0)$ and $(0, T(f))$ respectively are equivalent to the algebras Φ and

$T(L_1(R))$ with its L_1 norm we may sometimes write $\alpha I + T(f)$ instead of $(\alpha, T(f))$. Thus the symbols $(\alpha, T(f))$ and $\alpha I + T(f)$, with f in $L_1(R)$ have two norms, one as elements of $\mathfrak{A}$, the other as elements of $\mathfrak{A}_1$. These two norms are related by the inequality

$$|\alpha I + T(f)| \leq |\alpha I + T(f)|_1, \qquad f \in L_1(R).$$

Proofs for the preceding statements will be found at the beginning of Section 3.

The letters $\mathscr{M}$, $\mathscr{M}_1$ will be used for the structure spaces of $\mathfrak{A}$, $\mathfrak{A}_1$ respectively. The letter τ will be used for the natural homomorphic map of a commutative B-algebra into the space of continuous functions on its structure space (IX.2.9). Since $\mathfrak{A}$ is a B^*-algebra the map $\tau : \mathfrak{A} \to C(\mathscr{M})$ is an isometric *-isomorphism of $\mathfrak{A}$ onto all of $C(\mathscr{M})$ (IX.3.8). It should be recalled (IX.2.2) that there is a one-to-one correspondence between the non-zero complex valued homomorphisms on any commutative B-algebra $\mathfrak{A}$ and the maximal ideals in $\mathfrak{A}$. This correspondence is given by the equation

$$H(x) = (\tau x)(m), \qquad x \in \mathfrak{A},$$

where H is the homomorphism on $\mathfrak{A}$ corresponding to the maximal ideal m in $\mathfrak{A}$. Such homomorphisms are continuous (Corollary IX.2.3). Since every non-zero complex valued homomorphism H on either $\mathfrak{A}$ or $\mathfrak{A}_1$ is continuous and has $H(I) = 1$ it follows that H is completely determined by the values it takes on elements of the form $T(f)$ with f in $L_1(R)$. Thus each of the spaces $\mathscr{M}$ and $\mathscr{M}_1$ has a point at infinity p_∞ corresponding to the homomorphism H which is defined by the equations

$$H(I) = 1, \qquad H(T(f)) = 0, \qquad f \in L_1(R).$$

Thus if p is any point other than p_∞, $(\tau T(f))(p) \neq 0$ for some f in $L_1(R)$. As before it will sometimes be convenient to use the symbol $\mathscr{M}_0$ for $\mathscr{M} - \{p_\infty\}$.

1 Theorem. *The structure spaces of $\mathfrak{A}$ and $\mathfrak{A}_1$ are homeomorphic under the map which sends an ideal in $\mathfrak{A}$ into its intersection with $\mathfrak{A}_1$.*

Proof. The proof will be worded in terms of non-trivial complex valued homomorphisms rather than maximal ideals. If H is a homo-

morphism on $\mathfrak{A}$ its restriction $H_1 = H|\mathfrak{A}_1$ is a homomorphism on $\mathfrak{A}_1$. Thus the map $H \to H_1$ defines a map of $\mathscr{M}$ into $\mathscr{M}_1$. Since these homomorphisms are continuous (IX.2.8) and $\mathfrak{A}_1$ is dense in $\mathfrak{A}$ this map is one-to-one. To see that the range of this map is all of $\mathscr{M}_1$ let H_1 be any non-zero complex valued homomorphism on $\mathfrak{A}_1$. If $H_1(T(f)) = 0$ for f in $L_1(R)$ it is the restriction of the homomorphism H on $\mathfrak{A}$ defined by the equation $H(\alpha I + A) = \alpha$. If $H_1(T(f))$ does not vanish identically for f in $L_1(R)$ then , as was shown in the first part of the proof of Theorem 3.11, there is a continuous character h on R with

$$H_1(T(f)) = \int_R h(x)f(x)dx, \qquad f \in L_1(R).$$

The converse part of Theorem 3.11 shows that such a character determines a homomorphism on $\mathfrak{A}$ whose restriction to $\mathfrak{A}_1$ is H_1. Thus it has been shown that the map $H \to H_1 = H|\mathfrak{A}_1$ defines a one-to-one map of $\mathscr{M}$ onto all of $\mathscr{M}_1$. Now, by definition (IX.2.7), basic open sets in $\mathscr{M}$, $\mathscr{M}_1$ are defined in terms of a finite set of elements from $\mathfrak{A}$, $\mathfrak{A}_1$ respectively and so the continuity of the map $\mathscr{M} \to \mathscr{M}_1$ is an immediate consequence of the definition of the topology in $\mathscr{M}$ and $\mathscr{M}_1$. Since these spaces are compact (IX.2.8) it follows from Lemma I.5.8 that the map $\mathscr{M} \to \mathscr{M}_1$ is a homeomorphism. Q.E.D.

It follows from Theorem 1 and Theorem 3.15 that the structure space of $\mathfrak{A}_1$ is homeomorphic to the compactification $\hat{R} \cup \{p_\infty\}$ of the character group of R. The notation $\hat{R} \cup \{p_\infty\}$ is justified since under the established homeomorphisms between the three spaces $\mathscr{M}$, $\mathscr{M}_1$, and $\hat{R} \cup \{p_\infty\}$ the "points at infinity" in the three spaces correspond to each other. It will often be convenient to regard these spaces as being identified by these homeomorphisms.

If these spaces are identified by the above homeomorphisms then the function $(\tau T(f))(m)$ which, as shown in the preceding proof, is given by the integral $\int_R h(x)f(x)dx$ for some continuous character h on R, has the same value at the place m whether $T(h)$ is regarded as being an element of $\mathfrak{A}$ or $\mathfrak{A}_1$. Thus while there is still some ambiguity in the symbol $T(f)$, since its norm is only determined by specifying the algebra $\mathfrak{A}$ or $\mathfrak{A}_1$ to which it belongs, such specification is no longer necessary with regard to the symbol $\tau T(f)$. Since $\tau T(f)$ depends

only on f we shall sometimes find it convenient to denote it by the symbol τf or the symbol $\hat{f}$. Thus the map $f \to \hat{f}$ is an isomorphic map of the algebra $L_1(R)$ under convolution onto a subalgebra of the algebra $C(\mathscr{M}) = C(\mathscr{M}_1) = C(\hat{R} \cup \{p_\infty\})$.

2 Lemma. *Let the point $p \neq p_\infty$ be in the complement of the compact set C of $\mathscr{M}$. Then there is a point f in $L_1(R) \cap L_2(R)$ and a neighborhood N of p with*

$$0 \leq \hat{f}(m) \leq 1, \qquad m \in \mathscr{M};$$
$$\hat{f}(m) = 0, \qquad m \in C; \qquad \hat{f}(m) = 1, \qquad m \in N.$$

Proof. We regard $\mathscr{M}$ as being identified with the compactification $\hat{R} \cup \{p_\infty\}$ of the character group. Suppose first that $p = 0$ and choose the neighborhood W of 0 so that its closure is compact, does not contain p_∞ and is disjoint from C. Let V be a neighborhood of 0 with $-V = V$ and $V+V+V \subseteq W$. Let g_1, g_2 be the characteristic functions of V, $V+V$ respectively, and μ the measure introduced in Lemma 3.6. It follows from this lemma that $\mu(V) < \infty$ and hence that g_1 and g_2 are in $L_2(\hat{R}, \mathscr{B}, \mu)$ where $\mathscr{B}$ is the family of Borel sets in $\hat{R}$. It follows from the Plancherel theorem (8.9) that the functions $h_i = \tau^{-1}g_i$, $i = 1, 2$, are in $L_2(R)$ so that the function h defined on R by the equation $h(x) = h_1(x)\overline{h_2(x)}$ is in $L_1(R)$. It follows from Theorem 8.9(b) that $E(V)h_i = h_i$ and hence it is seen from Lemma 3.5 that $\hat{h}$ is a bounded continuous function and therefore in $L_2(R)$. Now

$$\hat{h}(m) = \int_R \overline{[x, m]}h_1(x)\overline{h_2(x)}dx.$$

Let $g_{1,m}$ be the translation of g_1 defined by the equation $g_{1,m}(q) = g_1(q+m)$. Then it is seen from Lemma 3.18 that

$$(\tau^{-1}g_{1,m})(x) = [x, -m](\tau^{-1}g_1)(x) = \overline{[x, m]}h_1(x)$$

and thus, since τ^{-1} is a unitary map

$$\hat{h}(m) = \int_R (\tau^{-1}g_{1,m})(x)\overline{(\tau^{-1}g_2)(x)}dx = \int_R g_1(q+m)\overline{g_2(q)}\mu(dq).$$

If m is in V then $g_1(q+m)\overline{g_2(q)} = g_1(q+m)$ and so it follows from Theorem 8.19 that $(\tau h)(m) = \mu(V)$ for m in V; whereas if m is not

in $V+V+V$ the above integrand vanishes and therefore $\hat{h}(m) = 0$ for such m and in particular $\hat{h}(m) = 0$ for m in C. For all q and m we have $g_1(q+m)\overline{g_2(q)} \leq g_1(q+m)$ and since μ is invariant

$$\hat{h}(m) \leq \int_{\hat{R}} g_1(q+m)\mu(dq) = \mu(V).$$

If f is defined as $h/\mu(V)$ the lemma has been proved in the case $p = 0$. The general case follows from this and the identity

$$(\tau f)(m-p) = (\tau\{[\cdot, p]f(\cdot)\})(m)$$

which was proved in Lemma 8.18. Q.E.D.

3 LEMMA. *Let C_1, C_2 be disjoint compact subsets of $\mathscr{M}$ with $p_\infty \notin C_2$. Then there is a point f in $L_1(R) \cap L_2(R)$ with*

$$0 \leq \hat{f}(m) \leq 1, \qquad m \in \mathscr{M};$$
$$\hat{f}(m) = 0, \qquad m \in C_1; \qquad \hat{f}(m) = 1, \qquad m \in C_2.$$

PROOF. According to the preceding lemma there is, for each p in C_2, a neighborhood N_p of p and a function f_p in $L_1(R) \cap L_2(R)$ for which $\hat{f}_p$ vanishes on C_1, is identically 1 on N_p, and has $0 \leq \hat{f}_p(m) \leq 1$ for all m. Since C_2 is compact there are a finite number of elements $f_1, \ldots, f_n$ in $L_1(R) \cap L_2(R)$ whose corresponding neighborhoods $N_1, \ldots, N_n$ cover C_2. The point $f = f_1 + f_2 - f_1 * f_2$ in $L_2(R) \cap L_2(R)$ has the properties

$$\hat{f}(m) = \hat{f}_1(m) + \hat{f}_2(m) - \hat{f}_1(m)\hat{f}_2(m) = 1, \qquad m \in N_1 \cup N_2,$$
$$\hat{f}(m) = 0, \qquad m \in C_1; \qquad 0 \leq \hat{f}(m) \leq 1, \qquad m \in \mathscr{M}.$$

It is clear that this process may be iterated a finite number of times to obtain the desired function. Q.E.D.

4 THEOREM. *Let f and g be integrable on R and let C be a compact set in $\mathscr{M}$ not containing the point at infinity. If $\hat{g}$ vanishes on the complement of C and $\hat{f}$ vanishes at no point of C then there is an integrable function h with $g = f * h$.*

PROOF. Since τ is a *-isomorphism on $\mathfrak{A}$ it follows from Lemma 8.1(c) that $\tau f^* = \overline{\tau f}$ so that $\tau(f * f^*)(m) = |f(m)|^2$. Thus, for some positive

number ε, $\tau(f * \tilde{f})(m) \geq \varepsilon$ for m in a neighborhood N of C. It follows from the preceding lemma that there is a function k in $L_1(R) \cap L_2(R)$ with

$$0 \leq \hat{k}(m) \leq 1, \qquad m \in \mathscr{M};$$
$$\hat{k}(m) = 0, \qquad m \in C; \qquad \hat{k}(m) = 1, \qquad m \notin N.$$

Thus $\hat{k}(m)\hat{g}(m) = 0$ for every m in $\mathscr{M}$ and, since the map $f \to \hat{f}$ is an isomorphism, it follows that $k * g = 0$. Since $\tau(k + f * \tilde{f})(m) > 0$ for every m in $\mathscr{M}$ the operator $T(k + f * \tilde{f})$ is contained in no maximal ideal in $\mathfrak{A}_1$ and hence it follows from Lemma IX.1.12(e) that it has an inverse $\alpha I + T(a)$ in $\mathfrak{A}_1$. Thus for every m in $\mathscr{M}$,

$$\begin{aligned}\hat{g}(m) &= (\alpha + \hat{a}(m))(\hat{k}(m) + \hat{f}(m)\hat{\tilde{f}}(m))\hat{g}(m) \\ &= (\alpha + \hat{a}(m))\hat{f}(m)\hat{\tilde{f}}(m)\hat{g}(m)\end{aligned}$$

from which it follows that the function h defined by the equation $h = \alpha\tilde{f} * g + a * \tilde{f} * g$ has the property that $\hat{g}(m) = \hat{h}(m)\hat{f}(m)$ for all m in $\mathscr{M}$. Since the map $f \to \hat{f}$ is an isomorphism this shows that $g = h * f$. Q.E.D.

5 Lemma. *The set of functions f in $L_1(R)$ for which $\hat{f}$ vanishes in a neighborhood of infinity is dense in $L_1(R)$.*

Proof. It follows from Lemma 3.6 that the set of all functions in $L_2(R, \mathscr{B}, \mu)$ which vanish outside of compact sets is dense in this space, and from the Plancherel theorem that the set of all f in $L_2(R)$ for which $\hat{f}$ vanishes except on a compact set in R is dense in $L_2(R)$. Since the map $[f, g] \to fg$ takes $L_2(R) \times L_2(R)$ onto all of $L_1(R)$ and is continuous it follows that the set of products fg where f, g are in $L_2(R)$ and their transforms $\hat{f}$, $\hat{g}$ vanish outside of compact sets in $\hat{R}$ is dense in $L_1(R)$. Thus the proof may be completed by showing that if f, g are in $L_2(R)$ with $\hat{f}$, $\hat{g}$ vanishing outside a compact set C in $\hat{R}$ with $C = -C$ then $\tau(fg)$ vanishes outside the compact set $C + C$. From the Plancherel theorem it is seen that τ is a unitary map and thus Lemma 3.18 shows that

$$\begin{aligned}\tau(fg)(m) &= \int_R \overline{[x, m]} f(x) \overline{\overline{g(x)}} dx \\ &= \int_{\hat{R}} \hat{f}(p + m) \overline{\hat{\overline{g}}(p)} \mu(dp).\end{aligned}$$

Since the integrand vanishes unless m is a point of $C+C$ the proof is complete. Q.E.D.

It should be recalled that for y in R the y translate f_y of a function f on R is defined by the equation $f_y(x) = f(x-y)$. A set of functions on R is said to be closed under translations if, for every y in R, f_y belongs to the set whenever f does.

6 LEMMA. *The closed linear manifold determined by the translates of the functions in a set $S \subseteq L_1(R)$ coincides with the closed ideal determined by S.*

PROOF. First note that if $f, g \in L_1(R)$ and $\varphi \in L_\infty(R)$ then the function $\varphi(x)f(x-y)g(y)$ is $\lambda\times\lambda$-integrable and so

$$\text{(i)} \qquad \int_R \varphi(x)(f*g)(x)dx = \int_R \varphi(x)\left\{\int_R f(x-y)g(y)dy\right\}dx$$
$$= \int_R g(y)\left\{\int_R \varphi(x)f(x-y)dx\right\}dy.$$

Now let $\mathfrak{L}$ be the closed linear manifold determined by the translates of the functions in S, and $\mathfrak{J}$ the smallest closed ideal in $L_1(R)$ containing S. It follows from Lemma 3.1(f) that $\mathfrak{L}$ is closed under translations. Let F be a continuous linear functional on $L_1(R)$ which vanishes on $\mathfrak{L}$, let φ be the bounded measurable function representing F (IV.8.5), and let f be any point in $\mathfrak{L}$. Then, since $0 = Ff_y = \int_R \varphi(x)f(x-y)dx$ equation (i) shows that $F(f*g) = 0$ for every g in $L_1(R)$. It follows from Corollary II.3.13 that $f*g$ is in $\mathfrak{L}$ for every g in $L_1(R)$, which shows that $\mathfrak{L}$ is an ideal and thus $\mathfrak{L} \supseteq \mathfrak{J}$. Conversely let f be in the closed ideal $\mathfrak{J}$ in $L_1(R)$ and suppose that the bounded linear functional F vanishes on $\mathfrak{J}$. Then if the function φ represents F we have

$$\int_R \varphi(x)(f*g)(x)dx = 0, \qquad g \in L_1(R).$$

The equation (i) shows that

$$\int_R \varphi(x)f(x-y)dx = 0$$

for almost all y in R. However, this function is continuous in y (Lemma 3.1(d)) and so it vanishes for all y in R. Thus Corollary II.3.13 shows that all the translates of f are in $\mathfrak{J}$ and thus proves

that $\mathfrak{J}$ is invariant under translations. This shows that $\mathfrak{J} \supseteq \mathfrak{L}$ and completes the proof of the lemma. Q.E.D.

→ 7 THEOREM. (*Wiener L_1-closure theorem*). *Linear combinations of the translates of a function f in $L_1(R)$ are dense in $L_1(R)$ if and only if its transform $\hat{f}$ does not vanish on the character group of R.*

PROOF. Let $\mathfrak{L}$ be the closed linear subspace spanned by the translates of the point of f in $L_1(R)$ and suppose that for every m in $\mathscr{M}$ other than $m = p_\infty$ we have $\hat{f}(m) \neq 0$. Then $\mathfrak{L}$ is closed under translations and by the preceding lemma is an ideal. If g is a point of $L_1(R)$ with $\hat{g}$ vanishing on the complement of some compact set then, by Theorem 4, there is a point h in $L_1(R)$ with $g = f * h$. This shows that $\mathfrak{L}$ contains every such g. It follows from Lemma 5 that $\mathfrak{L} = L_1(R)$.

Conversely suppose that linear combinations of the translates of the point f in $L_1(R)$ are dense in $L_1(R)$ and suppose that for some point $m \neq p_\infty$ we have $\hat{f}(m) = 0$. By Lemma 6, functions of the form $h = f * g$ with g in $L_1(R)$ are dense in $L_1(R)$. For such functions h we have $\hat{h}(m) = \hat{f}(m)\hat{g}(m) = 0$ and thus since the map $h \to \hat{h}$ is continuous, it follows that $\hat{h}(m) = 0$ for every h in $L_1(R)$. Since $m \neq p_\infty$ this contradicts Lemma 2 and completes the proof. Q.E.D.

8 THEOREM. *If $\mathfrak{L}$ is a proper closed linear subspace of $L_1(R)$ which is invariant under translation, then there is a point m in $\hat{R}$ for which $\hat{f}(m) = 0$ for every f in $\mathfrak{L}$.*

PROOF. Suppose the conclusion is false. Let C be a compact set in $\hat{R}$ and g a function in $L_1(R)$ whose transform $\hat{g}$ vanishes on the complement of C. For each p in C there is, by assumption, a point f_p in $\mathfrak{L}$ with $\hat{f}_p(p) \neq 0$. Since $\mathfrak{L}$ is an ideal (Lemma 6) $f_p * \tilde{f}_p$ is in $\mathfrak{L}$ and since $\tau(f_p * \tilde{f}_p)(m) = |\hat{f}_p(m)|^2$ by Lemma 8.1(c) we may and shall assume that f_p has been chosen so that $\hat{f}_p(m) \geqq 0$ for all m in $\hat{R}$. Let N_p be a neighborhood of p on which $\hat{f}_p$ is positive. Since C is compact, a finite number $N_{p_1}, \ldots, N_{p_n}$ of these neighborhoods cover C. The function $f = f_{p_1} + \ldots + f_{p_n}$ is in $\mathfrak{L}$ and has $\hat{f}(m) > 0$ on C. By Theorem 4 there is a function h in $L_1(R)$ with $g = f * h$. It follows from Lemma 6 that g is in $\mathfrak{L}$. Since g and C were arbitrary it follows from Lemma 5 that $\mathfrak{L} = L_1(R)$ which is the desired contradiction. Q.E.D.

The next result may be regarded as being a result dual to that just proved.

9 THEOREM. *A non-zero linear subspace of $L_\infty(R)$ which is invariant under translations and closed in the $L_1(R)$ topology of $L_\infty(R)$ contains at least one character on R.*

PROOF. Let $\mathfrak{K}$ be the subspace of $L_\infty(R)$ with the stated properties and let $\mathfrak{L}$ be the conjugate-orthogonal complement of $\mathfrak{K}$, i.e., the set of all h in $L_1(R)$ with

$$\text{(i)} \qquad \int_R \overline{h(x)}\varphi(x)dx = 0$$

for all φ in $\mathfrak{K}$. Since $\mathfrak{K}$ contains non-zero vectors, $\mathfrak{L}$ is a proper subspace of $L_1(R)$. The invariance of $\mathfrak{K}$ implies that of $\mathfrak{L}$ for if h is in $\mathfrak{L}$ and φ in $\mathfrak{K}$ then φ_{-y} is in $\mathfrak{K}$ and

$$\begin{aligned} 0 &= \int_R \overline{h(x)}\varphi_{-y}(x)dx = \int_R \overline{h(x)}\varphi(x+y)dx \\ &= \int_R \overline{h(x-y)}\varphi(x)dx = \int_R \overline{h_y(x)}\varphi(x)dx \end{aligned}$$

which shows that h_y is in $\mathfrak{L}$ and proves that $\mathfrak{L}$ is invariant under translations. The preceding theorem shows that there is a point m_0 in $\hat{R}$ with $\hat{h}(m_0) = 0$ for every h in $\mathfrak{L}$. Furthermore, since $\mathfrak{K}$ is closed in the $L_1(R)$ topology of $L_\infty(R)$, it follows from Corollary V.3.12 that $\mathfrak{K}$ is the conjugate-orthogonal complement of $\mathfrak{L}$, i.e., if equation (i) holds for some φ in $L_\infty(R)$ and every h in $\mathfrak{L}$ then φ is in $\mathfrak{K}$. Thus, if $h_{m_0} = \overline{[\cdot, m_0]}$ corresponds to m_0 as in Theorem 3.11 we have

$$0 = \hat{h}(m_0) = \int_R h(x)\overline{[x, m_0]}dx = \int_R \overline{h(x)}[x, m_0]dx$$

for every h in $\mathfrak{L}$ which shows that the character $[\cdot, m_0]$ is in $\mathfrak{K}$. Q.E.D.

The result just proved shows that if the bounded measurable function φ on R is not zero almost everywhere there is at least one character of R in the L_1-closed linear manifold $\mathfrak{K}(\varphi)$ of $L_\infty(R)$ which is determined by the translates of φ. The problem of *spectral synthesis* for functions, posed by A. Beurling, is to determine whether φ is in the L_1-closed linear manifold of $L_\infty(R)$ which is spanned by the characters in $\mathfrak{K}(\varphi)$. While this is not always the case, it is, as will be seen, sometimes true. The following definition of a spectral set

will introduce a basic notion in the study of the problem of spectral synthesis.

10 DEFINITION. The *spectral set* $\sigma(\varphi)$ of a bounded measurable function φ on R is the set of all characters on R which are contained in the L_1-closed subspace $\mathfrak{K}(\varphi)$ in $L_\infty(R)$ determined by the translates of φ.

A number of the elementary properties of spectral sets which will be used in the discussion of spectral synthesis are contained in the following lemma.

11 LEMMA. *Let φ and ψ be in $L_\infty(R)$, α be a complex number, $\varphi_y(x) = \varphi(x-y)$, and $\tilde{\varphi}(x) = \overline{\varphi(-x)}$ for x and y in R. Then*

(a) *The spectral set $\sigma(\varphi)$ is a closed subset of $\hat{R}$ which is void if and only if $\varphi = 0$.*

(b) $\sigma(\alpha\varphi) = \sigma(\varphi)$, $\qquad \alpha \neq 0.$

(c) $\sigma(\varphi_y) = \sigma(\varphi)$, $\qquad \sigma(\tilde{\varphi}) = \sigma(\varphi)$.

(d) *if $[\cdot, m]$ is a character on R then*

$$\sigma([\cdot, m]\varphi(\cdot)) = \sigma(\varphi) + m.$$

(e) $\sigma(\varphi+\psi) \subseteq \sigma(\varphi) \cup \sigma(\psi)$.

PROOF. It follows from Theorem 9 that $\sigma(\varphi)$ is void if and only if $\varphi = 0$. To see that $\sigma(\varphi)$ is closed let $\mathfrak{L}(\varphi)$ be the conjugate-orthogonal complement in $L_1(R)$ to $\mathfrak{K}(\varphi)$. Then, by Corollary V.8.12, a character m in $\hat{R}$ is in $\sigma(\varphi)$ if and only if

$$\text{(i)} \qquad \hat{f}(m) = \int_R \overline{[x, m]} f(x)\,dx = 0, \qquad f \in \mathfrak{L}(\varphi).$$

Since $\hat{f}$ is continuous the set $\{m \mid m \in \hat{R}, \hat{f}(m) = 0\}$ is closed and thus $\sigma(\varphi)$, which is the intersection of all these sets with f in $\mathfrak{L}(\varphi)$, is also closed. This proves (a). The statements (b) and $\sigma(\varphi_y) = \sigma(\varphi)$ are immediate for it follows from the definition of $\mathfrak{K}(\varphi)$ that $\mathfrak{K}(\varphi) = \mathfrak{K}(\alpha\varphi) = \mathfrak{K}(\varphi_y)$. An elementary calculation shows that $\tilde{f} \in \mathfrak{L}(\tilde{\varphi})$ if and only if $f \in \mathfrak{L}(\varphi)$, and, since τf and $\tau\tilde{f}$ have the same zeros, this fact together with (i) completes the proof of (c). Now let $\psi(x) = [x, m]\varphi(x)$ so that $f \in \mathfrak{L}(\psi)$ if and only if $\overline{[\cdot, m]}f(\cdot)$ is in $\mathfrak{L}(\varphi)$. Thus, using condition (i), it follows that m_1 is in $\sigma(\varphi)$ if and only if

$$\int_R \overline{[x, m_1]}\,\overline{[x, m]} f(x)\,dx = 0, \qquad f \in \mathfrak{L}(\psi).$$

This is clearly the case if and only if $m_1+m \in \sigma(\psi)$ which shows that $\sigma(\varphi)+m = \sigma(\psi)$ and proves (d).

We shall make an indirect proof of (e) by supposing that there is an m in $\sigma(\varphi+\psi)$ but in neither of the spectral sets $\sigma(\varphi)$ or $\sigma(\psi)$. It follows from (i) that there are functions f, g in $\mathfrak{L}(\varphi), \mathfrak{L}(\psi)$ respectively with $\hat{f}(m) \neq 0, \hat{g}(m) \neq 0$. Let $h = f * g$ so that $\hat{h}(m) = \hat{f}(m)\hat{g}(m) \neq 0$. Since f is in $\mathfrak{L}(\varphi)$ we have

$$\begin{aligned}\int_R h(x)\overline{\varphi(x-y)}dx &= \int_R\int_R f(x-z)g(z)\overline{\varphi(x-y)}dx\,dz\\ &= \int_R\int_R f(x)\varphi(x+z-y)g(z)dx\,dz\\ &= \int_R 0\cdot g(z)dz = 0.\end{aligned}$$

Thus h is in $\mathfrak{L}(\varphi)$. Similarly h is seen to be in $\mathfrak{L}(\psi)$ and thus it follows that h is in $\mathfrak{L}(\varphi+\psi)$. Since m is in $\sigma(\varphi+\psi)$, condition (i) shows that $\hat{h}(m) = 0$ which is a contradiction. Q.E.D.

The next lemma relates the notion of a spectral set with that of convolution.

12 Lemma. *Let φ be in $L_\infty(R)$ and f in $L_1(R)$*

(a) *A character m is in $\sigma(\varphi)$ if and only if $\hat{g}(m) = 0$ for all g in $L_1(R)$ for which $g * \varphi = 0$.*

(b) *$\sigma(f * \varphi) \subseteq \sigma(\varphi)$.*

(c) *The spectral set $\sigma(f * \varphi)$ does not intersect any open set in $\hat{R}$ upon which $\hat{f}$ vanishes.*

Proof. The function $f * \varphi$ is bounded and continuous (Lemma 3.1(d)) and so the statements to be proved are meaningful. To prove (a) note first that $g * \varphi = 0$ if and only if the function $\tilde{g}$ is in $\mathfrak{L}(\varphi)$, the conjugate-orthogonal complement in $L_1(R)$ to $\mathfrak{R}(\varphi)$. It was observed in the proof of the preceding theorem that a character m is in $\sigma(\varphi)$ if and only if $\hat{g}(m) = 0$ for every g in $\mathfrak{L}(\varphi)$. What is stated in part (a) above is that m is in $\sigma(\varphi)$ if and only if $\hat{g}(m) = 0$ for every g such that $\tilde{g}$ is in $\mathfrak{L}(\varphi)$. Since $(\tau\tilde{g})(m) = \overline{\hat{g}(m)}$, these two statements are readily seen to be equivalent.

Now let $\tilde{g} \in \mathfrak{L}(\varphi)$. Then, as was shown in the preceding paragraph, $g * \varphi = 0$. Consequently $g * f * \varphi = 0$ and so $\tilde{g} \in \mathfrak{L}(f * \varphi)$. This proves that $\mathfrak{L}(\varphi) \subseteq \mathfrak{L}(f * \varphi)$. Since $m \in \sigma(\varphi)$ if and only if $\hat{g}(m) = 0$

for every $\tilde{g}$ in $\mathfrak{L}(\varphi)$, and $m \in \sigma(f * \varphi)$ if and only if $\hat{g}(m) = 0$ for every $\tilde{g}$ in $\mathfrak{L}(f * \varphi)$, the statement (b) follows immediately.

Finally let $\hat{f}(m) = 0$ for m in the open set N. Let p be in N and, using Lemma 3, choose a point h in $L_1(R)$ such that $\hat{h}(p) = 1$ and $\hat{h}(m) = 0$ on the complement of N. Then $\tau(h * f) = \hat{h}\hat{f} = 0$, and, since τ is an isomorphism, $h * f = 0$. Thus $h * (f * \varphi) = 0$. Since $\hat{h}(p) \neq 0$ it follows from part (a) that $p \notin \sigma(f * \varphi)$. Q.E.D.

13 Theorem. *A bounded measurable function φ on R is in the L_1-closed linear subspace of $L_\infty(R)$ which is determined by the characters in any neighborhood of its spectral set. Conversely, if φ is in the L_1-closed linear manifold determined by the characters in some closed set F in R then $\sigma(\varphi) \subseteq F$.*

Proof. Let N be a neighborhood of $\sigma(\varphi)$ and suppose that φ is not in the L_1-closed subspace determined by the characters in N. By Corollary V.3.12 there is an f in $L_1(R)$ such that $\hat{f}(m) = 0$ for m in N and $\int_R f(x)\overline{\varphi(x)}dx \neq 0$. Since $(\tau\tilde{f})(m) = \overline{\hat{f}(m)}$ it follows that $(\tau\tilde{f})(m) = 0$ for m in N and thus it is seen from Lemma 12(c) that $\sigma(\tilde{f} * \varphi) \cap N$ is void. By Lemma 12(b), $\sigma(\tilde{f} * \varphi) \subseteq \sigma(\varphi) \subseteq N$, and so $\sigma(\tilde{f} * \varphi)$ is void and, by Lemma 11(a), we have $\tilde{f} * \varphi = 0$. By Lemma 3.1(d) $\tilde{f} * \varphi$ is a continuous function and so it vanishes identically, in particular

$$0 = (\tilde{f} * \varphi)(0) = \int_R \tilde{f}(x)\varphi(-x)dx = \int_R \overline{f(x)}\varphi(x)dx,$$

a contradiction.

To prove the converse suppose that there is a character p in $\sigma(\varphi)$ but not in F. By Lemma 2 there is a function f in $L_1(R)$ with $\hat{f}(p) = 1$ and $\hat{f}(m) = 0$ for m in F. Then

$$\int_R f(x)\overline{[x, m]}dx = \hat{f}(m) = 0, \qquad m \in F,$$

and so $\int_R f(x)\overline{\psi(x)}dx = 0$ for ψ in the L_1-closed subspace $\mathfrak{Z}$ spanned by the characters in F. Since $[x+y, m] = [x, m][y, m]$ it follows that the L_1-closed linear manifold $\mathfrak{R}$ determined by the characters in F is invariant under translation. Thus, since φ is in $\mathfrak{R}$ by hypothesis, all the translates of φ are in $\mathfrak{R}$. Since $\mathfrak{R}$ is L_1-closed it follows from Definition 10 that $[\cdot, p]$ is in $\mathfrak{R}$. But then $\hat{f}(p) = \int_R f(x)\overline{[x, p]}dx = 0$

which is the desired contradiction. Q.E.D.

Theorem 13 will be sharpened in Theorem 20 by showing that φ is in the subspace spanned by the characters in $\sigma(\varphi)$ provided that the boundary of its spectral set contains no non void perfect subset. Before proving this it will be useful to obtain some preliminary information. To this end we introduce a linear map Φ from $L_2(R)$ into $L_2(\hat{R})$ as follows. Let φ be in $L_\infty(R)$, so that for every f in $L_2(R)$ the function φf is also in $L_2(R)$ and $|\varphi f|_2 \leqq |\varphi|_\infty |f|_2$. The map Φ from $L_2(R)$ into $L_2(\hat{R})$ is defined, for f in $L_2(R)$, by the equation $\Phi(f) = \tau(\varphi f)$. The Plancherel theorem shows that Φ is continuous. It follows from Lemma 3.18 that if φ is a character m_0 in $\hat{R}$, i.e., $\varphi(x) = [x, m_0]$, $x \in R$, then $\Phi(f)$ is the translate $(\tau f)_{m_0}$ of τf which is given by the formula $(\Phi f)(m) = \hat{f}(m-m_0)$. In the following we shall be concerned with the special case of a function φ in $L_\infty(R)$ whose spectral set $\sigma(\varphi) = \{0\}$.

14 Lemma. *Let φ be in each of the spaces $L_1(R)$, $L_2(R)$ and $L_\infty(R)$ and have $\sigma(\varphi) = \{0\}$. Then $\Phi(f)$ vanishes on every open set in $\hat{R}$ upon which $\hat{f}$ vanishes.*

Proof. Let m be an arbitrary point in the open set N upon which $\hat{f}$ vanishes and let V be a neighborhood of the identity in $\hat{R}$ with $V = -V$ and $m+V \subseteq N$. There is, by Theorem 13, a generalized sequence the elements of which are linear combinations $\varphi_\alpha(x) = \sum_{i=1}^{r} c_i[x, m_i]$ of characters with m_i in V and such that

$$\text{(i)} \qquad \int_R \varphi(x)g(x)dx = \lim_\alpha \int_R \varphi_\alpha(x)g(x)dx, \qquad g \in L_1(R).$$

Since $m+V = m-V \subseteq N$, $m-m_i$ is in N and it follows that the linear operator Φ_α corresponding to φ_α has $(\Phi_\alpha f)(m) = 0$ for every α. It follows from equation (i) that

$$\begin{aligned}(\Phi f)(m) &= \int_R \overline{[x, m]}\varphi(x)f(x)dx \\ &= \lim_\alpha \int_R \overline{[x, m]}\varphi_\alpha(x)f(x)dx = \lim_\alpha (\Phi_\alpha f)(m) = 0,\end{aligned}$$

which shows that Φf vanishes at the arbitrary point m in N. Q.E.D.

15 Lemma. *Let the function φ in $L_\infty(R)$ have $\sigma(\varphi) = \{0\}$. Then, for some complex number α,*

$$\Phi(\chi_V) = \alpha\chi_V$$

for every open set V in $\hat{R}$ whose closure is compact.

Proof. It follows from Lemma 3.6(i) that $\mu(V)$ is finite and thus, as was observed in the note following the proof of that lemma, for every positive integer n there is an open set $U_n \subset V$ with $\overline{U}_n \subseteq V$ and $\mu(V \cap U_n') < 1/n$. We may and shall assume that $U_n \subseteq U_{n+1}$, $n = 1, 2, \ldots$. By Lemma 3 there are functions f_n in $L_1(R) \cap L_2(R)$ with $\hat{f}_n$ vanishing on the complement of V, $\hat{f}_n(m) = 1$ for m in $\overline{U}_n$, and $\hat{f}_n$ having all their values between zero and one. It is clear that the sequence $\{\hat{f}_n\}$ approaches χ_V in the norm of $L_2(\hat{R})$ and so $\Phi(f_n)$ approaches $\Phi(\tau^{-1}\chi_V)$ in this same space. By Corollary III.6.13(a) a subsequence converges almost everywhere and so we may and shall assume that Φf_n converges almost everywhere to $\Phi(\tau^{-1}\chi_V)$.

Since $\hat{f}_n$ vanishes on the complement of V it follows from the preceding lemma that Φf_n does likewise. It will next be shown that Φf_n is constant on U_n. To see this, let $m_1, m_2 \in U_n$. It follows from Lemma 3.18 that the functions g_n and g_n' defined for x in R by the equations $g_n(x) = \overline{[x, m_1]}f_n(x)$ and $g_n' = \overline{[x, m_2]}f_n(x)$ have the transforms $\hat{g}_n(m) = \hat{f}_n(m+m_1)$ and $\hat{g}_n'(m) = \hat{f}_n(m+m_2)$. The same lemma and the remarks preceding Lemma 14 show that $(\Phi g_n)(m) = (\Phi f_n)(m+m_1)$ and $(\Phi g_n')(m) = (\Phi f_n)(m+m_2)$ for all m. Thus, we have

$$\tau(g_n - g_n')(m) = \hat{f}_n(m+m_1) - \hat{f}_n(m+m_2) = 0$$

if m belongs to the intersection W of $U_n - m_1$ and $U_n - m_2$. Plainly, W is a neighborhood of the identity in $\hat{R}$. Thus, Lemma 14 shows that $(\Phi g_n)(m) = (\Phi g_n')(m)$ for m in W, which means that $(\Phi f_n)(m+m_1) = (\Phi f_n)(m+m_2)$ for m in W, and hence, *a fortiori*, $(\Phi f_n)(m_1) = (\Phi f_n)(m_2)$. It has now been shown that Φf_n vanishes on the complement of V and is a constant on U_n.

Since $(f_{n+1} - f_n)$ vanishes on U_n it follows from Lemma 14 that $\Phi f_{n+1} = \Phi f_n$ on U_n and thus that there is a complex number α_V which may depend upon V but not upon n such that for every m in $\bigcup_{n=1}^{\infty} U_n$, $(\Phi f_n)(m) = \alpha_V$ for all sufficiently large values of n. Thus

$$(\Phi\tau^{-1}\chi_V)(m) = \lim_n (\Phi f_n)(m) = \alpha_V\chi_V(m),$$

for almost all m.

It remains to be proved that the number α_V is independent of the open set V. If f is in $L_1(R) \cap L_2(R)$, $\hat{f}$ vanishes on the complement of V, and $f(m) = 1$ for m in an open subset V_0 of V, then the above proof shows that $(\Phi f)(m) = \alpha_V$ for every m in V_0, from which it follows that $\alpha_{V_0} = \alpha_V$. Now let V_1 be an arbitrary open subset of $\hat{R}$ with compact closure. Then it follows from what has just been demonstrated that $\alpha_{V_1} = \alpha_{V \cup V_1} = \alpha_V$, i.e., α_V is independent of V. Q.E.D.

16 THEOREM. *If the bounded measurable function φ has its spectral set consisting of the single point m then, for some complex number α, $\varphi(x) = \alpha[x, m]$ for almost all x in R*

PROOF. In view of Lemma 11(d) it suffices to prove the theorem in the case $m = 0$. In this case the preceding lemma gives an α such that for every open set V in $\hat{R}$ with compact closure we have

$$\text{(i)} \qquad \tau\varphi\tau^{-1}\chi_V = \chi_V.$$

Since every compact set in $\hat{R}$ is contained in an open set with compact closure it follows from Lemma 3.6 that equation (i) holds for any open set with finite measure. It follows from the regularity of μ that (i) is valid for every Borel set in $\hat{R}$ with finite measure. Since $\tau\varphi\tau^{-1}f$ is linear and continuous in f it follows that $\tau\varphi\tau^{-1}f = \alpha f$ and thus $\varphi\tau^{-1}f = \alpha\tau^{-1}f$ for all f in $L_2(R)$. Hence $\varphi(x) = \alpha$ for almost all x in R. Q.E.D.

17 COROLLARY. *If the transform τf of the function f in $L_1(R)$ vanishes at the point m_0 then f is the limit in $L_1(R)$ of a sequence $\{f_n\}$ each of whose elements has a transform τf_n vanishing in a neighborhood of m_0.*

PROOF. Let $\mathfrak{L}$ be the closure of the set of all functions h in $L_1(R)$ whose transform τh vanishes on a neighborhood of m_0. It will be shown that f is in $\mathfrak{L}$.

If g is in $\mathfrak{L}$ there is a sequence $\{g_n\}$ converging to g such that $\hat{g}_n$ vanishes in a neighborhood of m_0. Hence for every h in $L_1(R)$, $h * g_n \to h * g$ and since $\tau(h * g_n) = \hat{h}\hat{g}_n$ it follows that $h * g$ is in $\mathfrak{L}$ and that $\mathfrak{L}$ is an ideal. It follows from Lemma 6 that $\mathfrak{L}$ is closed under

translation. Thus $\mathfrak{K}$, the conjugate-orthogonal complement in $L_\infty(R)$ of $\mathfrak{L}$, is also closed under translation. Now $[\cdot, m_0]$ is the only character in $\mathfrak{K}$ since, if $m \neq m_0$, Lemma 3 shows that there is a function g in $L_1(R)$ whose transform $\hat{g}$ vanishes in a neighborhood of m_0 and $\hat{g}(m) \neq 0$. Thus $[\cdot, m]$ is not in $\mathfrak{K}$ and $[\cdot, m_0]$ is the only character in $\mathfrak{K}$. It follows from the preceding theorem that $\mathfrak{K}$ consists of scalar multiples of $[\cdot, m_0]$. Since $\hat{f}(m_0) = 0$ the function f is in the conjugate-orthogonal complement of $\mathfrak{K}$, which by Theorem V.3.12, is $\mathfrak{L}$. Q.E.D.

18 Corollary. *Let m_0 be a point of $\hat{R}$. Then there is a generalized sequence $\{h_\alpha\}$ in $L_1(R)$ with $\hat{h}_\alpha(m_0) = 1$, $|h_\alpha|_1 = 1$ for all α and for which $h_\alpha * f \to 0$ in $L_1(R)$ for every f in $L_1(R)$ with $\hat{f}(m_0) = 0$.*

Proof. Let V be a neighborhood of the origin in R with compact closure and let W be a neighborhood of the origin with the properties $W = -W$, $W + W \subseteq V$. Then the characteristic function χ_W is in $L_2(R)$ and has norm $|\chi_W|_2 = \mu(W)^{\frac{1}{2}}$. It follows from the Plancherel theorem that the positive function $p_V = (\tau^{-1}\chi_W)\overline{(\tau^{-1}\chi_W)}\mu(W)^{-1}$ has its $L_1(R)$ norm $|p_V|_1 = 1$. Since τ is a unitary map it follows from Lemma 3.18 that

$$\hat{p}_V(m_1) = \frac{1}{\mu(W)}\int_{\hat{R}} \chi_W(m+m_1)\overline{\chi_W(m)}\mu(dm),$$

and so $\hat{p}_V(0) = 1$ and $\hat{p}_V(m) = 0$ for $m \notin V$. Let $h_V(x) = [x, m_0]p_V(x)$ for x in R. Then h_V is in $L_1(R)$, $|h_V| = 1$, and $\hat{h}_V(m_0) = p_V(0) = 1$. The directed set $\{E, \leqq\}$ to be used in defining the generalized sequence of our corollary will be the family of neigborhoods of the origin in $\hat{R}$ with compact closures, ordered by defining $V \leqq U$ to mean that $U \subseteq V$. The required generalized sequence is then $\{h_V\}$ as defined above. It remains to be shown that $h_V * f \to 0$ in $L_1(R)$ provided that the function f in $L_1(R)$ has $\hat{f}(m_0) = 0$. It follows from Lemma 3.18 that $\hat{h}_V(m) = 0$ if $m \notin m_0 + V$. Now, if $\varepsilon > 0$, there is, by the preceding corollary, a function g in $L_1(R)$ such that $\hat{g}$ vanishes on some neighborhood U of m_0 and $|f-g|_1 < \varepsilon$. There is a V_ε in $\{E, \leqq\}$ such that any V in this set with $V_\varepsilon \leqq V$ has $m_0 + V \subseteq U$. Hence $\tau(h_V * g) = \hat{h}_V\hat{g}$ vanishes identically and $h_V * g = 0$. This, together with Lemma 3.1(b), shows that

$$|h_V * f|_1 - |h_V * f - h_V * g| \leq |f - g|_1 < \varepsilon, \qquad V_\varepsilon \leq V,$$

which completes the proof. Q.E.D.

19 LEMMA. *If φ and f are in $L_\infty(R)$ and $L_1(R)$ respectively and if $\hat{f}(m) = 0$ for every m in the spectral set $\sigma(\varphi)$, then $\sigma(f * \varphi)$ contains no isolated points.*

PROOF. We shall make an indirect proof by supposing that m_0 is an isolated point of $\sigma(f * \varphi)$. Since $\sigma(f * \varphi) \subseteq \sigma(\varphi)$ by Lemma 12, it is seen that $\hat{f}(m_0) = 0$. Let h be in $L_1(R)$ with $\hat{h}(m_0) = 1$ and $\hat{h}$ vanishing on an open set containing the remainder of $\sigma(f * \varphi)$. It follows from Lemma 12 that the set $(h * f * \varphi)$ contains at most the single point m_0 and hence, from Theorem 16 and Lemma 8.1(d), that there is a number α with $(h * f * \varphi)(x) = \alpha[x, m_0]$ for all x in R. To see that $\alpha = 0$ let $\{h_V\}$ be the generalized sequence of Corollary 18. Then, since $\hat{f}(m_0) = 0$, it follows from that corollary that $h_V * f \to 0$ in $L_1(R)$ and therefore $h_V * (h * f * \varphi) \to 0$. On the other hand

$$\begin{aligned} h_V * (h * f * g)(x) &= \alpha \int_R [x-y, m_0] h_V(y) dy \\ &= \alpha[x, m_0] \int_R \overline{[y, m_0]} h_V(y) dy = \alpha[x, m_0], \end{aligned}$$

which is independent of V. Thus $\alpha = 0$ and $(h * f * \varphi)(x) = \alpha[x, m_0] = 0$ for all x. Since $\hat{h}(m_0) = 1$ it follows from Lemma 12(a) that $m_0 \notin \sigma(f * \varphi)$ which is a contradiction. Q.E.D.

Making use of these preliminary results it will now be shown that if the spectral set of the bounded measurable function φ is such that every non-void closed subset of its boundary contains an isolated point, then φ is the limit, in the L_1-topology of $L_\infty(R)$, of a generalized sequence of linear combinations of the characters in $\sigma(\varphi)$.

20 THEOREM. *If the boundary of the spectral set of the bounded measurable function φ contains no non-void perfect subset then φ is in the L_1-closed linear subspace of $L_\infty(R)$ determined by the characters in $\sigma(\varphi)$.*

PROOF. If φ is not in the L_1-closed subspace spanned by the characters in $\sigma(\varphi)$ then it follows from Corollary V.3.12 that there is a function f in $L_1(R)$ with $\hat{f}(m) = 0$ for every m in $\sigma(\varphi)$ and

$$\int_R f(x)\overline{\varphi(x)}dx = 1.$$

Since $\tilde{f} * \varphi$ is continuous by Lemma 8.1(d) it follows from the above equation that $\tilde{f} * \varphi \neq 0$. From Lemma 12(b) it is seen that $\sigma(\tilde{f} * \varphi) \subseteq \sigma(\varphi)$ and from Lemma 12(c) and the equation $\tau\tilde{f} = \overline{\tau f}$ it follows that $\sigma(\tilde{f} * \varphi)$ contains no interior point of $\sigma(\varphi)$. Hence $\sigma(\tilde{f} * \varphi)$ is a closed subset of the boundary of $\sigma(\varphi)$. Since $\tilde{f} * \varphi \neq 0$ it follows from Lemma 11(a) that $\sigma(\tilde{f} * \varphi)$ is not void. Thus, by hypothesis, $\sigma(f * \varphi)$ contains an isolated point which contradicts Lemma 19. Q.E.D.

The next result shows in a striking manner the relations between the study of spectral synthesis and the original L_1 closure theorem of N. Wiener.

21 Theorem. *Let f and g be in $L_1(R)$ and let $\hat{f}$ vanish at every point in $\hat{R}$ where $\hat{g}$ vanishes. Then, if the boundary of the set of zeros of $\hat{g}$ contains no non-void perfect subset, it follows that f is in the closed linear subspace of L_1 which is spanned by the translates of g.*

Proof. Let $\mathfrak{L}$ be the closed linear subspace of $L_1(R)$ which is spanned by the translates of g, and let $\mathfrak{K}$ be the conjugate-orthogonal complement in $L_\infty(R)$ of $\mathfrak{L}$. Then $\mathfrak{K}$ is closed under translation. Moreover, the set of characters in $\mathfrak{K}$ consists precisely of those characters $[\cdot, m]$ with m in the set σ of zeros of g. Now if f is not in $\mathfrak{L}$ it follows from Corollary II.3.13 that there is a functional x^* vanishing on $\mathfrak{L}$ with $x^*f = 1$. If $\bar{\varphi}$ is the bounded measurable function representing x^* as in Theorem IV.8.5 then, since $x^*\mathfrak{L} = 0$, it follows that φ is in $\mathfrak{K}$ and $\int_R f(x)\overline{\varphi(x)}dx = 1$. Thus, since the convolution $\tilde{f} * g$ is a continuous function (Lemma 8.1(d)), $\tilde{f} * g \neq 0$. Now the inclusion $\sigma(\varphi) \subseteq \sigma$ follows from Definition 10 since φ is in $\mathfrak{K}$. Since $\tau\tilde{f} = \overline{\tau f}$, it follows from Lemma 12 and the hypothesis that $\sigma(\tilde{f} * \varphi)$ is a closed subset of the boundary of σ. Since the boundary of σ contains no non-void perfect subset and since $\tilde{f} * g \neq 0$ it follows from Lemma 11(a) that $\sigma(\tilde{f} * g)$ contains isolated points which contradicts Lemma 19. Q.E.D.

22 Theorem. *If the spectral set of a bounded measurable function φ is finite then φ is a linear combination of the characters in its spectral set.*

PROOF. Let $\sigma(\varphi)$ consist of the characters $[\cdot, m_1], \ldots, [\cdot, m_r]$. According to Theorem 20, φ is in the L_1-closure of the linear manifold in $L_\infty(R)$ spanned by these characters. It follows from Corollary V.3.12 that if

$$\int_R f(x)[x, m_i]dx = 0, \qquad i = 1, \ldots, r,$$

for some function f in $L_1(R)$ then $\int_R f(x)\varphi(x)dx = 0$. Lemma V.3.10 then shows that φ is a linear combination of the characters $[\cdot, m_1], \ldots, [\cdot, m_r]$. Q.E.D.

The next result, which gives a characterization of the spectral set of a bounded measurable function, is of particular interest since it can be generalized to apply to an unbounded function. We shall employ the notation, introduced just before Lemma 14, which associates with each function φ in $L_\infty(R)$ the linear map Φ of $L_2(R)$ into $L_2(\hat{R})$ defined by the equation $\Phi(f) = \tau(\varphi f)$. Since φf is in $L_1(R) \cap L_2(R)$ whenever f is, it follows that Φ maps each function in $L_1(R) \cap L_2(R)$ into a continuous function on $\hat{R}$ which vanishes at p_∞.

23 THEOREM. *Let φ be a bounded measurable function on R. Then a point m_0 in $\hat{R}$ is in the complement of the spectral set of φ if and only if there are neighborhoods V of the identity in $\hat{R}$ and U of m_0 such that the transform $\tau(\varphi f)$ vanishes on U for every f in $L_1(R) \cap L_2(R)$ whose transform vanishes on the complement of V.*

PROOF. If $m_0 \notin \sigma(\varphi)$ then there is a neighborhood V of the identity in $\hat{R}$ and a neighborhood U of m_0 such that $U \cap (\sigma(\varphi) + V + V)$ is void. Let f be a function in $L_1(R) \cap L_2(R)$ whose transform τf vanishes on the complement of V and let $\mathfrak{A}$ be the linear manifold in $L_\infty(R)$ of elements of the form

$$\varphi_V(x) = \sum_{i=1}^{n} c_i[x, m_i]$$

where $m_i \in \sigma(\varphi) + V$. For each such element φ_V let the map Φ_V from $L_2(R)$ to $L_2(\hat{R})$ be defined by the equation $\Phi_V(f) = \tau(\varphi_V f)$ and let $\Phi(f) = \tau(\varphi f)$. Then

$$(\Phi_V f)(m) = \sum_{i=1}^{n} c_i \int_R [x, m-m_i]f(x)dx = \sum_{i=1}^{n} c_i \hat{f}(m-m_i).$$

This shows that $\Phi_V f$ vanishes on U, for if m is in U then $m-m_i$ is in the complement of V. According to Theorem 13, φ is the limit in the L_1-topology of $L_\infty(R)$ of a generalized sequence $\{\varphi_\alpha\}$ in $\mathfrak{A}$. Thus, for m in U,

$$\begin{aligned}(\Phi f)(m) &= \int_R \overline{[x, m]}\varphi(x)f(x)dx \\ &= \lim_\alpha \int_R \overline{[x, m]}\varphi_\alpha(x)f(x)dx = \lim_\alpha\,(\Phi_\alpha f)(m) = 0\end{aligned}$$

which shows that $\tau(\varphi f)$ vanishes on U.

Conversely, let f be a function in $L_1(R) \cap L_2(R)$, whose existence is assured by Lemma 2, with $\hat{f}(0) \neq 0$ and $\hat{f}(m) = 0$ for m in the complement of V, and let $g = f_y$ be a translate of f. Then Corollary 3.17 shows that $\hat{g}(0) \neq 0$ and $\hat{g}(m) = 0$ for m in the complement of V. Thus, by hypothesis, the transform $\Phi(g) = \tau(\varphi g)$ vanishes on U. Hence, for m in U,

$$\begin{aligned}0 &= [y, m](\Phi g)(m) = [y, m]\int_R \overline{[x, m]}f(x-y)\varphi(x)dx \\ &= \int_R \overline{[x-y, m]}f(x-y)\varphi(x)dx.\end{aligned}$$

Now let $h(x) = [x, m_0]f(-x)$ for x in R. Then, since m_0 is in U, the preceding equations show that $h * \varphi = 0$. Now

$$\begin{aligned}\hat{h}(m_0) &= \int_R \overline{[x, m_0]}h(x)dx = \int_R \overline{[x, m_0]}[x, m_0]f(-x)dx \\ &= \int_R f(x)dx = \hat{f}(0) \neq 0,\end{aligned}$$

and so it follows from Lemma 12(a) that m_0 is not in $\sigma(\varphi)$. Q.E.D.

In case R is the additive group $(-\infty, \infty)$ of real numbers a characterization of the spectral set of a bounded measurable function may be stated in terms of analytic function theory. Such a characterization makes it possible to extend the notion of the spectral set of a bounded measurable function to certain unbounded functions. In giving this characterization the letter t will be used for the general element of the character group $\hat{R} = (-\infty, \infty)$ and $\hat{R}$ will be regarded as embedded in the complex z-plane in the usual way.

24 THEOREM. *Let φ be a complex valued essentially bounded*

measurable function of a real variable and let f be the complex valued function of a complex variable defined by the equations

$$(*)\qquad \begin{aligned} f(z) &= \int_0^{\infty} e^{-izx}\varphi(x)dx, && \mathscr{I}(z) < 0, \\ &= -\int_{-\infty}^{0} e^{-izx}\varphi(x)dx, && \mathscr{I}(z) > 0. \end{aligned}$$

Then the spectral set of φ consists of those real numbers t for which there is no analytic extension of f into a neighborhood of t.

Proof. Suppose that f has an analytic extension into a neighborhood of the real number t_0. For $\varepsilon > 0$ let

$$\varphi_\varepsilon(x) = \varphi(x)e^{-\varepsilon|x|}, \qquad x \in R,$$

so that $\varphi_\varepsilon(x) \to \varphi(x)$ boundedly as ε approaches zero. Now

$$\begin{aligned} \hat{\varphi}_\varepsilon(t) &= \int_{-\infty}^{\infty} e^{-ixt}e^{-\varepsilon|x|}\varphi(x)dx \\ &= \int_{-\infty}^{0} e^{-ix(t+i\varepsilon)}\varphi(x)dx + \int_0^{\infty} e^{-ix(t-i\varepsilon)}\varphi(x)dx \\ &= -f(t+i\varepsilon)+f(t-i\varepsilon). \end{aligned}$$

Since f is analytic near t_0 it is uniformly continuous in a compact neighborhood U of t_0 and thus $\hat{\varphi}_\varepsilon(t)$ converges to zero as ε approaches zero and uniformly for t in U. According to Lemma 2 there is a function h in $L_1(R) \cap L_2(R)$ such that $\hat{h}(t_0) \neq 0$ and $\hat{h}(t) = 0$ for t in the complement of U. Since $\varphi_\varepsilon(x) \to \varphi(x)$ boundedly we have

$$(\tilde{h} * \varphi)(y) = \int_{-\infty}^{\infty} \overline{h(x-y)}\varphi(x)dx = \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \overline{h(x-y)}\varphi_\varepsilon(x)dx.$$

It is seen from Corollary 3.17 that $\hat{h}_y(t) = e^{-iyt}\hat{h}(t)$, where $h_y(x) = h(x-y)$. Thus, since $\hat{h}$ vanishes in the complement of U, since $\hat{h}(t)$ vanishes on U, and since φ_ε is in $L_2(R)$, the Plancherel theorem may be applied to give

$$(\tilde{h} * \varphi)(y) = \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} e^{iyt}\overline{(\tau h)(t)}(\tau\varphi_\varepsilon)(t)dt = 0.$$

Since $h * \varphi = 0$ and $(\tau\tilde{h})(t_0) = \overline{(\tau h)(t_0)} \neq 0$ it follows from Lemma 12 that t_0 is not in the spectral set $\sigma(\varphi)$.

Conversely, suppose that $t_0 \notin \sigma(\varphi)$. Let U be a neighborhood of

t_0 whose closure does not intersect $\sigma(\varphi)$. Then, by Theorem 18, φ is in the L_1-closed linear subspace of $L_\infty(-\infty, \infty)$ which is spanned by the characters e^{itx} with t in the complement of U. For each ψ in $L_\infty(-\infty, \infty)$ let f_ψ be defined in terms of ψ by the same equations (*) that define f in terms of φ, and let $\mathfrak{S}(U)$ be the linear manifold of all ψ in $L_\infty(-\infty, \infty)$ for which f_ψ has an analytic extension to an open set N_ψ in the complex plane which contains U. An elementary calculation shows that the function f_ψ corresponding to the function $\psi(x) = e^{itx}$ is given by the equation $f_\psi(z) = i(t-z)^{-1}$. Hence for t in the complement of U this particular ψ is in $\mathfrak{S}(U)$. Since φ is in the L_1-closure of the linear manifold in $L_\infty(-\infty, \infty)$ spanned by the functions e^{itx}, $t \notin U$, and since these functions are in $\mathfrak{S}(U)$, in order to show that φ is in $\mathfrak{S}(U)$ it will suffice to prove that $\mathfrak{S}(U)$ is L_1-closed. The Krein-Šmulian theorem shows that it suffices to demonstrate that the intersection of $\mathfrak{S}(U)$ with every positive multiple of the closed unit sphere in $L_\infty(-\infty, \infty)$ is L_1-closed. Since $L_1(-\infty, \infty)$ is separable it follows from Theorem V.5.1 that it is sufficient to prove that every bounded sequence in $\mathfrak{S}(U)$ which converges in the L_1-topology has its limit also in $\mathfrak{S}(U)$.

To prove this let $\{\psi_n\}$ be a bounded sequence in $\mathfrak{S}(U)$ which converges in the L_1-topology to the function ψ, and let $f_n = f_{\psi_n}$. It must be shown that f_ψ may be extended analytically into a neighborhood containing U. For each fixed z with $\mathscr{I}(z) < 0$ the function h defined by the equations

$$\begin{aligned} h(x) &= 0, && x < 0 \\ &= e^{-izx} && x \geqq 0, \end{aligned}$$

is in $L_1(-\infty, \infty)$ and thus, since ψ_n approaches ψ in the L_1-topology of $L_\infty(-\infty, \infty)$, it may be concluded that $f_n(z) \to f_\psi(z)$ uniformly on each compact subset of the half-plane $\mathscr{I}(z) < 0$. A similar argument shows that $f_n(z) \to f_\psi(z)$ uniformly on each compact subset on the half-plane $\mathscr{I}(z) > 0$. If $\{f_n\}$ were known to be uniformly convergent in a neighborhood of U, the analyticity of its limit f_ψ would be clear. Unfortunately it is not clear that the sequence f_n is uniformly convergent on any region containing an interval of the real axis and so an additional argument is needed.

Let U be the open interval (a, b) and Q the rectangle with

vertices $a \pm i$, $b \pm i$. It is clear that f_n converges uniformly on any portion of Q whose closure contains neither a nor b. Let M be a bound for the sequence ψ_n so that

$$|f_n(a+is)| \leq M \int_0^{\infty} |e^{-iax}| e^{sx} dx = \frac{M}{|s|}, \qquad -1 \leq s < 0.$$

In the same way it is seen that $|f_n(a+is)| \leq M|s|^{-1}$ when $0 < s \leq 1$. Similar estimates may be obtained for the values of $f_n(z)$ for $z = b+is$. Consequently the sequence $\{g_n\}$ defined by the equations

$$\begin{aligned} g_n(z) &= f_n(z)(z-a)^2(z-b)^2, \qquad & z \neq a, b, \\ &= 0 \qquad\qquad\qquad\qquad , & z = a, b, \end{aligned}$$

converges uniformly on Q to the function g given by the equations

$$\begin{aligned} g(z) &= f_\psi(z)(z-a)^2(z-b)^2, \qquad & z \neq a, b, \\ &= 0 \qquad\qquad\qquad\qquad , & z = a, b. \end{aligned}$$

It follows from the maximum modulus principle that g_n converges uniformly in the interior of Q to an analytic function G which, at every point in this interior other than those on the real axis, satisfies the equation

$$G(z) = f_\psi(z)(z-a)^2(z-b)^2.$$

Thus $G(z)(z-a)^2(z-b)^{-2}$ is an analytic extension of f_ψ to the interior of Q. This shows that ψ is in $\mathfrak{S}(U)$ and completes the proof. Q.E.D.

5. Exercises

A. *Exercises on Almost Periodic Functions*

1 Show that if F is in AP, and $\inf_{-\infty<x<+\infty}|F(x)| > 0$, then the function $1/F(\cdot)$ is in AP.

2 If F is in AP, the limit

$$M(F) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} F(x)dy$$

exists. Moreover, $\lim_{T\to\infty}(1/2T)\int_{-T}^{+T} F(x+a)dx = M(F)$ uniformly in a.

3 Let F be in AP, and, for $-\infty < \lambda < +\infty$, let F_λ be defined

by the equation $F_\lambda(x) = e^{\lambda x}F(x)$, and let $g(x) = F(x)\bar{F}(x)$. Let $a(\lambda) = M(f_\lambda)$ where M is defined as in Exercise 2. Show that $a(\lambda) = 0$ except for at most a countable infinity of values λ_i, $i = 1, 2, \ldots$, and that $M(g) = \sum_{i=1}^{\infty} |a(\lambda_i)|^2$.

4 If f is a non-negative function in AP, and $M(f) = 0$ (in the notation of Exercise 2) then $f = 0$.

5 A continuous function f of two real variables $x = (x_1, x_2)$ is called almost periodic if for each $\varepsilon > 0$ there exists a number $L(\varepsilon)$ such that each circle in the plane of radius $L(\varepsilon)$ contains a vector y such that $|f(x)-f(x+y)| < \varepsilon$. Show that every such almost periodic function may be approximated uniformly by linear combinations of functions of the form $\exp i(t_1x_1+t_2x_2)$.

6 A continuous function f on a topological group G is called almost periodic if for each $\varepsilon > 0$ there exists a compact set $K(\varepsilon)$ such that for each g in G there exists an h in $K(\varepsilon)g$ such that $|f(x)-f(xh)| < \varepsilon$. Show that the set of continuous functions x whose translates $x(\cdot g)$ belong to a finite dimensional space are fundamental in the space of almost periodic functions, this space being normed with the norm,

$$|f| = \sup_{x \in G} |f(x)|.$$

If G is Abelian, show that every almost periodic function may be approximated uniformly by linear combinations of continuous functions x satisfying the identities

$$|x(g)| = 1, \qquad x(g_1g_2) = x(g_1)x(g_2).$$

B. *Two Exercises on Haar Measure*

7 Let $\mathring{U}_2$ denote the group of all unitary transformations of determinant 1 in 2-dimensional complex Hilbert space. Show that the matrix of each u in $\mathring{U}_2$ has the form

$$\begin{pmatrix} x_1+iy_1 & x_2+iy_2 \\ x_2-iy_2 & -x_1+iy_1 \end{pmatrix}$$

with $x_1^2+y_1^2+x_2^2+y_2^2 = 1$. Show that the mapping $\varphi : u \to (x_1, y_1, x_2, y_2)$ is a homeomorphism of $\mathring{U}_2$ with the surface S of the unit sphere in 4-dimensional space. Show that for each Borel set $e \subseteq S$, the Haar

measure of the set $\varphi^{-1}(e)$ is K times its hyperarea as a subset of the surface S; and evaluate the absolute constant K.

8 Let U_n denote the group of all unitary transformations of n-dimensional complex Hilbert space E^n. Show that the set F of all u in U_n for which $\det(u+I) = 0$ is of Haar measure zero. Show that the mapping

$$\varphi : u \to i\frac{u-I}{u+I}$$

is a homeomorphism of $U_n - F$ with the set Σ of all Hermitian operators in E^n. Find a one-to-one linear homeomorphism ψ of Σ with n^2-dimensional real Euclidean space E^{n^2}. Then find an explicit formula for the measure v defined by the equation $v(e) = \mu((\psi\varphi)^{-1}e)$, where e denotes an arbitrary Borel subset of E^{n^2}, and μ denotes the Haar measure on the group U_n.

C. *The Wiener Closure Theorem as a Tauberian Theorem*

9 (Wiener Tauberian Theorem). Let f be in $L_\infty(-\infty, +\infty)$ and let φ be a function in $L_1(-\infty, +\infty)$ with a nowhere vanishing Fourier transform. Suppose that for some constant α,

$$\lim_{x\to\infty} (\varphi * f)(x) = \alpha \int_{-\infty}^{+\infty} \varphi(t)dt.$$

Then

$$\lim_{x\to\infty} (\psi * f)(x) = \alpha \int_{-\infty}^{+\infty} \psi(t)dt$$

for every ψ in L_1.

10 Let φ be in $L_1(0, \infty)$ and suppose that

$$\int_0^\infty \varphi(t)t^{ix}dt \neq 0 \qquad -\infty < x < +\infty.$$

Suppose that for some constant α,

$$\lim_{x\to\infty} \frac{1}{x}\int_0^\infty \varphi\left(\frac{t}{x}\right) f(t)dt = \alpha \int_0^\infty \varphi(t)dt.$$

Then we have

$$\lim_{x\to\infty} \frac{1}{x}\int_0^\infty \psi\left(\frac{t}{x}\right) f(t)dt = \alpha\int_0^\infty \psi(t)dt$$

for each ψ in $L_1(0, \infty)$. (Hint: Use Exercise 9.)

11 (Hardy-Littlewood Generalization of Tauber's Theorem; Continuous Case). Let f be in $L_\infty(0, \infty)$ and

$$\lim_{x\to\infty} \frac{1}{x}\int_0^\infty e^{-t/x} f(t)dt = A.$$

Then

$$\lim_{x\to\infty} \frac{1}{x}\int_0^x f(t)dt = A.$$

12 Let f be measurable, bounded on each bounded subset of the positive real axis, and non-negative. Then if the integral $\int_0^\infty e^{-t/x} f(t)dt$ exists for all $x > 0$, and

$$\lim_{x\to\infty} \frac{1}{x}\int_0^\infty e^{-t/x} f(t)dt = A,$$

it follows that

(i) $$\frac{1}{x}\int_0^x f(t)dt \text{ is bounded for } x > 0,$$

(ii) $$\lim_{x\to\infty} \frac{1}{x}\int_0^\infty \frac{t}{x} e^{-t/x} F(t)dt = A,$$

where

$$F(t) = t^{-1}\int_0^t f(s)ds.$$

Moreover,

(iii) $$\lim_{x\to\infty} \frac{\varepsilon}{x^\varepsilon}\int_0^x t^{\varepsilon-1} F(t)dt = A, \qquad \varepsilon > 0,$$

(iv) $$\lim_{x\to\infty} \frac{\varepsilon}{\varepsilon-1}\int_0^1 (1-t^{\varepsilon-1}) f(xt)dt = A, \qquad \varepsilon > 0,$$

(v) $$\lim_{x\to\infty} \int_0^1 f(xt)dt = A.$$

(vi) $$\lim_{x\to\infty} \frac{1}{x}\int_0^x f(t)dt = A.$$

13 Let f be real, measurable, bounded below, and let f be bounded on each bounded subset of the positive real axis. Then, if the integral $\int_0^\infty e^{-t/x} f(t)dt$ exists for each $x > 0$, and

$$\lim_{x\to\infty} \frac{1}{x} \int_0^\infty e^{-t/x} f(t)dt = A,$$

it follows that

$$\lim_{x\to\infty} \frac{1}{x} \int_0^x f(t)dt = A.$$

(Hint: Use Exercise 12. Compare the hypothesis and conclusion with Exercise 11.)

14 Let f be in $L_\infty(0, \infty)$. If

$$\lim_{t\to\infty} \int_{c_i t}^{t} \frac{f(x)}{x}\, dx = A \log \frac{1}{c_i}, \qquad i = 1, 2,$$

for two numbers c_1, c_2 such that $0 < c_1 < c_2 < 1$, and such that $\log c_1$ is not a rational multiple of $\log c_2$, then

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(x)dx = A.$$

If $\log c_1$ is a rational multiple of $\log c_2$, this conclusion is no longer valid.

15 Let $b(x)$ be measurable, non-negative, and bounded on each bounded subset of $(0, \infty)$. Suppose that

$$\int_1^x b(y) \left(\frac{1}{y} - \frac{1}{x}\right) dy = \alpha_1 \log x + \alpha_2 + o(1) \text{ as } x \to \infty.$$

Show that

(i) $$\frac{1}{x} \int_{x/2}^{x} b(y)dy \text{ is bounded for } x \geq 2.$$

(ii) $$B(x) = \frac{1}{x} \int_1^x b(y)dy \text{ is bounded for } x \geq 1,$$

(iii) $$\int_1^x \frac{B(y)}{y}\, dy = \alpha_1 \log x + \alpha_2 + o(1).$$

(iv) $$\lim_{x\to\infty} \frac{\varepsilon}{x^{\varepsilon}} \int_0^x t^{\varepsilon-1} B(t)\,dt = \alpha_1, \qquad \varepsilon > 0.$$

(v) $$\lim_{x\to\infty} \frac{\varepsilon}{\varepsilon-1} \int_0^1 (1-t^{\varepsilon-1}) b(xt)\,dt = \alpha_1, \qquad \varepsilon > 0.$$

(vi) $$\lim_{x\to\infty} \int_0^1 b(xt)\,dt = \alpha_1 .$$

(vii) $$\lim_{x\to\infty} \frac{1}{x} \int_0^x b(t)\,dt = \alpha_1 .$$

(Hint: Use Exercise 14.)

16 Let b be a real measurable bounded function on $(0, \infty)$. Then if

$$\lim_{x\to\infty} \int_1^x b(y) \left(\frac{1}{y} - \frac{1}{x}\right) dy = \alpha,$$

it follows that

$$\lim_{x\to\infty} \int_1^x \frac{b(y)}{y}\, dy = \alpha.$$

(Hint: Use Exercise 15.)

17 (Hardy-Littlewood). Let b be measurable and real on $[0, \infty)$. Suppose that b is bounded on every bounded subset of $[0, \infty)$, and that $xb(x)$ is bounded on the interval $0 \leq x < \infty$. Then if

$$\lim_{x\to\infty} \frac{1}{x} \int_0^x \left\{\int_0^t b(s)\,ds\right\} dt = \alpha,$$

it follows that

$$\lim_{t\to\infty} \int_0^t b(s)\,ds = \alpha.$$

(Hint: Use Exercise 16.)

18 (Hardy-Littlewood). Let b be measurable and real on $[0, \infty)$. Suppose that b is bounded on every bounded subset of $[0, \infty)$, that $xb(x)$ is bounded on the interval $0 \leq x < \infty$, and that the integral

$$\int_0^\infty e^{-t/x} b(t)\,dt$$

exists for all $x > 0$. Then, if

$$\lim_{x\to\infty}\int_0^\infty e^{-t/x} b(t)dt = \alpha,$$

it follows that

$$\lim_{x\to\infty}\int_0^x b(t)dt = \alpha.$$

(Hint: Show that $\int_0^x b(t)dt$ is bounded by the method of Exercise II.4.58, and use Exercises 13 and 17.)

19 (Hardy-Littlewood). Let the sequence a_n be real, bounded below, and let

$$\sum_{n=0}^{\infty} a_n x^n$$

converge for $x < 1$. Then, if

$$\lim_{x\to 1-} (1-x)\sum_{n=0}^{\infty} a_n x^n = A,$$

it follows that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{m=0}^{n-1} a_m = A.$$

(Hint: Let $f(t) = a_n$ for $n \leqq t < n+1$, and use Exercise 18.)

20 (Hardy-Littlewood). Let the sequence a_n be real, and let na_n be bounded. Let

$$\sum_{n=0}^{\infty} a_n x^n$$

converge for $x < 1$. Then, if

$$\lim_{x\to 1-} \sum_{n=0}^{\infty} a_n x^n = A,$$

it follows that

$$\lim_{n\to\infty} \sum_{m=0}^{n} a_m = A.$$

(Hint: Let $f(t) = a_n$ for $n \leqq t < n-1$, and use Exercise 18. This is Hardy and Littlewood's famous extension of Tauber's theorem, Exercise II.4.54. Compare the present hypothesis with the hypothesis of Tauber, and compare the method of proof.)

21 Let f be real, measurable, and bounded on the interval $[0, \infty)$. Suppose that for each $s > 0$, the integral

$$\int_0^\infty \frac{f(t)}{(s+t)^2}\,dt$$

exists. Then, if

$$\int_0^\infty \frac{f(t)}{(s+t)^2}\,dt \sim \frac{A}{s} \qquad \text{as} \qquad s \to 0+,$$

it follows that

$$\int_0^\infty f(t)dt \sim As \qquad \text{as} \qquad s \to 0+,$$

while if

$$\int_0^\infty \frac{f(t)}{(s+t)^2}\,dt \sim \frac{A}{s} \qquad \text{as} \qquad s \to \infty,$$

it follows that

$$\int_0^\infty f(t)dt \sim As \qquad \text{as} \qquad s \to \infty.$$

22 Let f be measurable and real on $[0, \infty)$, and suppose that $xf(x)$ is bounded on the interval $0 \leqq x < \infty$. Suppose that f is bounded on every bounded subset of $0 \leqq x < \infty$, and that

$$\int_0^\infty \frac{f(t)}{s+t}\,dt$$

exists for each positive s. Then, if

$$\int_0^\infty \frac{f(t)}{s+t}\,dt \sim \alpha s^{-1} \qquad \text{as} \qquad s \to \infty,$$

it follows that

$$\lim_{s\to\infty} \int_0^s f(t)dt = \alpha.$$

(Hint: Adapt the method of proof of Exercise 18.)

23 Let f be defined on $[0, \infty)$ and have two continuous derivatives. Suppose that $f''(x) = O(x^{-3})$ as $x \to \infty$, and that $xf(x) \to A$ as $x \to \infty$. Then $x^2f'(x) \to -A$ as $x \to \infty$.

(Hint: Let $p(t) = t^3 f''(t)$ for $t \geqq 1$, and apply Exercise 10 to the bounded function p.)

24 Let f be defined on $[0, \infty)$, and have two continuous derivatives. Let $\alpha > 2$, and let $f''(x) = O(x^{-\alpha})$ as $x \to \infty$, and $x^{\alpha-2}f(x) \to A$ as $x \to \infty$. Then $x^{\alpha-1}f'(x) \to (2-\alpha)A$ as $x \to \infty$.

6. Hilbert-Schmidt Operators

In this section the theory of operators of the Hilbert-Schmidt type will be developed and rather deep and fundamental completeness theorems for the eigenfunctions of such operators and associated unbounded operators will be proved. These results will be based upon a strong inequality due to T. Carleman which will also be derived in this section. Some of these results will be used in later chapters. In Section 8 applications are made in a series of exercises which develop the classical Fredholm theory of integral operators in a general form.

The formal definition of the class of Hilbert-Schmidt operators on a Hilbert space will follow, but for the purposes of introduction it may be stated here that if the Hilbert space is represented as a space $L_2(S, \Sigma, \mu)$ with positive measure μ, then the Hilbert-Schmidt operators are those operators K having a representation in the form

$$(Kf)(s) = \int_S k(s, t)f(t)\mu(dt), \qquad f \in L_2(S, \Sigma, \mu),$$

where

$$\int_S\int_S |k(s, t)|^2 \mu(ds)\mu(dt) < \infty.$$

These are compact operators that have important properties not shared by all compact operators. In some treatments of the Hilbert-Schmidt theory it is assumed that the kernel k is Hermitian symmetric so that the corresponding operator K is self adjoint. No such restriction is made here and the completeness theorems obtained will be applicable to certain classes of nonselfadjoint boundary value problems.

In most of the discussion that follows it will be more convenient to work with an abstract Hilbert space rather than one of its rep-

resentations as an L_2-space and in this setting the class of Hilbert-Schmidt operators may be defined as follows.

1 DEFINITION. Let $\{x_\alpha, \alpha \in A\}$ be a complete orthonormal set in the Hilbert space $\mathfrak{H}$. A bounded linear operator T is said to be a *Hilbert-Schmidt operator* in case the quantity $\|T\|$ defined by the equation

$$\|T\| = \{\sum_{\alpha\in A} |Tx_\alpha|^2\}^{\frac{1}{2}}$$

is finite. The number $\|T\|$ is sometimes called the *Hilbert-Schmidt norm* or the *double-norm* of T. The class of all Hilbert-Schmidt operators on $\mathfrak{H}$ will be denoted by HS.

In this definition of the class HS a particular orthonormal sequence was used. The following lemma shows that the class HS depends only upon the Hilbert space and not upon the basis.

2 LEMMA. *The Hilbert-Schmidt norm is independent of the orthonormal basis used in its definition. If T is in HS and U is a unitary operator in $\mathfrak{H}$, then $U^{-1}TU$ is in HS and $\|T\| = \|U^{-1}TU\|$. In addition, $|T| \leq \|T\|$ and $\|T\| = \|T^*\|$.*

PROOF. Let $\|T\|_A$, $\|T\|_B$ be the double norms of an operator when defined in terms of the complete orthonormal systems $\{x_\alpha, \alpha \in A\}$, $\{y_\beta, \beta \in B\}$ respectively. By using the identity $|x|^2 = \sum_\beta |(x, y_\beta)|^2$, which was proved in Theorem IV.4.13, it is seen that

$$\begin{aligned}\|T\|_A^2 &= \sum_\alpha |Tx_\alpha|^2 = \sum_\alpha \sum_\beta |(Tx_\alpha, y_\beta)|^2 \\ &= \sum_\beta \sum_\alpha |(x_\alpha, T^* y_\beta)|^2 = \sum_\beta |T^* y_\beta|^2 = \|T^*\|_B^2.\end{aligned}$$

If the two orthonormal systems are taken to be the same this identity shows that $\|T^*\|_B = \|T\|_B$ and thus $\|T\|_A = \|T^*\|_B = \|T\|_B$ which proves the first and last statements of the lemma.

If U is a unitary operator, then the set $\{Ux_\alpha, \alpha \in A\}$ is also a complete orthonormal set in $\mathfrak{H}$ and since $|x| = |U^{-1}x|$, we have

$$\|U^{-1}TU\|^2 = \sum_{\alpha\in A} |U^{-1}TUx_\alpha|^2 = \sum_{\alpha\in A} |TUx_\alpha|^2 = \|T\|^2,$$

which shows that $U^{-1}TU$ is a Hilbert-Schmidt operator if T is.

Finally, if $\varepsilon > 0$ let x_0 be an element of unit norm such that $|T|^2 < |Tx_0|^2 + \varepsilon$. Since there is a complete orthonormal set containing the element x_0 we clearly have $|T|^2 \leq \|T\|^2 + \varepsilon$ and hence $|T| \leq \|T\|$. Q.E.D.

3 COROLLARY. *If T is in HS and $\{x_\alpha, \alpha \in A\}$ is any complete orthonormal set in $\mathfrak{H}$, then*

$$\|T\| = \{\sum_{\alpha, \beta \in A} |(Tx_\alpha, x_\beta)|^2\}^{\frac{1}{2}}.$$

PROOF. This follows because $|Tx_\alpha|^2 = \sum_{\beta \in A} |(Tx_\alpha, x_\beta)|^2$, and, since the terms are positive, the double sum exists. Q.E.D.

4 THEOREM. *The set HS of all Hilbert-Schmidt operators is a B-space under the Hilbert-Schmidt norm. In addition, HS is an algebra with $\|TS\| \leq \|T\| \cdot \|S\|$ for every S and T in HS.*

PROOF. It is evident that if T is in HS and a is a scalar, then $\|aT\| = |a|\,\|T\|$. Let T, S be in HS and $\{x_\alpha, \alpha \in A\}$ be a complete orthonormal set in $\mathfrak{H}$. It follows from Corollary 3 and Minkowski's inequality that

$$\begin{aligned} \|T+S\| &= \{\sum_{\alpha,\beta} |((T+S)x_\alpha, x_\beta)|^2\}^{\frac{1}{2}} \\ &\leq \{\sum_{\alpha,\beta} |(Tx_\alpha, x_\beta)|^2\}^{\frac{1}{2}} + \{\sum_{\alpha,\beta} |(Sx_\alpha, x_\beta)|^2\}^{\frac{1}{2}} \\ &= \|T\| + \|S\|, \end{aligned}$$

so that $T+S$ is in HS. To prove that HS is complete, let $\{T_n\}$ be a sequence in HS with $\|T_n - T_m\| \to 0$. It follows from Lemma 2 that $|T_n - T_m| \to 0$ and so there is a bounded linear operator T in $\mathfrak{H}$ with $|T - T_n| \to 0$. To see that T is in HS, let k be an upper bound for the sequence $\{\|T_n\|\}$. If A_1 is any finite subset of A, then

$$\sum_{\alpha \in A_1} |Tx_\alpha|^2 = \lim_{n \to \infty} \sum_{\alpha \in A_1} |T_n x_\alpha|^2 \leq k^2,$$

and hence $\|T\|^2 = \sum_{\alpha \in A} |Tx_\alpha|^2 \leq k^2$, showing that T is in HS. Let $m(\varepsilon)$ be chosen so that $\|T_n - T_m\| < \varepsilon$ for $n, m \geq m(\varepsilon)$. Then, for $m > m(\varepsilon)$,

$$\begin{aligned} \sum_{\alpha \in A_1} |(T - T_m)x_\alpha|^2 &= \lim_{n \to \infty} \sum_{\alpha \in A_1} |(T_n - T_m)x_\alpha|^2 \\ &< \limsup \|T_n - T_m\|^2 < \varepsilon^2, \end{aligned}$$

from which it follows that $\|T - T_m\| \leq \varepsilon$ for $m > m(\varepsilon)$ and completes the proof that HS is a B-space under the Hilbert-Schmidt norm.

Finally, let T be in HS and let B be any bounded linear operator in H. Then

$$\|BT\|^2 = \sum_{\alpha\in A} |BTx_\alpha|^2 \leq |B|^2 \sum_{\alpha\in A} |Tx_\alpha|^2 = |B|^2\|T\|^2,$$

$$\|TB\| = \|(TB)^*\| = \|B^*T^*\| \leq |B|\,\|T\|.$$

In particular if S is in HS, then since $|S| \leq \|S\|$, we have $\|ST\| \leq |S|\,\|T\| \leq \|S\|\,\|T\|$. Q.E.D.

5 Corollary. *The set of Hilbert-Schmidt operators is a two-sided ideal in the B-algebra of all bounded linear operators in Hilbert space. Moreover, if T is in HS and B is a bounded operator,* $\|TB\| \leq \|T\|\,|B|$ *and* $\|BT\| \leq |B|\,\|T\|$.

Proof. It was seen during the proof of the theorem that HS is a subalgebra of $B(\mathfrak{H})$, and the final paragraph of the proof shows that it is a two-sided ideal, and that the above inequalities are valid. Q.E.D.

6 Theorem. *Every Hilbert-Schmidt operator is compact and is the limit in the Hilbert-Schmidt norm of a sequence of operators with finite dimensional range.*

Proof. Let $\{x_\alpha, \alpha \in A\}$ be a complete orthonormal set in $\mathfrak{H}$ and let T be in HS. Since

$$\|T\|^2 = \sum_{\alpha\in A} |Tx_\alpha|^2 < \infty,$$

only a countable number of elements $|Tx_\alpha|^2$ can be different from zero. Also, for every integer n there is a finite subset $A_n \subseteq A$ such that

$$\sum_{\alpha\notin A_n} |Tx_\alpha|^2 < \frac{1}{n^2}.$$

For each n let the linear operator T_n be defined by the equations $T_n x_\alpha = Tx$ if $\alpha \in A_n$ and $T_n x_\alpha = 0$ if $\alpha \notin A_n$. Then the range of T_n is finite dimensional.

Furthermore,

$$\|T-T_n\|^2 = \sum_{\alpha \notin A_n} |Tx_\alpha|^2 < \frac{1}{n^2},$$

and so $|T-T_n| \leq \|T-T_n\| < 1/n$. Hence T is the limit in HS and in the uniform operator topology of the sequence $\{T_n\}$. It follows from Lemma VI.5.3 that T is compact. Q.E.D.

Not every compact operator is in HS, however. For example, if $\{x_n\}$ is an orthonormal set in a separable Hilbert space, let T be the operator determined by the equations $Tx_n = n^{-\frac{1}{2}}x_n$, $n = 1, 2, \ldots$. The operator T is compact (cf. Exercise X.8.5), but it is not in HS.

It has been noted in the preceding discussion that the class of Hilbert-Schmidt operators forms a Banach algebra (without identity) under the norm $\|\cdot\|$. It may readily be shown that in this algebra the inner product defined by

$$((S, T)) = \sum_\alpha (Sx_\alpha, Tx_\alpha),$$

where $\{x_\alpha\}$ is a complete orthonormal system, has the properties required of an inner product on Hilbert space and $((T, T)) = \|T\|^2$. Thus the algebra HS is a Hilbert space in which the involution $S \to S^*$ satisfies the identity

$$((ST, R)) = ((T, S^*R)).$$

Such algebras, which are known as H^*-algebras, have been discussed by W. Ambrose [1] who has shown that conversely, every H^*-algebra is topologically and algebraically isomorphic to an algebra of Hilbert-Schmidt operators on some Hilbert space.

7 Theorem. *If T is a Hilbert-Schmidt operator and f is a single valued analytic function on its spectrum which vanishes at zero, then $f(T)$ is a Hilbert-Schmidt operator and the map $T \to f(T)$ of HS into itself is continuous. Furthermore if $\{f_n\}$ is a sequence of such functions having as common domain a neighborhood N of the spectrum of T and if $f_n(\lambda) \to f(\lambda)$ uniformly for λ in N, then $f_n(T) \to f(T)$ in HS.*

Proof. If the Hilbert space $\mathfrak{H}$ is finite dimensional, the result is trivial and so it will be assumed that $\mathfrak{H}$ is infinite dimensional. It was seen in Theorem 4 that HS is a B-space and an algebra in which $\|TS\| \leq \|T\|\,\|S\|$. A unit may be adjoined by the method

described in Section IX.1 resulting in a B-algebra consisting of all pairs $[\alpha, T]$ with α a scalar and T an operator in HS. The norm in this algebra is $\|[\alpha, T]\| = |\alpha|+\|T\|$. The algebra so obtained by adjoining a unit to HS will be denoted by HS^+.

Since $\mathfrak{H}$ is infinite dimensional it follows from Theorems 6 and IV.8.5 that the identity operator is not in HS and thus HS has no unit.

First we note an element $[\alpha, T]$ in HS^+ has an inverse if and only if $\alpha I+T$ has an inverse in the algebra $B(\mathfrak{H})$ of bounded operators in $\mathfrak{H}$. For if $[\beta, S] = [\alpha, T]^{-1}$, then $[1, 0] = [\alpha, T][\beta, S] = [\alpha\beta, \alpha S+\beta T+TS]$ and hence $\beta = \alpha^{-1}$ and $\alpha S+\beta T+TS = 0$. An easy calculation shows that $\beta I+S = (\alpha I+T)^{-1}$. Conversely, let $B = (\alpha I+T)^{-1}$. Since T is compact and $\mathfrak{H}$ is infinite dimensional, we cannot have $\alpha = 0$ for, if $\alpha = 0$, Lemmas 5 and 6 would imply that the identity $I = BT$ is compact which contradicts Theorem IV.3.5. Let $S = B - \alpha^{-1}I$. Then $\alpha^{-1}BT = \alpha^{-1}B(T+\alpha I-\alpha I) = \alpha^{-1}(I-\alpha B) = -S$. Hence $S = -\alpha^{-1}BT$ and Corollary 5 shows that S is in HS. This proves that if $(\alpha I+T)^{-1}$ exists as a bounded operator, then $[\alpha, T]^{-1}$ exists in HS^+ and is equal to $[\alpha^{-1}, S]$.

Hence the spectrum of an element T in HS, when regarded as an element of the B-algebra HS^+, is the same as the spectrum of T when regarded as an element of the algebra $B(\mathfrak{H})$ of all bounded operators in $\mathfrak{H}$. Since the operation of inversion is continuous in any B-algebra (IX.1.8) the mapping $\lambda \to [\lambda, -T]^{-1}$ is continuous for $\lambda \notin \sigma(T)$. If θ is the mapping of HS^+ into $B(\mathfrak{H})$ which sends $[\alpha, T]$ into $\alpha I+T$, then θ is continuous and $\theta([\lambda, -T]^{-1}) = R(\lambda; T)$.

Since $[\lambda, -T]^{-1}$ is a continuous function for λ in the complement of $\sigma(T)$, it follows that the integral

$$[*] \qquad \frac{1}{2\pi i}\int_C f(\lambda)[\lambda, -T]^{-1}\,d\lambda,$$

where C is a positively oriented rectifiable Jordan curve contained in the domain of f and containing $\sigma(T)$, exists in the norm of HS^+. If the integral in [*] is the element $[\mu, U]$ in HS^+, then from Theorem III.2.19, it is seen that

$$\mu I+U = \theta[\mu, U] = \frac{1}{2\pi i}\int_C f(\lambda)R(\lambda; T)\,d\lambda = f(T).$$

To show that $\mu = 0$, let η be the multiplicative linear functional on HS^+ defined by the equation $\eta\{[\alpha, T]\} = \alpha$. Since η is a homomorphism it follows that $\eta([\lambda, T]^{-1}) = \lambda^{-1}$ and since η is continuous it follows from Theorem III.2.19(c) that

$$\mu = \eta\{[\mu, U]\} = \frac{1}{2\pi i}\int_C f(\lambda)\lambda^{-1}d\lambda = f(0).$$

By hypothesis $f(0) = 0$ and so $f(T) = U$ and is therefore in HS.

If $\lim T_n = T$ in the norm of HS it follows from Lemma VII.6.5 that the contour C of the integral in [*] contains $\sigma(T_n)$ for all sufficiently large n. From Corollary VII.6.3 it is seen that, in the norm of HS^+,

$$\lim_{n\to\infty}[\lambda, -T_n]^{-1} = [\lambda, -T]^{-1}$$

uniformly for λ in C. Thus it follows from Theorem III.2.19 that

$$\begin{aligned}\lim_{n\to\infty} f(T_n) &= \lim_{n\to\infty}\theta\left\{\frac{1}{2\pi i}\int_C f(\lambda)[\lambda, -T_n]^{-1}d\lambda\right\}\\ &= \frac{1}{2\pi i}\int_C f(\lambda)R(\lambda, T)d\lambda = f(T),\end{aligned}$$

the limit being in the norm of HS^+.

In proving the final part of the present theorem, we may clearly suppose that the contour C of (*) lies entirely within the set N. Then, we have

$$\begin{aligned}f(T) &= \theta\left\{\frac{1}{2\pi i}\int_C f(\lambda)[\lambda, -T]^{-1}d\lambda\right\}\\ &= \lim_{n\to\infty}\theta\left\{\frac{1}{2\pi i}\int_C f_n(\lambda)[\lambda, -T]^{-1}d\lambda\right\}\\ &= \lim_{n\to\infty} f_n(T)\end{aligned}$$

in the norm of HS. Q.E.D.

As has been shown in the above discussion, the algebra of Hilbert-Schmidt operators is generated from the algebra of operators with finite dimensional ranges by taking closures relative to a larger norm. It is therefore natural to conjecture that some of the

properties of finite dimensional operators, which in the case of a general linear operator in Hilbert space are irrevocably lost, will be retained by Hilbert-Schmidt operators. To show that this is indeed the case we need to derive a variety of inequalities for operators in finite dimensional Hilbert spaces. Outstanding among these are the well-known "Hadamard determinant inequality," the discovery of which, at the beginning of the present century, opened the gateway to the understanding of integral operators, and the remarkable inequality of Carleman given in Theorem 15.

The next few theorems will deal with finite dimensional Hilbert spaces and give some elementary information about the notion of the trace of an operator.

8 Definition. Let $\{x_1, \ldots, x_n\}$ be a basis for the finite dimensional complex Hilbert space E^n. Let A be an operator in E^n and suppose that $Ax_i = \sum_{j=1}^n a_{ij}x_j$, $i = 1, \ldots, n$. The *trace* of the operator A, denoted by $\operatorname{tr}(A)$, is defined to be

$$\operatorname{tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

It will be shown in the following lemma that the trace of A is independent of the basis $\{x_i\}$ used to define it. It is quite evident that the trace is a linear function of A.

9 Lemma. *The trace of an operator in E^n is independent of the basis used to define it. In addition, if A and B are any two operators in E^n, then* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Proof. We shall prove the second assertion first. Let $Bx_i = \sum_{j=1}^n b_{ij}x_j$. Then

$$ABx_i = \sum_{j=1}^{n} \sum_{k=1}^{n} b_{ij}a_{jk}x_k, \qquad BAx_i = \sum_{j=1}^{n} \sum_{k=1}^{n} a_{ij}b_{jk}x_k.$$

Hence

$$\operatorname{tr}(AB) = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}a_{ji} = \operatorname{tr}(BA).$$

To establish the first assertion, let $\{y_1, \ldots, y_n\}$ be any other basis for E^n. Then the linear operator C defined by $y_i = Cx_i$, $i = 1, \ldots, n$, is a one-to-one map of E^n onto all of itself. We will

calculate the trace of A relative to the basis $y_1, \ldots, y_n$. Note that

$$AC^{-1}y_i = Ax_i = \sum_{j=1}^{n} a_{ij}x_j = C^{-1}\sum_{j=1}^{n} a_{ij}y_j,$$

and so,

$$CAC^{-1}y_i = \sum_{j=1}^{n} a_{ij}y_j.$$

From this it follows that the trace of CAC^{-1}, calculated relative to the basis $\{y_1, \ldots, y_n\}$, is $\sum_{i=1}^{n} a_{ii}$. By what has already been proved the trace of CAC^{-1} is the same as the trace of $A = C^{-1}CA$, both traces being calculated with respect to the basis $\{y_1, \ldots, y_n\}$. This proves the first statement. Q.E.D.

We recall that the *characteristic polynomial* of an operator A in E^n is found by representing A as a matrix with respect to any convenient basis for E^n and calculating the determinant of $\lambda I - A$:

$$\Delta(\lambda) = \det(\lambda I - A).$$

Using elementary properties of determinants it is easy to prove that $\Delta(\lambda)$ is independent of the choice of the basis.

10 LEMMA. (a) *The trace of an operator in E^n is equal to the negative of the coefficient of λ^{n-1} in the characteristic polynomial of the operator.*

(b) *The trace of an operator in E^n is equal to the sum of the numbers in the spectrum of the operator, if each number is counted according to its multiplicity as a root of the characteristic polynomial.*

(c) *The trace of a nilpotent operator in E^n is zero.*

PROOF. If $\{x_1, \ldots, x_n\}$ is a basis for E^n and $Ax_i = \sum_{j=1}^{n} a_{ij}x_j$,

$$\Delta(\lambda) = \det \begin{vmatrix} \lambda - a_{11} & -a_{12} \ldots & -a_{1n} \\ -a_{21} & \lambda - a_{22} \ldots & -a_{2n} \\ \vdots & & \\ -a_{n1} & -a_{n2} \ldots & \lambda - a_{nn} \end{vmatrix}.$$

By considering the expansion of this determinant it is seen to be a polynomial $\Delta(\lambda) = \lambda^n - c_1\lambda^{n-1} + \ldots \pm c_n$ with $c_1 = \operatorname{tr}(A)$.

To prove (b) it is sufficient to recall that $\sigma(A)$ consists of all λ for which $\lambda I - A$ is singular, i.e., all λ for which $\Delta(\lambda) = 0$. Since the

sum of the roots of the equation $\Delta(\lambda) = 0$, counting an m-fold root m times, is c_1, the assertion is proved. Finally, if A is nilpotent, then $\sigma(A) = \{0\}$ and so $\operatorname{tr}(A) = 0$ by (b). Q.E.D.

11 Lemma. *If A is a linear operator in the space E^n, if $\sigma(A) = \{\lambda_1, \ldots, \lambda_k\}$, and if $m_i = \dim E(\lambda_i; A)E^n$, $i = 1, \ldots, k$; then $\operatorname{tr}(A) = \sum_{i=1}^{k} m_i \lambda_i$.*

Proof. If λ_i is in $\sigma(A)$, it is seen in Theorem VII.1.7 that $A - \lambda_i I$ is nilpotent on $E(\lambda_i; A)E^n$. Hence $AE(\lambda_i; A) = \lambda_i E(\lambda_i; A) + N_i$, where N_i is a nilpotent operator in E^n. Since $I = \sum_{i=1}^{k} E(\lambda_i; A)$ it follows that

$$A = \sum_{i=1}^{k} \lambda_i E(\lambda_i; A) + \sum_{i=1}^{k} N_i.$$

Now the trace is a linear function, and if P is a projection operator, then by selecting a basis for E^n which is the set-theoretic union of a basis for PE^n and a basis for $(I - P)E^n$, it is seen immediately that $\operatorname{tr}(P) = \dim PE^n$. It follows from these remarks and Lemma 10(c) that $\operatorname{tr}(A) = \sum_{i=1}^{k} m_i \lambda_i$. Q.E.D.

Remark. The number $\dim E(\mu; A)E^n$ is the multiplicity of μ as a root of the characteristic polynomial of A. This result is contained in the Exercise VII.2.8 and so Lemma 11 is a corollary of that exercise. This notion of the multiplicity of μ is different from the dimension of the manifold $\{x | x \in E^n, Ax = \mu x\}$ of eigenvectors of A corresponding to μ. For self adjoint or Hermitian symmetric matrices A the two notions coincide. If A is an operator in E^n, we shall say that $\lambda_1, \ldots, \lambda_n$ are the *eigenvalues of A repeated according to multiplicities* if each λ_i is an eigenvalue of A and every eigenvalue μ of A occurs in this listing m times, where $m = \dim E(\mu; A)E^n$. The same terminology will be later employed for Hilbert-Schmidt operators.

12 Theorem. *(Hadamard's inequality) If (a_{ij}) is an $n \times n$ matrix of complex numbers, then*

[*] $$|\det(a_{ij})| \leq \prod_{j=1}^{n} \{\sum_{i=1}^{n} |a_{ij}|^2\}^{\frac{1}{2}}.$$

Note that this inequality may be interpreted as stating that the

volume of any n-sided parallelepiped is never larger than the volume of a rectangular parallelepiped with sides of the same lengths.

Proof. The inequality is obvious for $n = 1$ and may be easily checked for $n = 2$. We shall suppose the inequality to be known for $n - 1$, and proceed by induction. If (a_{ij}) is an $n \times n$ matrix, let $u_j = [a_{1j}, a_{2j}, \ldots, a_{nj}]$, $j = 1, \ldots, n$, define a set of n elements of n-dimensional space E^n. If $u_1 = 0$, then both sides of the inequality [$*$] vanish and the result is trivial. If $|u_1| = \{\sum_{j=1}^n |a_{j1}|^2\}^{\frac{1}{2}} \neq 0$, then since both sides of the inequality [$*$] are homogeneous in u_1, it may be assumed that $|u_1| = 1$. Let $v_1, v_2, \ldots, v_n$ be an orthonormal basis for E^n with $v_1 = u_1$ and let W be the unitary operator defined by the equations $Wv_1 = [1, 0, 0, \ldots, 0]$, $Wv_2 = [0, 1, 0, \ldots, 0], \ldots, Wv_n = [0, 0, \ldots, 0, 1]$. Since a unitary operator in E^n has a determinant of absolute value unity, we have

$$\begin{aligned}|\det(a_{ij})| &= |\det(u_1, u_2, \ldots, u_n)| = |\det(W)\det(u_1, u_1, \ldots, u_n)| \\ &= |\det(Wu_1, Wu_2, \ldots, Wu_n)|.\end{aligned}$$

Let the coordinates of Wu_k be $[w_{1k}, \ldots, w_{nk}]$. Then, since $Wu_1 = [1, 0, 0, \ldots, 0]$, it follows that

$$\begin{aligned}\det(Wu_1, Wu_2, \ldots, Wu_n) &= \det \begin{vmatrix} 1 & w_{12} & \ldots & w_{1n} \\ 0 & w_{22} & \ldots & w_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & w_{n2} & \ldots & w_{nn} \end{vmatrix} \\ &= \det \begin{vmatrix} w_{22} & \ldots & w_{2n} \\ \vdots & & \vdots \\ w_{n2} & \ldots & w_{nn} \end{vmatrix}.\end{aligned}$$

Using the induction hypothesis, we conclude that

$$|\det(a_{ij})| \leq \prod_{j=2}^{n} \{\sum_{i=2}^{n} |w_{ij}|^2\}^{\frac{1}{2}}.$$

But since

$$\{\sum_{i=2}^{n} |w_{ij}|^2\}^{\frac{1}{2}} \leq \{\sum_{i=1}^{n} |w_{ij}|^2\}^{\frac{1}{2}} = |Wu_j| = |u_j|, \quad j = 1, \ldots, n,$$

and $|u_1| = 1$, this proves the present lemma. Q.E.D.

Hadamard's inequality will be used in the following way. Let

(a_{ij}) be the matrix of an operator A in E^n relative to the orthonormal basis $\delta_1 = [1, 0, \ldots, 0], \ldots, \delta_n = [0, \ldots, 0, 1]$. Let A_{ij} denote the cofactor of the element a_{ij}, i.e., A_{ij} is $(-1)^{i+j}$ times the determinant of the $(n-1)\times(n-1)$ matrix obtained by deleting the ith row and the jth column in (a_{ij}). Then $\det(A) = \sum_{i=1}^n a_{ij}A_{ij}$ and $\sum_{i=1}^n a_{ij}A_{ik} = 0$ if $j \neq k$. Assuming that A is one-to-one, Cramer's rule for A^{-1} asserts that the matrix of $\det(A)A^{-1}$, relative to the basis $\delta_1, \ldots, \delta_n$, is the transpose of the matrix (A_{ij}). Consequently, if $x = [\xi_1, \ldots, \xi_n]$ and $y = [\zeta_1, \ldots, \zeta_n]$, then

$$\det(A)(A^{-1}x, y) = \sum_{i,j=1}^{n} A_{ij}\xi_i\bar{\zeta}_j.$$

On the other hand, by expanding according to the first row and first column, it is seen that

$$\det(A)(A^{-1}x, y) = -\det\begin{vmatrix} 0 & \bar{\zeta}_1 & \ldots & \bar{\zeta}_n \\ \xi_1 & a_{11} & \ldots & a_{1n} \\ \vdots & \vdots & & \vdots \\ \xi_n & a_{n1} & \ldots & a_{nn} \end{vmatrix}.$$

Hadamard's inequality will be employed to estimate this determinant.

13 Lemma. *Let (a_{ij}) be the matrix of a one-to-one operator A in E^n relative to the basis $\delta_1 = [1, 0, \ldots, 0], \ldots, \delta_n = [0, \ldots, 0, 1]$. Then the Hilbert-Schmidt norm of A is*

$$\|A\| = \{\sum_{i,j=1}^{n} |a_{ij}|^2\}^{\frac{1}{2}},$$

Furthermore, if $x = [\xi_1, \ldots, \xi_n]$ and $y = [\zeta_1, \ldots, \zeta_n]$ are two vectors in E^n, then

$$[\dagger] \qquad \left|\det\begin{vmatrix} 0 & \bar{\zeta}_1 & \ldots & \bar{\zeta}_n \\ \xi_1 & a_{11} & \ldots & a_{1n} \\ \vdots & \vdots & & \vdots \\ \xi_n & a_{n1} & \ldots & a_{nn} \end{vmatrix}\right| \leq \frac{|x|\,|y|\,\|A\|^{n-1}}{(n-1)^{(n-1)/2}}.$$

Proof. The first conclusion follows from Corollary 8. Now if $x = 0$ the inequality $[\dagger]$ is trivial, so by the homogeneity of $[\dagger]$ in x it is sufficient to treat the case $|x| = 1$. As in the proof of Theorem 12, there is a unitary operator W in E^n with $x = W\delta_n$. It follows from the

preceding remarks that the final conclusion of the lemma is equivalent to the statement that

$$|\det(A)(A^{-1}x, y)| \leq |y| \, \|A\|^{n-1}(n-1)^{-(n-1)/2}.$$

Now, since W is unitary, $\det(A) = \det(W^{-1}AW)$ and $(A^{-1}x, y) = (A^{-1}W\delta_n, y) = (W^{-1}A^{-1}W\delta_n, W^{-1}y)$. By setting $B = W^{-1}AW$ we have $B^{-1} = W^{-1}A^{-1}W$ and it follows from Lemma 2 that $\|B\| = \|A\|$. Also, since W is unitary, the vector $z = W^{-1}y$ has the norm $|z| = |y|$. Hence the statement to be established may be written as

$$|\det(B)(B^{-1}\delta_n, z)| \leq |z| \, \|B\|^{n-1}(n-1)^{-(n-1)/2}.$$

By expressing this statement in terms of determinants, it is seen that it suffices to prove the lemma in the particular case where $0 = \xi_1 = \ldots = \xi_{n-1}$ and $1 = \xi_n$. Thus the determinant to be estimated is

$$\det \begin{bmatrix} \bar{\zeta}_1 & \ldots \bar{\zeta}_n \\ a_{11} & \ldots a_{1n} \\ \vdots & \vdots \\ a_{n-1,1} & \ldots a_{n-1,n} \end{bmatrix} = \det \begin{vmatrix} \bar{\zeta}_1 & a_{11} \ldots a_{n-1,1} \\ \vdots & \vdots \quad\quad \vdots \\ \bar{\zeta}_n & a_{1n} \ldots a_{n-1,n} \end{vmatrix},$$

Let D denote the absolute value of this determinant. Then Hadamard's inequality shows that

$$[*] \qquad D \leq |y| \prod_{i=1}^{n-1} \Big\{ \sum_{j=1}^{n} |a_{ij}|^2 \Big\}^{\frac{1}{2}}.$$

Since the geometric mean of a finite collection of positive numbers is at most equal to their arithmetic mean (cf, VI.11.84), if follows that

$$\prod_{i=1}^{n-1} \sum_{j=1}^{n} |a_{ij}|^2 \leq \left[\frac{1}{n-1} \sum_{i=1}^{n-1} \sum_{j=1}^{n} |a_{ij}|^2 \right]^{n-1} \leq \left[\frac{\|A\|^2}{n-1} \right]^{n-1}.$$

By taking the square root of both sides of this inequality and combining it with [*] it is seen that

$$D \leq \frac{|y| \, \|A\|^{n-1}}{(n-1)^{(n-1)/2}},$$

which is the desired result. Q.E.D.

14 Lemma. *Let A be an operator in an n-dimensional Hilbert space E^n, and suppose that* $\operatorname{tr}(A) = 0$. *Then there is an orthonormal basis $\{\varphi_1, \ldots, \varphi_n\}$ for E^n, with $(A\varphi_i, \varphi_i) = 0$, $1 \leq i \leq n$.*

Proof. If $n = 1$ then, since $\operatorname{tr}(A) = 0$, $A = 0$ and the statement is obvious. It will next be shown by induction that there is some non-zero vector φ in E^n with $(A\varphi, \varphi) = 0$. To do this we first consider the case $n = 2$ and suppose that an orthonormal basis has been chosen for which the matrix of A has the subdiagonal form

$$\begin{pmatrix} a & 0 \\ b & -a \end{pmatrix},$$

Let $\varphi = [1, z]$, so that $(A\varphi, \varphi) = a(1-|z|^2)+b\bar{z}$. If $a = 0$ we place $z = 0$. If $a \neq 0$ let $z = re^{i\theta}$ where θ is chosen so that $c = ba^{-1}e^{-i\theta}$ is real and where r is the positive root of the equation $r^2-cr-1 = 0$. In either case it is readily seen that $(A\varphi, \varphi) = 0$.

Next suppose that $n > 2$ and that the statement to be proved is false. Then

$$\min_{|\varphi|=1} |(A\varphi, \varphi)| > 0.$$

Since the unit sphere in E^n is compact, this minimum is attained at some unit vector φ_1. Let $m = (A\varphi_1, \varphi_1)$. After choosing an orthonormal basis $\{\varphi_1, \ldots, \varphi_n\}$ we have, by hypothesis,

$$\operatorname{tr}(A) = m + \sum_{i=2}^{n} (A\varphi_i, \varphi_i) = 0.$$

This equality can be rewritten in the form

$$\sum_{i=2}^{n} \left(E\left(A + \frac{m}{n-1} I\right) \varphi_i, \varphi_i \right) = 0,$$

where E is the self adjoint projection of E^n onto the subspace S spanned by $\varphi_2, \ldots, \varphi_n$. The operator $E(A+m/(n-1)I)$ evidently maps S into itself. Hence it follows from the induction hypothesis that there is a unit vector φ in S with

$$\left(E\left(A + \frac{m}{n-1} I\right) \varphi, \varphi \right) = 0.$$

That is, since $E\varphi = \varphi$,

$$(A\varphi, \varphi) = -\frac{m}{n-1}.$$

Thus

$$|(A\varphi, \varphi)| = \frac{m}{n-1} < |(A\varphi_1, \varphi_1)|$$

contrary to the definition of φ_1. Hence $(A\varphi_1, \varphi_1) = 0$.

The proof of the present lemma can now be completed by induction. Let φ be a vector of norm one with $(A\varphi, \varphi) = 0$. Let S_0 be the orthocomplement of the one dimensional space spanned by φ, and E_0 be the orthogonal projection of E^n onto S_0. It has been observed that the lemma is true in the case that $n = 1$ and we now assume that it is known to be true for $n-1$ dimensional space. Then, by this induction hypothesis, it is seen that there is an orthonormal basis $\{\psi_2, \ldots, \psi_n\}$, whose span is the subspace S_0, and such that $(E_0 A\psi_i, \psi_i) = (A\psi_i, \psi_i) = 0$, $2 \leqq i \leqq n$. Then $\{\varphi, \psi_2, \ldots, \psi_n\}$ is the required basis for E^n. Q.E.D.

15 THEOREM. *Let A be an operator in E^n and $\lambda_1, \ldots, \lambda_n$ its eigenvalues repeated according to multiplicities, and let $\lambda \neq 0$ be a complex number not in the spectrum of A. Then*

$$\left|\prod_{i=1}^{n}\left(1-\frac{\lambda_i}{\lambda}\right) e^{\lambda_i/\lambda}(\lambda I - A)^{-1}\right| \leq |\lambda| \left\{\exp \tfrac{1}{2}\left(1+\frac{\|A\|^2}{|\lambda|^2}\right)\right\}.$$

PROOF. Let $B = A/\lambda$ so that by Theorems VII.3.11 and VII.3.19 we have $\sigma(B) = \{\lambda_1/\lambda, \ldots, \lambda_n/\lambda\}$ and $E(\lambda_i/\lambda; B) = E(\lambda_i; A)$. Also $\mathrm{tr}(B) = \sum_{i=1}^{n} \lambda_i/\lambda = \mathrm{tr}(A)/\lambda$. Let N be any integer with $N > |\mathrm{tr}(B)|$. For each such N define the operator B_N in $E^n \oplus E^N$ by the equation $B_N[x, y] = [Bx, (-1/N)\,\mathrm{tr}(B)y]$. It is evident from Definition 8 that $\mathrm{tr}(B_N) = 0$, and that the eigenvalues of $I - B_N$ are

$$1-\frac{\lambda_1}{\lambda}, \ldots, 1-\frac{\lambda_n}{\lambda}, \qquad 1+\frac{\mathrm{tr}(B)}{N}, \ldots, 1+\frac{\mathrm{tr}(B)}{N}.$$

Hence $\det(I - B_N)$ is numerically equal to the product of these numbers, i.e.,

$$\text{(i)}\quad |\det(I-B_N)| = \left|\left(1+\frac{\operatorname{tr}(B)}{N}\right)^N \prod_{i=1}^{n}\left(1-\frac{\lambda_i}{\lambda}\right)\right|.$$

Since $(1/N)|\operatorname{tr}(B)| < 1$ and $\lambda \neq \lambda_k$, the inverse operator $(I-B_N)^{-1}$ exists and it is readily seen that

$$(I-B_N)^{-1}[x, y] = \left[(I-B)^{-1}x,\ \left(1+\frac{1}{N}\operatorname{tr}(B)\right)^{-1} y\right].$$

Therefore $|(I-B)^{-1}| \leqq |(I-B_N)^{-1}|$ and so

$$\text{(ii)}\quad |\det(I-B_N)||(I-B)^{-1}| \leq |\det(I-B_N)||(I-B_N)^{-1}|.$$

From Lemma 18 it follows that

$$\text{(iii)}\quad |\det(I-B_N)||(I-B_N)^{-1}| \leqq \frac{\|I-B_N\|^{N+n-1}}{(N+n-1)^{(N+n-1)/2}}.$$

Now, by Lemma 14, since $\operatorname{tr}(B_N) = 0$ there is an orthonormal basis $z_1, \ldots, z_{n+N}$ in $E^n \oplus E^N$ such that $(B_N z_k, z_k) = 0$. Relative to the basis $z_1, \ldots, z_{n+N}$ the matrix of the operator $I-B_N$ has 1 along the principal diagonal and the negative of the coefficients in the matrix of B_N elsewhere. Consequently,

$$\text{(iv)}\quad \|I-B_N\|^2 = N+n+\|B_N\|^2 = N+n+N^{-1}|\operatorname{tr}(B)|^2+\|B\|^2.$$

By combining formulas (i) to (iv) it is seen that

$$\left|\left(1+\frac{\operatorname{tr}(B)}{N}\right)^N \prod_{i=1}^{n}\left(1-\frac{\lambda_i}{\lambda}\right)(I-B)^{-1}\right|$$

$$\leqq \frac{(N+n+N^{-1}|\operatorname{tr}(B)|^2+\|B\|^2)^{(N+n-1)/2}}{(N+n-1)^{(N+n-1)/2}} = \frac{\left(1+\dfrac{|\operatorname{tr}(B)|^2}{N(N+n)}+\dfrac{\|B\|^2}{N+n}\right)^{(N+n}}{\left(1-\dfrac{1}{N+n}\right)^{(N+n-1)/2}}$$

This inequality is valid for all sufficiently large N, and therefore, by letting N increase without bound, it is seen that

$$\left|e^{\operatorname{tr}(B)} \prod_{i=1}^{n}\left(1-\frac{\lambda_i}{\lambda}\right)(I-B)^{-1}\right| \leqq \exp(\tfrac{1}{2}(1+\|B\|^2)).$$

By recalling that $B = A/\lambda$ and that $\operatorname{tr}(B) = \sum_{i=1}^{n}\lambda_i/\lambda$, the conclusion

of the present lemma follows immediately. Q.E.D.

Having established these preliminary theorems on finite dimensional spaces, we now return to the study of Hilbert space. It is desired to generalize the notion of trace to certain operators in Hilbert space and at first glance it may appear that this notion is immediately available for Hilbert-Schmidt operators. However, this is not true, as the following example shows. Let $\{x_n\}$ be an orthonormal basis for a Hilbert space $\mathfrak{H}$, and let T be a linear operator defined by the equations $Tx_n = x_n/n$. Then the series

$$\sum_{n=1}^{\infty} |Tx_n|^2 = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

converges, so that T is a Hilbert-Schmidt operator, while the series

$$\sum_{n=1}^{\infty} (Tx_n, x_n) = \sum_{n=1}^{\infty} \frac{1}{n},$$

which we might hope to use to define the trace $\operatorname{tr}(T)$, diverges. Thus, a trace cannot be defined in this way for every operator of the class HS. It will be seen, however, that if $T = UV$, where *both U and V belong to the class HS*, then the series

$$\sum_{n=1}^{\infty} (Tx_n, x_n) = \sum_{n=1}^{\infty} (Vx_n, U^*x_n)$$

converges, and defines a useful notion of "trace." With this slight change in approach as compared to the finite dimensional case, enough of that theory may be carried over to generalize Theorem 15 to arbitrary Hilbert-Schmidt operators.

16 Lemma. *If S and T are Hilbert-Schmidt operators in Hilbert space $\mathfrak{H}$ and if $\{x_\alpha\}$ is a complete orthonormal basis for $\mathfrak{H}$, then the series $\sum_\alpha (Sx_\alpha, T^*x_\alpha)$ converges absolutely to a limit which is independent of the basis.*

Proof. Let $\{x_\alpha\}$ and $\{y_\beta\}$ be any two orthonormal bases for $\mathfrak{H}$. By the Schwarz inequality and Theorem IV.4.13,

$$\begin{aligned}\sum_{\alpha,\beta}\left|(Sx_\alpha, y_\beta)\overline{(T^*x_\alpha, y_\beta)}\right| &\leq \{\sum_{\alpha,\beta}|(Sx_\alpha, y_\beta)|^2\}^{\frac{1}{2}}\{\sum_{\alpha,\beta}|(T^*x_\alpha, y_\beta)|^2\}^{\frac{1}{2}} \\ &= \{\sum_{\alpha}|Sx_\alpha|^2\}^{\frac{1}{2}}\{\sum_{\alpha}|T^*x_\alpha|^2\}^{\frac{1}{2}} = \|S\|\,\|T^*\|.\end{aligned}$$

Thus the double series

$$[\dagger] \qquad \sum_{\alpha, \beta} (Sx_\alpha, y_\beta)\overline{(T^* x_\alpha, y_\beta)}$$

converges absolutely, and hence the corresponding iterated series exist and are equal. By Theorem IV.4.13,

$$[\dagger\dagger] \qquad \begin{aligned} \sum_\alpha (Sx_\alpha, T^* x_\alpha) &= \sum_\alpha \sum_\beta (Sx_\alpha, y_\beta)\overline{(T^* x_\alpha, y_\beta)} \\ &= \sum_\beta \sum_\alpha (Ty_\beta, x_\alpha)\overline{(S^* y_\beta, x_\alpha)} \\ &= \sum_\beta (Ty_\beta, S^* y_\beta). \end{aligned}$$

The existence and equality of the iterated limits corresponding to [†] implies the existence and equality of the single limits just written. Taking $\{y_\beta\}$ to be the same as $\{x_\alpha\}$ it is seen that $\sum_\alpha(Sx_\alpha, T^*x_\alpha) = \sum_\alpha(Tx_\alpha, S^*x_\alpha)$, so that this expression is symmetric in S and T. The independence of the basis used in the calculation now follows by another application of [††]. Q.E.D.

17 Definition. If S and T are Hilbert-Schmidt operators in $\mathfrak{H}$, then the *trace of S and T* is defined to be

$$\operatorname{tr}(S, T) = \sum_\alpha (Sx_\alpha, T^* x_\alpha),$$

where $\{x_\alpha\}$ is any orthonormal basis for $\mathfrak{H}$.

18 Theorem. *The trace function is a symmetric bilinear function defined on the product of HS with itself. In addition, if S and T are in HS, then*

$$|\operatorname{tr}(S, T)| \leqq \|S\|\,\|T\|, \qquad \operatorname{tr}(T, T^*) = \|T\|^2.$$

Proof. The symmetry of the trace function and the inequality $|\operatorname{tr}(S, T)| \leq \|S\|\,\|T\|$ were established during the proof of the preceding lemma. The bilinearity and the fact that $\operatorname{tr}(T, T^*) = \|T\|^2$ follow immediately from Definitions 1 and 17. Q.E.D.

It follows readily from the results stated above that HS forms a Hilbert space under the inner product $[S, T] = \operatorname{tr}(S, T^*)$. Although this fact is interesting, it will not prove to be of much use in the following. However, the trace function itself is a useful tool.

19 Corollary. *The function* $\operatorname{tr}(S, T)$ *is continuous as a mapping from* $HS \oplus HS$ *into the field of scalars.*

20 Lemma. *Let* T *be a linear operator in Hilbert space* $\mathfrak{H}$ *having a finite-dimensional range. Let* $\mathfrak{N}$ *be the nullspace of* T, *and let* E *be the orthogonal projection onto a finite dimensional subspace of* $\mathfrak{H}$ *containing* $\mathfrak{N}^{\perp}$. *Then:*

(a) *the spectra of the operators* T *and* ET *coincide.*

(b) *Let* f *be an analytic function of class* $\mathscr{F}(T)$ (*cf. Chapter* VII.3) *such that* $f(0) = 0$. *Then* $f(ET) = Ef(T)$, $f(T) = f(T)E$, $\operatorname{tr}(f(T), T) = \operatorname{tr}(f(ET), ET)$, *and* $\operatorname{tr}(f(ET), ET)$ *coincides with the trace of the restriction of the operator* $ETf(T)$ *to the finite dimensional space* $E\mathfrak{H}$.

Proof. (a) Since $\mathfrak{H}$ is infinite dimensional the origin belongs to the spectrum of both T and ET. Suppose that $\lambda \neq 0$ belongs to the spectrum of T. Since T is compact, Theorem VII.4.5 shows that λ is an eigenvalue and hence for some non-zero x in $\mathfrak{H}$ we have $Tx = \lambda x$, and hence, since $T = TE$, we have $(ET)(Ex) = \lambda Ex$. Hence λ belongs to the spectrum of ET. Conversely, suppose that a non-zero scalar λ belongs to the spectrum of ET. Then, for some non-zero x in $E\mathfrak{H}$, we have $ETx = \lambda x$. Then $Tx = \lambda x + y$, where y belongs to the subspace $(I-E)\mathfrak{H}$, and hence to the nullspace of T. Let $u = \lambda^{-1}y$. Then $T(x+u) = \lambda(x+u)$, hence λ is an eigenvalue of T.

(b) Suppose that λ is in the resolvent set of T. From the identity

$$\lambda I - T = (\lambda I - ET)\lambda^{-1}(I-E)(\lambda I - T) + (\lambda I - ET)E,$$

which is easily verified by expanding the right hand side into monomial terms and using the identity $T = TE$, one obtains by multiplying by $(\lambda I - ET)^{-1}$ on the left and by $(\lambda I - T)^{-1}$ on the right

$$[*] \qquad (\lambda I - ET)^{-1} = \lambda^{-1}(I-E) + E(\lambda I - T)^{-1}.$$

Suppose that the analytic function f is in the class $\mathscr{F}(T)$, and hence, by (a), in the class $\mathscr{F}(ET)$, and that $f(0) = 0$. Using [*], taking a suitable contour of integration C and using the definition of $f(T)$ given in Section VII.3, it is seen that

$$f(ET) = \frac{1}{2\pi i}\int_C f(\lambda)(\lambda I - ET)^{-1}d\lambda = \frac{1}{2\pi i}\left\{\int_C \frac{f(\lambda)}{\lambda}\,d\lambda\right\}(I-E)$$

$$+\frac{1}{2\pi i}E\int_C f(\lambda)(\lambda I - T)^{-1}d\lambda = f(0)(I-E)+Ef(T) = Ef(T).$$

In much the same way it may be proved that $f(T)E = f(TE)$, which, since $T = TE$, shows that $f(T)E = f(T)$.

Let $\{x_\alpha, \alpha \in A\}$ be an orthonormal basis for $\mathfrak{H}$. Since $E\mathfrak{H}$ is finite dimensional we may suppose without loss of generality that there is a finite subset B of A such that $\{x_\alpha, \alpha \in B\}$ is an orthonormal basis for $E\mathfrak{H}$, and $\{x_\alpha, \alpha \in A-B\}$ is an orthonormal basis for $(I-E)\mathfrak{H}$. Then, since $T = TE$, we have $T^* = ET^*$ and

$$
\begin{aligned}
\mathrm{tr}(f(ET), ET) &= \mathrm{tr}(Ef(T), ET)\\
&= \sum_{\alpha\in A}(Ef(T)x_\alpha, (ET)^*x_\alpha)\\
[\dagger]\qquad &= \sum_{\alpha\in B}(f(T)x_\alpha, T^*x_\alpha)\\
&= \sum_{\alpha\in B}(ETf(T)x_\alpha, x_\alpha)\\
&= \mathrm{tr}(ETf(T)|E\mathfrak{H}).
\end{aligned}
$$

Since $f(T)E = f(T)$, we have $f(T)(I-E) = 0$, $f(T)x_\alpha = 0$ for $x_\alpha \in A-B$, and it follows from the third line of [†] that

$$
\begin{aligned}
\mathrm{tr}(f(T), T) &= \sum_{\alpha\in A}(f(T)x_\alpha, T^*x_\alpha)\\
&= \sum_{\alpha\in B}(f(T)x_\alpha, T^*x_\alpha)\\
&= \mathrm{tr}(f(ET), ET),
\end{aligned}
$$

which completes the proof. Q.E.D.

21 Lemma. *Let T be a linear operator in the finite-dimensional Hilbert space E^n, and let $\lambda_1, \ldots, \lambda_n$ be its eigenvalues repeated according to multiplicities. Then there is an orthonormal basis $\{x_i\}$ for E^n for which*

$$(Tx_i, x_i) = \lambda_i, \qquad i = 1, \ldots, n.$$

Proof. It will be shown that there is an orthonormal basis $\{x_i\}$ for E^n in terms of which the matrix (a_{ij}) of T has "subdiagonal form,"

i.e., $a_{ij} = 0$ for $j > i$. This will be proved by induction on n. If $n = 1$, the result is obvious. Next let $n > 1$, and let $\hat{\lambda}$ be an eigenvalue of T. Then $T-\hat{\lambda}I$ maps E^n into a proper subspace S_0. Let S be an $n-1$ dimensional subspace of E^n such that $S \supseteqq S_0$. Then, since S is necessarily invariant under T, there exists by the inductive hypothesis, an orthonormal basis $\{x_1, \ldots, x_{n-1}\}$ for S with $((T-\hat{\lambda}I)x_i, x_j) = 0$ for $j > i$. Let x_n be orthogonal to S and have norm one so that $\{x_1, \ldots, x_n\}$ is an orthonormal basis for E^n. Then the matrix of $T-\hat{\lambda}I$ in terms of $\{x_1, \ldots, x_n\}$ is $((T-\hat{\lambda}I)x_i, x_j)$ and has $((T-\hat{\lambda}I)x_i, x_j) = 0$ for $j > i$. This completes the construction of the desired orthonormal basis.

Thus the determinant $\det(\lambda I-T)$ is

$$\det(\lambda I-T) = (\lambda-(Tx_1, x_1)) \ldots (\lambda-(Tx_n, x_n)).$$

Hence it follows from the remark preceding Theorem 12 that the sequence (Tx_i, x_i), $i = 1, \ldots, n$, is the sequence of eigenvalues of T repeated according to multiplicities. Q.E.D.

Remark. Using the power series for the exponential function of an operator, it is easy to see that if A is an operator in E^n whose matrix relative to a basis $\{x_1, \ldots, x_n\}$ is subdiagonal and has diagonal elements $\lambda_1, \ldots, \lambda_n$, then the matrix of e^A relative to this same basis is also subdiagonal and has diagonal elements $e^{\lambda_1}, \ldots, e^{\lambda_n}$, and thus has determinant

$$e^{\lambda_1} \ldots e^{\lambda_n} = e^{(\lambda_1+\cdots\lambda_n)} = e^{\operatorname{tr}(A)}.$$

That is, $\det e^A = e^{\operatorname{tr}(A)}$. Since, according to the preceding proof, every matrix can be put into subdiagonal form relative to some basis for E^n, this identity is valid in general. If A is an operator in E^n such that A^{-1} exists, so that $\log A$ can be defined, we can write this identity as

$$\det A = \det e^{\log A} = e^{\operatorname{tr}(\log A)}.$$

This identity will be useful below.

In the next lemma and corollary, we make the essential technical steps needed to carry over the finite dimensional results to the infinite dimensional case.

22 Lemma. *Let λ, z be complex numbers with $\lambda z \neq 1$ and let*

$$f(\lambda, z) = z^{-1}[\log(1-\lambda z)+\lambda z].$$

Let $\{x_\alpha\}$ be an orthonormal basis for Hilbert space, and let T be a Hilbert-Schmidt operator whose spectrum does not include the number λ^{-1}. Then for any finite subset B of $\{x_\alpha\}$ the following inequality holds:

$$
\begin{aligned}
&\exp\left[\operatorname{tr}\big(f(\lambda, T), T\big)\right] \\
&\leq \exp\left\{\tfrac{1}{2}\sum_{\alpha\notin B}|\lambda T x_\alpha|^2\right\}\left[\exp\left\{\tfrac{1}{2}\sum_{\alpha\in B}\mathscr{R}(\lambda T x_\alpha, x_\alpha)\right\}\right]\prod_{\alpha\in B}(1-2\mathscr{R}(\lambda T x_\alpha, x_\alpha) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad +|\lambda T x_\alpha|^2)^{\frac{1}{2}}.
\end{aligned}
$$

Proof. It has already been observed that $Tx_\alpha = 0$ except for at most a countable subset of $\{x_\alpha\}$. Let $\{x_{\alpha_i}\}$ be this subset, and put $x_i = x_{\alpha_i}$. We may and shall assume that $\{x_1, \ldots, x_n\} = B$. Let T_k be the linear operator defined as follows:

$$
\begin{aligned}
T_k x_i &= T x_i, && 1 \leq i \leq k \\
T_k x_i &= 0\ , && i > k \\
T_k x_\alpha &= 0\ , && x_\alpha \notin B,
\end{aligned}
$$

The operator T_k has finite dimensional range, and the sequence $\{T_k\}$ converges to T in Hilbert-Schmidt norm.

If $\lambda = 0$, the present lemma is trivial. Assuming henceforth that $\lambda \neq 0$, we have $f(\lambda, 0) = 0$. Let $\mathfrak{H}_k$ be the subspace of $\mathfrak{H}$ spanned by $x_1, \ldots, x_k$, let E_k be the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{H}_k$, and let $\hat{T}_k = E_k T_k|\mathfrak{H}_k$ be the restriction of $E_k T_k$ to $\mathfrak{H}_k$. By Lemma 20,

$$
\operatorname{tr}\big(f(\lambda, T_k), T_k\big) = \operatorname{tr}\big(\hat{T}_k f(\lambda, \hat{T}_k)\big).
$$

Thus, using Theorem 12 and the remark preceding the statement of the present theorem,

$$
\begin{aligned}
|e^{\operatorname{tr}(f(\lambda, T_k), T_k)}| &= |e^{\operatorname{tr}\,(\log((I-\lambda\hat{T}_k)+\lambda\hat{T}_k))}| \\
&= |e^{\operatorname{tr}\,(\log(I-\lambda\hat{T}_k))}\, e^{\lambda\operatorname{tr}(\hat{T}_k)}| \\
&= |\det(I-\lambda\hat{T}_k)|\, |e^{\lambda\operatorname{tr}(\hat{T}_k)}| \\
&\leq \prod_{i=1}^{k} |(I-\lambda\hat{T}_k)x_i|\, e^{\mathscr{R}(\lambda\operatorname{tr}(\hat{T}_k))} \\
&\leq \prod_{i=1}^{k} |(I-\lambda T_k)x_i|\, e^{\mathscr{R}(\lambda\operatorname{tr}(\hat{T}_k))}
\end{aligned}
$$

$$= \prod_{i=1}^{k} [1-2\mathscr{R}(\lambda T_k x_i, x_i)+|\lambda T_k x_i|^2]^{\frac{1}{2}}\, e^{\mathscr{R}(\lambda\,\mathrm{tr}(\hat{T}_k))}$$
$$= \prod_{i=1}^{k} [1-2\mathscr{R}(\lambda T x_i, x_i)+|\lambda T x_i|^2]^{\frac{1}{2}} \exp\{\sum_{i=1}^{k} \mathscr{R}(\lambda T x_i, x_i)\}.$$

Given an integer n, choose $k > n$. Then, if the inequality $1+\xi \leq e^{\xi}$ is used with $\xi = -2\mathscr{R}(\lambda T x_i, x_i)+|\lambda T x_i|^2$, the last expression in the above sequence of inequalities is not larger than

$$\prod_{i=1}^{n} [1-2\mathscr{R}(\lambda T x_i, x_i)+|\lambda T x_i|^2]^{\frac{1}{2}}\, e^{-\sum_{i=n+1}^{k}\mathscr{R}(\lambda T x_i, x_i)}$$
$$\cdot\, e^{\frac{1}{2}\sum_{i=n+1}^{k}|\lambda T x_i|^2} \cdot e^{\sum_{i=1}^{k}\mathscr{R}(\lambda T x_i, x_i)}$$
$$= \prod_{i=1}^{n} [1-2\mathscr{R}(\lambda T x_i, x_i)+|\lambda T x_i|^2]^{\frac{1}{2}}\, e^{\sum_{i=1}^{n}\mathscr{R}(\lambda T x_i, x_i)}$$
$$\cdot\, e^{\frac{1}{2}\sum_{i=n+1}^{k}|\lambda T x_i|^2}.$$

Thus, we have shown that for each $k > n$,

$$\exp[\mathrm{tr}(f(\lambda, T_k), T_k)] \leq \prod_{i=1}^{n} [1-2\mathscr{R}(\lambda T x_i, x_i)+|\lambda T x_i|^2]^{\frac{1}{2}}$$

[†]

$$\cdot \exp\left(\sum_{i=1}^{n}\mathscr{R}(\lambda T x_i, x_i)\right) \exp\left(\tfrac{1}{2}\sum_{i=n+1}^{k}|\lambda T x_i|^2\right).$$

By Theorem 7, $f(\lambda, T_k)$ converges to $f(\lambda, T)$ in the Hilbert-Schmidt norm. Hence, by the continuity of the trace function (cf. Corollary 19),

$$\mathrm{tr}(f(\lambda, T_k), T_k) \to \mathrm{tr}(f(\lambda, T), T).$$

By letting k tend to infinity in inequality [†], the desired inequality follows. Q.E.D.

23 Corollary. *For any positive ε we have*

$$\lim_{|\lambda|\to\infty} e^{-\varepsilon|\lambda|^2}\,|\exp[\mathrm{tr}(f(\lambda, T), T)]| = 0.$$

Proof. Choose the finite set B so large that

$$\sum_{\alpha\notin B}|Tx_\alpha|^2 < \varepsilon$$

and for simplicity let $M = \sum_{\alpha\in B}|(Tx_\alpha, x_\alpha)|$. It follows from Lemma 22 that

$$e^{-\varepsilon|\lambda|^2}|\exp[\operatorname{tr}(f(\lambda, T), T)]| \leq \prod_{\alpha\in B}[1 - 2\mathscr{R}(\lambda T x_\alpha, x_\alpha) + |\lambda T x_\alpha|^2]^{\frac{1}{2}} e^{(-\varepsilon|\lambda|^2+|\lambda|M)/2}$$

and the limit of the expression on the right is clearly zero, Q.E.D.

We are now in a position to obtain the fruits of our labors in the form of infinite dimensional generalizations of key finite dimensional results. These generalizations are given in the next two theorems, and in Theorem 27 below.

In the course of the next few theorems, it will be helpful to recall that an operator N in Hilbert space is quasi-nilpotent if $\lim_{n\to\infty}|N^n|^{1/n} = 0$. (See VII.5.12 and IX.2.5). By Lemma VII.8.4, this is equivalent to the condition that $\sigma(N) = \{0\}$.

24 Theorem. *Let N be a quasi-nilpotent Hilbert-Schmidt operator. Then* $\operatorname{tr}(N, N) = 0$.

Proof. Since $\sigma(N) = \{0\}$, the function $f(\lambda, N)$ of Lemma 22 is defined for all complex λ, and by Theorem 7 has as its value a Hilbert-Schmidt operator. Since

$$\lim_{\Delta\lambda\to 0}\frac{f(\lambda+\Delta\lambda, z)-f(\lambda, z)}{\Delta\lambda} = \frac{\partial f(\lambda, z)}{\partial\lambda}$$

for each λ, uniformly for z in some neighborhood of $z = 0$, it follows from Theorem 7 that

$$\lim_{\Delta\lambda\to 0}\frac{f(\lambda+\Delta\lambda, N)-f(\lambda, N)}{\Delta\lambda}$$

exists in the norm of HS and equals $\partial f(\lambda, N)/\partial\lambda$. Thus, the function $f(\lambda, N)$ is an analytic function with values in HS. It follows from Corollary 19 that the function g defined by the equation

$$g(\lambda) = \operatorname{tr}(f(\lambda, N), N)$$

is analytic for all λ. It is also clear from the definition of $f(\lambda, N)$ that $g(0) = 0$. The preceding lemma shows that $|\exp(g(\lambda))| = o(e^{\varepsilon|\lambda|^2})$ as $\lambda \to \infty$ for each positive ε. Thus,

$$[*] \qquad \limsup_{|\lambda|\to\infty} \mathscr{R}(g(\lambda)) - \varepsilon|\lambda|^2 \leq 0$$

for each $\varepsilon > 0$.

We now make use of a well-known and elementary complex function-theoretic inequality of Carathéodory (whose statement and proof are given below as Lemma 32) to deduce from this last inequality the fact that

$$\lim_{|\lambda|\to\infty} \frac{|g(\lambda)|}{|\lambda|^2} = 0.$$

Thus, the function $g(\lambda)/\lambda^2$ is analytic and vanishes at $\lambda = \infty$. It follows immediately that g has the Laurent expansion

$$g(\lambda) = a\lambda + b + \frac{c}{\lambda} + \ldots$$

in the neighborhood of $\lambda = \infty$. Consequently, the analytic function $g(\lambda) - a\lambda$ is analytic for all finite and infinite λ and vanishes at $\lambda = 0$. By Liouville's theorem, it follows that $g(\lambda) - a\lambda = 0$, i.e., $g(\lambda) = a\lambda$.

Now, since $f(\lambda, z) = z^{-1}[\log(1-\lambda z)+\lambda z]$, it is clear that

$$f(\lambda, z) = -\sum_{k=2}^{\infty} \frac{\lambda^k z^{k-1}}{k}$$

the series converging uniformly for sufficiently small λ and z. Thus, it follows from Theorem 7 that

$$f(\lambda, N) = -\sum_{k=2}^{\infty} \frac{\lambda^k N^{k-1}}{k},$$

the series converging in the norm of HS for all sufficiently small λ. Thus, by Corollary 19,

$$a\lambda - g(\lambda) = -\sum_{k=2}^{\infty} \frac{\lambda^k}{k} \operatorname{tr}(N^{k-1}, N).$$

It follows immediately that $\operatorname{tr}(N, N) = 0$. Q.E.D.

25 THEOREM. *Let T be a Hilbert-Schmidt operator, and let $\lambda_1, \lambda_2, \ldots$ be an enumeration of its non-zero eigenvalues, each repeated a number of times equal to its multiplicity. If f and g are functions*

analytic in a neighborhood of the spectrum of T and vanishing at the origin, then $f(T)$ and $g(T)$ are Hilbert-Schmidt operators, and

$$\operatorname{tr}\big(f(T), g(T)\big) = \sum_{i=1}^{\infty} f(\lambda_i)g(\lambda_i),$$

where the series on the right hand side is absolutely convergent.

Proof. It will first be shown that

$$[*] \qquad \sum_{i=1}^{\infty} |f(\lambda_i)|^2 < \infty.$$

It follows from Theorem 7 and the spectral mapping theorem (VII.3.19) that it is sufficient to consider the case $f(z) = z$, and to prove that

$$[**] \qquad \sum_{i=1}^{\infty} |\lambda_i|^2 < \infty.$$

Let $E(\lambda_i; T)$ be the projection defined in Section VII.3, and let T_n be the restriction of the operator T to the subspace

$$\mathfrak{H}_n = \sum_{i=1}^{n} E(\lambda_i; T)\mathfrak{H}.$$

Clearly the eigenvalues of T_n in $\mathfrak{H}_n$ are $\lambda_1, \ldots, \lambda_n$. Since furthermore $\|T_n\| \leq \|T\|$, and since from Lemma 21 and Corollary 3 we have

$$\sum_{i=1}^{n} |\lambda_i|^2 \leq \|T_n\|^2,$$

the conclusion [**] is immediate.

The absolute convergence of the series $\sum_{i=1}^{\infty} f(\lambda_i)g(\lambda_i)$ thus follows from [*] and Schwarz' inequality. Let $\mathfrak{H}'$ be the closure of the subspace $\sum_{i=1}^{\infty} E(\lambda_i; T)\mathfrak{H}$, and let $\mathfrak{H}''$ be the orthocomplement of $\mathfrak{H}'$. Let $\{\varphi_n\}$ be an orthonormal basis for $\mathfrak{H}'$ chosen so that $\{\varphi_1, \ldots, \varphi_{n_1}\}$ is a basis for $\mathfrak{H}_1$, $\{\varphi_1, \ldots, \varphi_{n_2}\}$ a basis for $\mathfrak{H}_2$, etc. Let $\{\psi_\alpha\}$ be an orthonormal basis for $\mathfrak{H}''$. Then it clearly follows from Definition 17 that

$$\operatorname{tr}\big(f(T), g(T)\big) = \sum_{i=1}^{\infty} (f(T)\varphi_i, g(T)^*\varphi_i) + \sum_{\alpha} (f(T)\psi_\alpha, g(T)^*\psi_\alpha).$$

We have

$$\begin{aligned}\sum_{i=1}^{\infty}(f(T)\varphi_i, g(T)^*\varphi_i) &= \lim_{j\to\infty}\sum_{i=1}^{n_j}(f(T)\varphi_i, g(T)^*\varphi_i)\\ &= \lim_{j\to\infty}\operatorname{tr}(gf(T)(\mathfrak{H}_j)) = \sum_{i=1}^{\infty} g(\lambda_i)f(\lambda_i)\end{aligned}$$

by Theorem VII.3.20, Theorem VII.3.19, and Lemma 10. The proof of the present theorem will consequently be completed once it is shown that $\sum_\alpha (f(T)\psi_\alpha, g(T)^*\psi_\alpha) = 0$.

Since $(f(T)\psi_\alpha, g(T)^*\psi_\alpha) = (g(T)\psi_\alpha, f(T)^*\psi_\alpha)$, the validity of this equation is evidently a consequence of the validity of the three equations

$$[\dagger] \qquad \begin{aligned}\sum_\alpha (f(T)\psi_\alpha, f(T)^*\psi_\alpha) &= 0,\\ \sum_\alpha (g(T)\psi_\alpha, g(T)^*\psi_\alpha) &= 0,\\ \sum_\alpha ((f+g)(T)\psi_\alpha, (f+g)(T)^*\psi_\alpha) &= 0.\end{aligned}$$

All these equations being of the same form, it is sufficient for us to demonstrate the first of them. By Theorem VII.3.20, $\mathfrak{H}'$ is mapped into itself by $f(T)$. Thus, $\mathfrak{H}''$ is mapped into itself by $f(T)^*$. Let $f(T)^*|\mathfrak{H}'' = S$. Then, by Theorem 7, Lemma 2, and Definition 1, S is a Hilbert-Schmidt operator. We have

$$(Pf(T)x, y) = (f(T)x, y) = (x, f(T)^*y), \qquad x, y \in \mathfrak{H}'',$$

P denoting the orthogonal projection of $\mathfrak{H}$ on $\mathfrak{H}''$. Thus $Pf(T)|\mathfrak{H}'' = S^*$. Hence, [†] is equivalent to the assertion

$$[\dagger\dagger] \qquad \operatorname{tr}(S, S) = 0.$$

It follows from Theorem 24 that to prove [††], it suffices to show that S is quasi-nilpotent. If this is not so, then by Theorem VII.4.5, there exists a non-zero complex number μ and a non-zero vector $x \in \mathfrak{H}''$ such that $Sx = \mu x$. Thus, by Theorem VII.4.5 again, $E(\mu; f(T)^*)\mathfrak{H}'' \neq \{0\}$. From the paragraph following Definition VII.3.17, from Lemma VI.2.10, and from Definition VII.3.9, it is seen that

$$E(\mu; f(T)^*) = E(\bar{\mu}; f(T))^*.$$

Hence, according to Theorem VII.3.20, there is a non-zero complex

number ν such that $E(\nu; T)^*\mathfrak{H}'' \neq 0$. However, since $(\mathfrak{H}'', E(\nu; T)\mathfrak{H}) = 0$ for every non-zero complex number ν by definition, we have a contradiction which proves the present lemma. Q.E.D.

26 THEOREM. *Let T be a Hilbert-Schmidt operator with non-zero eigenvalues $\lambda_1, \lambda_2, \ldots$ repeated according to multiplicities. Then the infinite product*

$$\varphi_\lambda(T) = \prod_{i=1}^{\infty} \left(1 - \frac{\lambda_i}{\lambda}\right) e^{\lambda_i/\lambda}$$

converges and defines a function analytic for $\lambda \neq 0$. For each fixed $\lambda \neq 0$, $\varphi_\lambda(T)$ is a continuous complex valued function on the B-space of all Hilbert-Schmidt operators.

PROOF. First note that if ζ is a complex number with $|\zeta| < 1$ then

$$(*) \qquad \log e^{\zeta}(1-\zeta) = \zeta - (\zeta + \tfrac{1}{2}\zeta^2 + \tfrac{1}{3}\zeta^3 + \ldots) = O(|\zeta|^2),$$

as $\zeta \to 0$. Let $f(\zeta) = \zeta^{-1} \log \{e^{\zeta}(1-\zeta)\}$ and $g(\zeta) = \zeta$, so that f and g are functions analytic except for $\zeta = 1$ and vanish for $\zeta = 0$. If T is a Hilbert-Schmidt operator with eigenvalues $\lambda_1, \lambda_2, \ldots$ all distinct from 1 and if $\lambda \neq 0$, then by VII.3.11, T/λ has eigenvalues $\lambda_1/\lambda, \lambda_2/\lambda, \ldots$ Applying Theorem 25 it is seen that

$$\operatorname{tr}\left(f\left(\frac{T}{\lambda}\right) \cdot \frac{T}{\lambda}\right) = \sum_{k=1}^{\infty} \log \left\{e^{\lambda_k/\lambda}\left(1 - \frac{\lambda_k}{\lambda}\right)\right\},$$

and that the series converges absolutely provided that $\lambda \neq \lambda_k$ for any k.

In view of the fact that $\lambda_k \to 0$ it follows from the estimate in $(*)$ that the series

$$\sum_{k=1}^{\infty} \log \left\{e^{\lambda_k/\lambda}\left(1 - \frac{\lambda_k}{\lambda}\right)\right\}$$

converges uniformly and absolutely for each compact set of numbers λ which contains neither 0 nor any of the elements λ_k. Therefore, taking exponentials, it follows that the product

$$\varphi_\lambda(T) = \prod_{k=1}^{\infty} e^{\lambda_k/\lambda}\left(1 - \frac{\lambda_k}{\lambda}\right)$$

converges uniformly for each such compact set of λ. Since this

product clearly converges to zero for $\lambda = \lambda_k$ it is readily seen that the function $\varphi_\lambda(T)$ is analytic for $\lambda \neq 0$ and vanishes only for λ in $\sigma(T)$.

It remains to show that if $\lambda \neq 0$, then $\varphi_\lambda(T)$ is continuous in T relative to the Hilbert-Schmidt norm in HS. To do this let $\{T_n\}$ be a sequence in HS with $\|T_n - T\| \to 0$. Then if C is a compact set in $\rho(T)$, it follows from the fact that $|T_n - T| \leq \|T_n - T\|$ and Lemma VII.6.8 that $C \subseteq \rho(T_n)$ for sufficiently large n. If f is the function introduced at the beginning of the proof, then for sufficiently large n the operators $f(T_n/\lambda)$ are defined for all λ in C and by Theorem 7 they are in HS. It follows from Theorem 7 that

$$\lim_{n\to\infty} \left\| f\left(\frac{T_n}{\lambda}\right) - f\left(\frac{T}{\lambda}\right)\right\| = 0$$

uniformly for λ in the compact set C. Thus, from Theorem 14, it is seen that

$$\operatorname{tr}\left(f\left(\frac{T}{\lambda}\right), \frac{T}{\lambda}\right) = \lim_{n\to\infty} \operatorname{tr}\left(f\left(\frac{T_n}{\lambda}\right), \frac{T_n}{\lambda}\right),$$

the limit being uniform for λ in C.

Now since $\varphi_\lambda(T) = \exp\{\operatorname{tr}(f(T/\lambda, T/\lambda)\}$ for λ in $\rho(T)$, it follows that

$$\varphi_\lambda(T) = \lim_{n\to\infty} \varphi_\lambda(T_n)$$

uniformly for λ in C. But for each n the function $\varphi_\lambda(T_n)$ is analytic for $\lambda \neq 0$. Hence if C is a contour not enclosing zero, the uniform convergence of $\{\varphi_\lambda(T_n)\}$ on C and the maximum modulus principle imply that the convergence is uniform inside C, even though C may enclose points of $\sigma(T)$. Hence $\varphi_\lambda(T_n) \to \varphi_\lambda(T)$ for all $\lambda \neq 0$, proving that the mapping $T \to \varphi_\lambda(T)$ is continuous on HS. Q.E.D.

The preceding results give a considerable amount of information concerning the distribution of the non-zero eigenvalues of a Hilbert-Schmidt operator. We have seen that there can be only countably many eigenvalues and they must converge to their sole limit point $\lambda = 0$ rapidly enough so that $\Sigma|\lambda_i|^2$ converges. Moreover, there is a function $\varphi_\lambda(T)$ analytic for $\lambda \neq 0$, whose zeros are precisely the eigenvalues of the operator, and this analytic function is of a rather elementary nature. Using the information about the function $\varphi_\lambda(T)$

which has been developed in Theorem 26, we are now prepared to extend Carleman's fundamental inequality, given in Theorem 15 for operators in finite dimensional spaces, to general Hilbert-Schmidt operators.

In stating the next result, we continue to use the notation

$$\varphi_\lambda(T) = \prod_{k=1}^{\infty}\left(1-\frac{\lambda_k}{\lambda}\right)e^{\lambda_k/\lambda},$$

and recall the fact that $\varphi_\lambda(T)$ is analytic for $\lambda \neq 0$ and vanishes precisely on $\sigma(T)$.

27 THEOREM. (Carleman). *If λ is in the resolvent set of the Hilbert-Schmidt operator T, then*

$$|\varphi_\lambda(T)(\lambda I-T)^{-1}| \leq |\lambda| \exp\left\{\tfrac{1}{2}\left(1+\frac{\|T\|^2}{|\lambda|^2}\right)\right\}.$$

PROOF. It follows from Theorems 26, 6, and Lemma VII.6.1, that it is sufficient to consider the case in which T has a finite dimensional range $\mathfrak{R}$. Let $\mathfrak{H}$ be the domain of T and let $\mathfrak{N} = \{x \in \mathfrak{H} | Tx = 0\}$. Then $\mathfrak{N}^\perp$ is mapped by T in a one-to-one fashion into $\mathfrak{R}$. Thus $\mathfrak{N}^\perp$ must be finite dimensional. Let $\mathfrak{S}$ be a one-dimensional subspace of $\mathfrak{N}$, $\mathfrak{M}_1 = \mathfrak{N}^\perp+\mathfrak{R}+\mathfrak{S}$, and $\mathfrak{M}_2 = \mathfrak{M}_1^\perp$. Then $T\mathfrak{M}_2 = 0$, and $T\mathfrak{M}_1 \subseteq \mathfrak{M}_1$. Put $T_1 = T|\mathfrak{M}_1$. Then clearly $\|T_1\| = \|T\|$, $\sigma(T_1) = \sigma(T)$, $\varphi_\lambda(T_1) = \varphi_\lambda(T)$. Moreover, if $m_1 \in \mathfrak{M}_1$ and $m_2 \in \mathfrak{M}_2$,

$$(\lambda I-T)^{-1}(m_1+m_2) = (\lambda I-T_1)^{-1}m_1+\lambda^{-1}m_2.$$

Thus

$$|(\lambda I-T)^{-1}| = \max(|\lambda^{-1}|, |(\lambda I-T_1)^{-1}|).$$

On the other hand, we cannot have

$$|\lambda|^{-1} > |(\lambda I-T_1)^{-1}|,$$

since Lemma VII.6.1 would then imply that T_1 had an inverse which is impossible since the eigenvectors in $\mathfrak{S}$ belong to its domain $\mathfrak{M}_1$. Thus

$$|(\lambda I-T)^{-1}| = |(\lambda I-T_1)^{-1}|.$$

Consequently, the present theorem follows immediately from Theorem 15. Q.E.D.

28 COROLLARY. *Let N be a quasi-nilpotent Hilbert-Schmidt operator. Then for every $\lambda \neq 0$ we have*

$$|R(\lambda; N)| \leq |\lambda| \exp\left\{\tfrac{1}{2}\left(1+\left\|\frac{N}{\lambda}\right\|^2\right)\right\}.$$

The effectiveness of Carleman's inequality in the study of the completeness properties of the eigenfunctions of Hilbert-Schmidt operators will be apparent from the next theorem, where a function-theoretic argument based upon the principle of Phragmén and Lindelöf will be used to establish an important completeness property in a very general case.

In the following theorem and in its corollaries the symbol $\mathrm{sp}(T)$ will denote the closed subspace spanned by all vectors x in Hilbert space $\mathfrak{H}$ which satisfy an equation of the form $(\lambda I-T)^n x = 0$ for some complex λ and for some non-negative integer n, T being a bounded or unbounded operator in Hilbert space $\mathfrak{H}$.

29 THEOREM. *Let $\gamma_1, \ldots, \gamma_5$ be non-overlapping differentiable arcs in the complex plane starting at the origin. Suppose that each of the five regions into which the plane is divided by these arcs is contained in a sector of angular opening less than $\pi/2$. Let $N > 0$ be an integer, and let T be a Hilbert-Schmidt operator in Hilbert space $\mathfrak{H}$ whose resolvent satisfies the inequality*

$$|R(\lambda; T)| = O(|\lambda|^{-N})$$

as $\lambda \to 0$ along any of the arcs γ_i. Then the subspace $\mathrm{sp}(T)$ *contains the subspace $T^N\mathfrak{H}$.*

PROOF. By the Hahn-Banach theorem (II.3.13) it suffices to show that every element y in $\mathfrak{H}$ satisfying the condition $(x, y) = 0$ for x in $\mathrm{sp}(T)$ also has $(T^N x, y) = 0$ for x in $\mathfrak{H}$.

Let y be such an element. By Theorem VII.4.5, the function $y(\lambda) = \lambda^N R(\lambda; T^*)y$ is analytic everywhere in the plane except at $\lambda = 0$ and at an isolated set of points λ_m such that $\lambda_m \to 0$; at the points λ_m the function $y(\lambda)$ may have a pole. For $\lambda \neq \lambda_m$ and λ in the neighborhood of λ_m, we have

$$\begin{aligned} y(\lambda) &= \lambda^N E(\lambda_m; T^*)R(\lambda; T^*)y + \lambda^N R(\lambda; T^*)(I-E(\lambda_m; T^*))y \\ &= \lambda^N E(\bar{\lambda}_m; T)^* R(\bar{\lambda}; T)^* y + y_1(\lambda) \end{aligned}$$

and Theorem VII.3.20 and Lemma VII.3.2 show that the function

$y_1(\lambda)$ is analytic even at $\lambda = \lambda_m$. It will now be shown that $y_2(\lambda) = \lambda^N E(\bar{\lambda}_m; T)^* R(\bar{\lambda}; T)^* y$ vanishes which will prove that $y(\lambda)$ is analytic at all the points $\lambda = \lambda_m$, so that $y(\lambda)$ can only fail to be analytic at the point $\lambda = 0$. To show this, note that

$$\begin{aligned}(y_2(\lambda), x) &= (\lambda^N E(\bar{\lambda}_m; T)^* R(\bar{\lambda}; T)^* y, x) \\ &= \lambda^N (y, E(\bar{\lambda}_m; T) R(\lambda; T) x).\end{aligned}$$

Now it follows from Theorem VII.4.5 that $E(\bar{\lambda}_m; T)R(\lambda; T)x$ is in $\mathrm{sp}(T)$. Since y is in $\mathrm{sp}(T)^\perp$, it follows that $(y_2(\lambda), x) = 0$ for each x in $\mathfrak{H}$ and thus (II.3.14) that $y_2(\lambda) = 0$. Thus $\lambda^N R(\lambda; T^*)y$ is analytic everywhere in the plane except at the origin. Suppose that this function is also known to be analytic at the origin. It follows from the identity

$$T^{*N} R(\lambda; T^*)y = \lambda^N R(\lambda; T^*)y - \lambda^{N-1}y - \lambda^{N-2}T^*y - \ldots - T^{*(N-1)}y$$

that the function $T^{*N}R(\lambda; T^*)y$ is analytic in the whole plane. Since this function is bounded, it follows from Liouville's theorem that $T^{*N}R(\lambda; T^*)y = 0$, and hence, from the power series expansion of the resolvent, that $T^{*N}y = 0$ (cf. VII.3.4). That is, $(y, T^N x) = 0$ for all x in $\mathfrak{H}$, as was to be shown.

The proof thus rests upon the assertion that the function $y(\lambda) = \lambda^N R(\lambda; T^*)y$ is analytic at the origin. To establish this assertion, we proceed as follows. Since $y(\lambda)$ has an isolated singularity at $\lambda = 0$, it must (cf. Section III.14) have a Laurent expansion

$$y(\lambda) = \ldots + a_{-s}\lambda^{-s} + \ldots$$

valid for $\lambda \neq 0$. Thus, for each x in $\mathfrak{H}$, $(y(\lambda), x)$ has the Laurent expansion

$$(y(\lambda), x) = \ldots + (a_{-s}, x)\lambda^{-s} + \ldots.$$

We will show below that this last function is analytic for each x in $\mathfrak{H}$. It will consequently follow that $(a_{-s}, x) = 0$ for each $s > 0$ and each x in $\mathfrak{H}$; so that by the Hahn Banach theorem $a_{-s} = 0$ for each $s > 0$. Consequently, $y(\lambda)$ is analytic at $\lambda = 0$. Thus, to conclude the present proof, we have only to prove that the function $f(\lambda) = (\lambda^N R(\lambda; T^*)y, x)$ is analytic at the origin for every x in $\mathfrak{H}$. Now, the orthocomplement $\mathrm{sp}(T)^\perp$ of the subspace $\mathrm{sp}(T)$ is mapped into itself

by the operator T^*. Let S be the restriction of T^* to $\mathrm{sp}(T)^\perp$. Assume that S is not quasi-nilpotent. Since, by Lemma 2 and Theorem 6, S is compact. it follows from Theorem VII.4.5 that there exists a non-zero complex number μ and a non-zero vector x in $\mathrm{sp}(T)^\perp$ such that $Sx = \mu x$. Thus, by Theorem VII.4.5 again, $E(\mu; T^*)(\mathrm{sp}(T)^\perp) \neq \{0\}$. It follows from the paragraph following Definition VII.3.17, from Lemma VI.2.10, and from Definition VII.3.9, that

$$E(\mu; T^*) = E(\bar{\mu}; T)^*.$$

It is seen from Theorem VII.4.5 that $E(\bar{\mu}; T)\mathfrak{H} \subseteq \mathrm{sp}(T)$ and thus that $(\mathrm{sp}(T)^\perp, E(\bar{\mu}; T)\mathfrak{H}) = 0$. Hence $(E(\mu; T^*)\mathrm{sp}(T)^\perp, \mathfrak{H}) = 0$ which contradicts the fact that $E(\mu; T^*)\mathrm{sp}(T)^\perp \neq \{0\}$. This shows that S is quasi-nilpotent. By Corollary 28, S satisfies the inequality

$$|R(\lambda; S)| = O(e^{\|S\|^2|\lambda|^2}) \qquad \text{as} \qquad \lambda \to 0.$$

The function f thus enjoys the following two properties:

(a) $|f(\lambda)| = |(\lambda^N R(\lambda; T)y, x)| = O(e^{\|S\|^2|\lambda|^2})$ as $\lambda \to 0$.

(b) $f(\lambda)$ is bounded on each of the five arcs γ_i, by the hypothesis of the theorem.

It follows from the principle of Phragmén and Lindelöf that f is analytic at the origin. Q.E.D.

For the convenience of the reader, a sketch of the Phragmén-Lindelöf principle is appended at the end of the present section.

30 Corollary. *Let the arcs $\gamma_1, \ldots, \gamma_5$ be chosen as in the preceding theorem and suppose that as λ tends to zero along any of these arcs the resolvent of the Hilbert-Schmidt operator T satisfies the inequality $|R(\lambda; T)| = O(|\lambda|^{-1})$. Then the subspace* $\mathrm{sp}(T)$ *coincides with the entire Hilbert space $\mathfrak{H}$.*

Proof. Since it follows from the preceding theorem that the subspace $\mathrm{sp}(T)$ contains the closure of the range of T, it suffices to show that the joint span of the range $\mathfrak{R}(T)$ and the nullspace $\mathfrak{N}(T)$ is the entire space $\mathfrak{H}$. Let $\{\lambda_n\}$ be a sequence of complex numbers converging to zero along one of the arcs γ_i and let x be an arbitrary point in $\mathfrak{H}$. Then, by hypothesis, the sequence $\{\lambda_n R(\lambda_n; T)x\}$ is bounded. It can then be assumed without loss of generality (cf.

IV.4.7) that this sequence is weakly convergent to an element y. The proof will be completed by showing (a), that $Ty = 0$; and (b), that $x-y$ belongs to the subspace $\overline{\Re(T)}$. To prove (a), note that for every z in $\mathfrak{H}$ we have

$$
\begin{aligned}
|(Ty, z)| = |(y, T^*z)| &= |\lim_{n\to\infty}(x, \bar{\lambda}_n R(\bar{\lambda}_n; T^*)T^*z)| \\
&= |\lim_{n\to\infty}[(x, \bar{\lambda}_n^2 R(\bar{\lambda}; T^*)z) - \lambda_n(x, z)]| \\
&\leq \lim_{n\to\infty}|\lambda_n|\,|x|\,|z|\,|\bar{\lambda}_n R(\bar{\lambda}_n; T^*)| + \lim_{n\to\infty}|\lambda_n|\,|x|\,|z| \\
&= \lim_{n\to\infty}|\lambda_n| O(1) = 0.
\end{aligned}
$$

To prove (b) note that for every z in the orthocomplement of $\Re(T)$ we have

$$
(x-y, z) = \lim_{n\to\infty}(x - \lambda_n R(\lambda_n; T)x, z) = -\lim_{n\to\infty}(TR(\lambda_n; T)x, z) = 0
$$

and hence $x-y$ belongs to $(\Re(T)^{\perp})^{\perp} = \overline{\Re(T)}$. Q.E.D.

31 Corollary. *Let T be a densely defined unbounded operator in Hilbert space $\mathfrak{H}$, with the property that for some λ_0 in the resolvent set of T, the operator $R(\lambda_0; T)$ is of Hilbert-Schmidt class. Let $\gamma_1, \ldots, \gamma_5$ be non-overlapping differentiable arcs having a limiting direction at infinity, and such that no adjacent pair of arcs form an angle as great as $\pi/2$ at infinity. Suppose that the resolvent $R(\lambda; T)$ satisfies an inequality $|R(\lambda; T)| = O(|\lambda|^{-N})$ as $\lambda \to \infty$ along each arc γ_i. Then the subspace* sp(T) *coincides with the entire Hilbert space $\mathfrak{H}$.*

Proof. Suppose for the sake of definiteness that $\lambda_0 = 0$ and thus that T^{-1} is a Hilbert-Schmidt operator. From the identity

$$
R(\lambda^{-1}; T^{-1}) = \lambda I - \lambda^2 R(\lambda; T)
$$

it follows that the operator T^{-1} satisfies the hypothesis of the preceding theorem. It follows readily from Theorem VII.9.8 that the set of all elements x which satisfy an equation of the form $(\lambda I - T)^n x = 0$ for some complex λ and for some non-negative integer n coincides with the set of all elements which satisfy an equation of the form $(\mu I - T^{-1})^m x = 0$ for some complex μ and some non-negative integer

m. From this and from the preceding theorem we conclude there is a positive integer N for which $\mathrm{sp}(T) \supseteq (T^{-1})^N \mathfrak{H}$. Because T has dense domain, $\overline{T^{-1}\mathfrak{H}} = \mathfrak{H}$, and hence $\overline{(T^{-1})^N \mathfrak{H}} = \mathfrak{H}$. Q.E.D.

We conclude the present section with the proof of the two well-known principles of complex function theory used above.

32 LEMMA. (Carathéodory) *Let f be a function analytic in the circle $|z| < R$ of the complex plane, and let $f(0) = 0$. Then, if $\mathscr{R}f(z) < M$ for $|z| = R$, we have $|f(z)| \leqq 2M$ in $|z| \leqq \frac{1}{2}R$.*

PROOF. Making a change of scale of independent and dependent variables, we may evidently suppose that $M = R = 1$. Let

$$g(z) = \frac{f(z)}{2-f(z)}.$$

Then, since $\zeta \to \zeta(2-\zeta)^{-1}$ maps the half plane $\mathscr{R}\zeta \leqq 1$ into the unit circle and maps $\zeta = 0$ onto itself, we have $|g(z)| < 1$ for $|z| = 1$, $g(0) = 0$. Applying the maximum-modulus principle to $g(z)/z$, we consequently find that

$$\frac{|f(z)|}{2+|f(z)|} \leqq \left|\frac{f(z)}{2-f(z)}\right| \leqq \tfrac{1}{2}$$

for $|z| \leqq \frac{1}{2}$. Thus $1-\frac{1}{2}|f(z)| \geqq 0$ for $|z| \leqq \frac{1}{2}$. Q.E.D.

33 LEMMA. (Phragmén-Lindelöf) *Let g be a function of the complex variable z defined and analytic in the interior of the smaller closed angular sector σ of the unit circle formed by a non-intersecting pair of differentiable Jordan arcs γ_1 and γ_2 running from the origin to the unit circle, and forming an angle of opening less than $\pi/2$ at the origin. Suppose that g is also analytic in a neighborhood of each of the half-open arcs $\gamma_1 - \{0\}$ and $\gamma_2 - \{0\}$, that g is bounded on each of these half-open arcs, and that*

$$|g(z)| = O(e^{|z|^{-2}})$$

as $z \to 0$, z remaining in the interior of σ. Then

$$|g(z)| = O(1)$$

as $z \to 0$, z remaining in the interior of σ.

PROOF. By rotating the complex plane, we may evidently suppose without loss of generality that σ is a subset of the angular sector

$$\sigma_1 = \left\{z \,\middle|\, |z| \leqq 1, |\arg z| < \frac{\pi}{4} - \delta\right\},$$

where δ is some positive number. Let M be a common bound for g on the half-open arcs $\gamma_1 - \{0\}$, $\gamma_2 - \{0\}$, and on the arc

$$\gamma_3 = \{z \in \sigma \,|\, |z| = \tfrac{1}{2}\}.$$

For each $\varepsilon > 0$ consider the function

$$h_\varepsilon(z) = \exp(-\varepsilon z^{-2-\delta})g(z).$$

Since $|\arg(z^{-2-\delta})| < (2+\delta)(\pi/4 - \delta) < \pi/2 - \delta$, for all z in σ_1, we have

[†] $$|\exp(-\varepsilon z^{-2-\delta})| \leqq 1, \qquad z \in \sigma_1$$

and even

[††] $$|\exp(-\varepsilon z^{-2-\delta})| \leqq \exp(-\varepsilon|z|^{-2-\delta} \sin \delta), \qquad z \in \sigma_1.$$

Thus, since $|g(z)| = O(e^{|z|^{-2}})$, it follows from [††] that $|h_\varepsilon(z)|$ converges to zero as z approaches zero, z remaining in σ. Consequently, it follows from [†] that $|h_\varepsilon(z)|$ is bounded by M on the whole boundary of

$$\sigma_2 = \{z \in \sigma \,|\, |z| \leqq \tfrac{1}{2}\}.$$

Using the maximum-modulus principle, we deduce that $|h_\varepsilon(z)|$ is bounded by M throughout σ_2. If $\varepsilon \to 0$, we have $h_\varepsilon(z) \to g(z)$. Thus, letting $\varepsilon \to 0$, we find that $g(z)$ is bounded by M in the interior of σ_2. Q.E.D.

7. The Hilbert Transform and the Calderón-Zygmund Inequality

In the present section, we shall consider certain important singular integral operators, and certain inequalities for those operators introduced in the one-dimensional case by Hilbert and M. Riesz, and in the n-dimensional case by Calderón and Zygmund. These operators and inequalities, interesting in themselves, will also be of particular use in the subsequent chapters.

The convolution integrals

$$(k*f)(x) = \int_{E^n} k(x-y)f(y)dy \tag{1}$$

will be considered as operators in $L_p(E^n)$, and conditions will be given under which it may be asserted that the linear mapping $T_k: f \to k*f$ is a bounded operator in $L_p(E^n)$. If $\int_{E^n} |k(y)|dy < \infty$, then it follows from Lemma 3.1 that the convolution integral (1) exists for almost all x, and defines a bounded mapping of $L_p(E^n)$ into itself, $1 \leqq p \leqq \infty$. For $p = 1$, $p = \infty$, and $p = 2$, the exact norm of this mapping may be determined. For $p = 2$, the n-dimensional analogue of Theorem 3.21(d) (cf. the discussion following 3.22) gives

$$|T_k|_2 = \sup_{y \in E^n} \left| \int_{E^n} e^{i(x,y)} k(x)dx \right|.$$

Using the fact that $L_1^* = L_\infty$, it follows from Theorem IV.8.5 that

$$|T_k|_\infty = \int_{E^n} |k(x)|dx.$$

Since $|T_k^*| = |T_k|$ it is seen that

$$|T_k|_1 = \int_{E^n} |k(x)|dx$$

also.

Since in some cases a meaning may be assigned to the integral

$$\int_{E^n} e^{i(x,y)} k(x)dx \tag{2}$$

even if $\int_{E^n} |k(x)|dx = \infty$ (for instance, by Plancherel's theorems XI.3.9 and XI.3.22 if $\int_{E^n} |k(x)|^2 dx < \infty$), this suggests that we try to define the operator T_k even in cases where $\int_{E^n} |k(x)|dx = \infty$. In such cases we may hope that the integral (1) exists in some suitable "mean" or "proper value" sense, and defines a bounded mapping of $L_2(E^n)$ into $L_2(E^n)$, provided only that the integral (2) exists in some related sense, and is bounded in y.

Consider, for instance, the case $n = 1$, and take $k(x) = 1/x$: in this case, the integral (1) becomes

$$\int_{-\infty}^{+\infty} \frac{f(y)}{x-y} dy, \tag{3}$$

an integral studied by Hilbert. The integral (2) may be interpreted in terms of a Cauchy principal value as

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{e^{ixy}}{x}\, dx &= \lim_{\varepsilon\to 0} \left\{\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right\} \frac{e^{ixy}}{x}\, dx \\ &= \lim_{\varepsilon\to 0} \int_{\varepsilon}^{\infty} \frac{e^{ixy}-e^{-ixy}}{x}\, dx \\ &= \lim_{\varepsilon\to 0} 2i \int_{\varepsilon}^{\infty} \frac{\sin xy}{x}\, dx \\ &= \lim_{\varepsilon\to 0} 2i \int_{\varepsilon y}^{\infty} \frac{\sin x}{x}\, dx \\ &= 2i \operatorname{sgn}(y) \int_{0}^{\infty} \frac{\sin x}{x}\, dx \\ &= \pi i \operatorname{sgn}(y).\end{aligned}$$

This is a bounded function. Thus there are reasons to expect that Hilbert's integral (3) defines a mapping of $L_2(-\infty, +\infty)$ into itself which is of norm exactly π. It will be shown presently that this is indeed the case. Since $\int_{-\infty}^{+\infty} |x|^{-1} dx = \infty$, the integral (3) does not define a bounded mapping in $L_1(-\infty, +\infty)$ (and similarly, it does not define a bounded mapping in $L_\infty(-\infty, +\infty)$). In the spaces $L_p(-\infty, \infty)$ with $1 < p < \infty$, M. Riesz has shown that Hilbert's integral (3) does define a bounded operator. His beautiful proof of this fact will be given later in this section. As is signalized by its failure for the limiting cases $p = 1$ and $p = \infty$, this inequality of M. Riesz is far deeper than the elementary inequality

$$\int_{-\infty}^{+\infty} \left|\int_{-\infty}^{+\infty} k(x-y)f(y)dy\right|^p dx \le \left\{\int_{-\infty}^{+\infty} |k(x)|dx\right\}^p \int_{-\infty}^{+\infty} |f(x)|^p\, dx,$$

valid if $\int_{-\infty}^{+\infty} |k(x)|dx < \infty$, which we have considered as its prototype.

It should be emphasized in connection with the Hilbert integral (3) that our proper value calculation is tenable only because $1/x$ takes on values for x negative exactly equal in magnitude and opposite in sign to the values which it takes on for x positive. This circumstance makes

$$\int_{-1}^{+1} \frac{dx}{x} = 0$$

(in the principal value sense), for instance. If we tried to take $|x|^{-1}$ as the convolution kernel, i.e., if we considered the integral

$$\int_{-\infty}^{+\infty} \frac{f(x)}{|x-y|}\, dx$$

instead of (8), all our considerations would fail.

In the multi-dimensional case the convolution integrals

$$\int_{-\infty}^{+\infty} \frac{\Omega(x-y)}{|x-y|^n} f(y)dy \tag{4}$$

of the form analyzed by Calderón and Zygmund will be considered. Here, Ω is a function giving the "angular" dependence of the quantity

$$\frac{\Omega(x)}{|x|^n}$$

on its independent variable, and is consequently taken to satisfy the condition $\Omega(x) = \Omega(tx)$, $t > 0$. In the particular case of Hilbert's integral (2) for instance, $\Omega(x) = \operatorname{sgn} x$. In the case that Ω is an odd function, i.e., $\Omega(-x) = -\Omega(x)$, it is easy to show, using M. Riesz's inequality, that the integral (4) defines a bounded mapping of $L_p(E^n)$ into itself, $1 < p < \infty$, if Ω has a finite (hypersurface) integral over the hypersurface of the unit sphere in E^n. This will be done below. But, even if $\Omega(-x) \neq -\Omega(x)$, it may still happen (for $n > 1$, but not for $n = 1$) that the hypersurface integral of Ω over the hypersurface of the unit sphere is zero. An example for $n = 2$ is the important integral

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{f(w)}{(z-w)^2}\, dudv$$

where we have taken $z = x+iy$, $w = u+iv$. This integral has the form of the integral (4) where Ω is the even function $\Omega(z) = (|z|/z)^2$ and the hypersurface integral of Ω over the hypersurface of the unit sphere in E^2 is

$$\int_0^{2\pi} e^{2i\theta}\,d\theta = 0.$$

If the hypersurface integral of Ω over the hypersurface of the unit sphere is zero, then, whether Ω is odd or not, (4) defines a bounded mapping of $L_p(E^n)$ into itself for $1 < p < \infty$. This statement, which is valid under mild smoothness restrictions on Ω, constitutes the inequality of Calderón and Zygmund. Most of the present section is devoted to its proof. The real difficulty, as has been indicated, comes for Ω even; this difficulty will be overcome by reducing the case in which Ω is even to the case in which Ω is odd.

Since we shall be considering kernels with a sort of "radial" symmetry, we will have considerable occasion in the course of this section to make computations in "spherical polar coordinates." For this reason, we shall now explain the way in which these coordinates can be established. Let E^n be Euclidean n-space, S the unit sphere in E^n, λ_n the Lebesgue measure in E^n, $E^n_0 = E^n-\{0\}$, and R the positive real axis. Since E^n_0 and E^n differ only by a single point, they may be regarded as being identical from the measure-theoretic point of view. Each point x in E^n_0 may be written uniquely as $x = r\omega$, where $r \in R$, $\omega \in S$, and the mapping $[r, \omega] \to x = r\omega$ is evidently a homeomorphism of $R \times S$ onto E^n_0. Thus the σ-field $\mathscr{B}_n$ of Borel subsets of E^n_0 is the product σ-field of the σ-field $\mathscr{B}_R$ of Borel subsets of R and the σ-field $\mathscr{B}_S$ of Borel subsets of S, in the sense of Definition III.11.3. We have

$$(1) \qquad \int_{E^n} f(tx)dx = |t|^{-n}\int_{E^n} f(x)dx, \qquad t \neq 0, \qquad f \in L_1(E^n).$$

Thus, the measure

$$\alpha(e) = \int_e \frac{dx}{|x|^{n-1}}, \qquad e \in \mathscr{B}_n,$$

has the property that

$$(2) \qquad \alpha(te) = |t|\alpha(e).$$

Define a measure μ on $\mathscr{B}_S$ by putting

$$\mu(e) = \alpha\big((1, 2]\times e\big), \qquad e \in \mathscr{B}_S.$$

The measure μ, for reasons that will appear later, is called *the measure*

of hypersurface on S. Using (2), we have

$$2^{-n}\mu(e) = \alpha\big((2^{-n}, 2^{-n+1}]\times e\big), \qquad e \in \mathscr{B}_S, \qquad n \geqq 0,$$

and adding all these equations we have

$$2\mu(e) = \alpha\big((0, 2]\times e\big), \qquad e \in \mathscr{B}_S.$$

Using (2) again we find

$$t\mu(e) = \alpha\big((0, t]\times e\big), \qquad e \in \mathscr{B}_S, \qquad t > 0,$$

so that also

$$(b-a)\mu(e) = \alpha\big((a, b]\times e\big), \qquad e \in \mathscr{B}_S, \qquad \infty > b > a > 0.$$

It now follows by standard measure-theoretic arguments that

$$\lambda_1(d)\mu(e) = \alpha(d\times e), \qquad d \in \mathscr{B}_R, \qquad e \in \mathscr{B}_S.$$

Thus the measure-space $(E^n_o, \mathscr{B}_n, \alpha)$ is the direct product measure-space $(R, \mathscr{B}_R, \lambda_1)\times(S, \mathscr{B}_S, \mu)$. Using Fubini's theorem III.11.9, Theorem III.11.14, and Theorem III.10.4, we find that for each λ_n-integrable function f, and also for each non-negative λ_n-measurable f,

$$\int_{E^n} f(x)dx = \int_0^\infty \int_S \{f(r\omega)r^{n-1}\}\, \mu(d\omega)dr.$$

Application of this formula, of Fubini's theorem in this connection, etc., will be called "writing the integral $\int_{E^n} f(x)dx$ in spherical polar coordinates." In the remainder of this section, we will use the rules for making such a "change of variables" freely and sometimes implicitly.

Two useful identities satisfied by the measure μ of hypersurface on S are

$$\mu(-e) = \mu(e), \qquad e \in \mathscr{B}_S,$$

$$\mu(e) = \mu(Ve), \qquad e \in \mathscr{B}_S,$$

where V is an arbitrary rotation of E^n. Moreover,

$$\mu(S) = \int_{1\leq|x|\leq 2} \frac{dx}{|x|^{n-1}} < \infty,$$

a fact which we shall use constantly.

We begin our formal development by considering a Lebesgue measurable function f defined on Euclidean n-space E^n, supposing that f has a finite number of "singularities" at which it is not Lebesgue integrable, and defining a certain Cauchy-type principal value integral for f.

1 DEFINITION. Let f be a Lebesgue measurable function defined on Euclidean n-space E^n. Suppose that there exist a finite number of points $p_1, \ldots, p_k$ in E^n such that for each $\varepsilon > 0$ and $R > 0$, f is Lebesgue integrable over the set

$$S(R, \varepsilon; p_1, \ldots, p_k) = \{x \in E^n \mid |x| < R, |x - p_i| \geqq \varepsilon, i = 1, \ldots, k\}.$$

Then, if $\lim_{\varepsilon \to 0, R\to\infty} \int_{S(R, \varepsilon; p_1, \ldots, p_k)} f(x)dx = \alpha$ exists, we will say that *f is integrable in the principal value sense*, and write

$$\mathscr{P}\int_{E^n} f(x)dx = \alpha.$$

The points $p_1, \ldots, p_k$ will be called *singularities of f*, it being assumed that f is not integrable in a neighborhood of any one of them.

2 LEMMA. *Let f and g be measurable functions defined on E^n, and suppose that f and g are integrable in the principal value sense. Then*

(i) *If α and β are scalars, $\alpha f + \beta g$ is integrable in the principal value sense, and*

$$\mathscr{P}\int_{E^n} \{\alpha f(x) + \beta g(x)\}dx = \alpha\mathscr{P}\int_{E^n} f(x)dx + \beta\mathscr{P}\int_{E^n} g(x)dx.$$

(ii) *For each real α, the function h defined by $h(x) = f(\alpha x)$ is integrable in the principal value sense, and*

$$\mathscr{P}\int_{E^n} h(x)dx = |\alpha|^{-n}\mathscr{P}\int_{E^n} f(x)dx.$$

(iii) *If U is a homogeneous linear isometric mapping of E^n into itself, the function h defined by $h(x) = f(Ux)$ is integrable in the principal value sense, and*

$$\mathscr{P}\int_{E^n} h(x)dx = \mathscr{P}\int_{E^n} f(x)dx.$$

(iv) *If $|f(x)| = o(|x|^{1-n})$ as $|x| \to \infty$, then, for each x_0 in E^n, the function h defined by $h(x) = f(x+x_0)$ is integrable in the principal value sense, and*

$$\mathscr{P}\int_{E^n} h(x)dx = \mathscr{P}\int_{E^n} f(x)dx.$$

PROOF. Statement (i) is evident from Definition 1. Statements (ii) and (iii) are evident consequences of Definition 1 and of the formulae

$$\int_{E^n} \varphi(x)dx = \int_{E^n} \varphi(Ux)dx,$$
$$\int_{E^n} \varphi(\alpha x)dx = |\alpha|^{-n} \int_{E^n} \varphi(x)dx,$$

which are valid for every Lebesgue integrable function φ. Statement (iv) will follow in the same way as soon as it is shown that $|f(x)| = o(|x|^{1-n})$ implies that

$$\lim_{R\to\infty} \int_{S(0,R)\Delta S(x_0,R)} f(x)dx = 0,$$

where $S(y, R) = \{x \in E^n \mid |x-y| < R\}$, and where $b\Delta e = (b \cup e)(b \cap e)'$ denotes the symmetric difference of the sets b and e. Since $S(0, R)\Delta S(x_0, R) \subseteq (S(0, R+|x_0|))(S(0, R-|x_0|))'$ and $|f(x)| = o(|x|^{1-n})$ as $|x| \to \infty$, it is clearly sufficient to prove that

$$\alpha_R = \lambda_n[S(0, R+|x_0|)(S(0, R-|x_0|))'] = O(R^{n-1})$$

as $R \to \infty$. But, transforming to spherical polar coordinates and letting ω_n denote the hypersurface measure of the unit sphere in E^n, we have

$$\begin{aligned}\alpha_R = \omega_n \int_{R-|x_0|}^{R+|x_0|} r^{n-1}\,dr &\le 2\omega_n|x_0|(R+|x_0|)^{n-1} \\ &= O(R^{n-1}).\end{aligned}$$ Q.E.D.

In the next lemma, we introduce for the first time the specific type of singular integrals with which we will be principally concerned.

3 LEMMA. *Let g be a measurable function defined on E^n. Suppose that the points $p_1, \ldots, p_k$ in E^n are such that for some $\varepsilon > 0$, g is Lebesgue integrable over the set $\{x \mid |x-p_i| > \varepsilon, i = 1, \ldots, k\}$. Suppose that in the ε-neighborhood of each of the points $p_1, \ldots, p_k$, g is of the form*

$$\frac{\Omega_i(x-p_i)}{|x-p_i|^n} f_i(x),$$

where $\Omega_t(x) = \Omega_i(tx)$, $t > 0$, *where* Ω_i *is integrable over the hypersurface of the unit sphere in* E^n *and has hypersurface integral zero, and where* f_i *is continuously differentiable. Then the principal value integral*

$$\mathscr{P}\int_{E^n} g(x)dx$$

exists.

Proof. If $\mu(e)$ denotes the hypersurface measure of the Borel subset e of the unit sphere S in E^n then for $i = 1, \ldots, k$,

$$\int_{\delta>|x-p_i|>\delta_1} \frac{\Omega_i(x-p_i)}{|x-p_i|^n} f_i(p_i)dx - f_i(p_i)\int_{\delta_1}^{\delta}\frac{dr}{r}\int_S \Omega_i(\omega)\mu(d\omega) = 0.$$

It follows that

$$\begin{aligned}\left|\int_{\delta>|x-p_i|>\delta_1} g(x)dx\right| &\leq \int_{\delta_1}^{\delta}\frac{dr}{r}\int_S |\Omega_i(\omega)|\, |f(x-p_i)-f(p_i)|\mu(d\omega)\\ &\leq K\int_{\delta_1}^{\delta}\frac{r\,dr}{r}\int_S |\Omega_i(\omega)|\mu(d\omega)\\ &= K(\delta-\delta_1)\int_S |\Omega_i(\omega)|\mu(d\omega),\end{aligned}$$

for some constant K and sufficiently small $\delta > \delta_1 > 0$. Thus

$$\lim_{\substack{\delta\to 0\\ \delta_1\to 0}}\int_{\delta>|x-p_i|>\delta_1} g(x)dx = 0, \qquad i = 1, \ldots, k.$$

The existence of the limit

$$\lim_{\delta\to 0}\int_{|x-p_i|>\delta} g(x)dx, \qquad i = 1, \ldots, k,$$

now follows from Cauchy's convergence criterion. Q.E.D.

4 Definition. Let $\Omega(x)$ be a function of x defined for $x \neq 0$ in E^n such that

(i) $\Omega(x) = \Omega\left(\dfrac{x}{|x|}\right), \qquad x \neq 0, \qquad x \in E^n.$

(ii) The hypersurface integral of Ω over the set S vanishes. Then the expression

$$K_\Omega(x, y) = \frac{\Omega(x-y)}{|x-y|^n}, \qquad x \neq y, \qquad x, y \in E^n$$

is said to be a *convolution kernel of Calderón-Zygmund type.*

5 LEMMA. *Let Ω_1 and Ω_2 be two infinitely differentiable functions having the properties* (i) *and* (ii) *of the preceding definition. Suppose that Ω_1 is odd, and that Ω_2 is even. Then there exists an odd function Ω_3 having the same properties* (i) *and* (ii), *and such that the integral*

$$\mathscr{P}\int_{E^n} \frac{\Omega_1(x-u)}{|x-u|^n} \frac{\Omega_2(u-y)}{|u-y|^n}\, du, \tag{1}$$

which exists for $x \neq y$ by Lemma 3, can be written in the form

$$\frac{\Omega_3(x-y)}{|x-y|^n}.$$

PROOF. Let the function defined for $x \neq y$, $x, y \in E^n$ by the principal value integral (1) be denoted by $K(x, y)$. Then, from Lemma 2, $K(x, y) = K(x-y, 0) = K_1(x-y)$, where we have written K_1 for the function $K_1(x) = K(x, 0)$. Since $\Omega_i(\alpha x) = \Omega_i(x)$, $i = 1, 2$, $\alpha > 0$, it follows from Lemma 2 that, for $\alpha > 0$, $K_1(\alpha x) = \alpha^{-n} K_1(x)$, so that the function $\Omega_3(x) = |x|^n K_1(x)$ satisfies condition (i) of the preceding definition, and we have

$$K(x, y) = \frac{\Omega_3(x-y)}{|x-y|^n}.$$

That Ω_3 is odd follows immediately by Lemma 2, since

$$\begin{aligned} K_1(-x) &= \mathscr{P}\int_{E^n} \frac{\Omega_1(-x-u)}{|x+u|^n} \frac{\Omega_2(u)}{|u|^n}\, du \\ &= -\mathscr{P}\int_{E^n} \frac{\Omega_1(x+u)}{|x+u|^n} \frac{\Omega_2(u)}{|u|^n}\, du \\ &= -\mathscr{P}\int_{E^n} \frac{\Omega_1(x-v)}{|x-v|^n} \frac{\Omega_2(v)}{|v|^n}\, dv \\ &= -K_1(x). \end{aligned}$$

Since $\Omega_3(x) = -\Omega_3(-x)$, it is clear that the hypersurface integral of

Ω_3 over the hypersurface of the unit sphere is zero, and thus the present lemma is fully proved. Q.E.D.

6 DEFINITION. If Ω_1, Ω_2 and Ω_3 are as in Lemma 5, then the convolution kernel K_{Ω_3} is said to be *the convolution-product* of the kernels K_{Ω_1} and K_{Ω_2}. We write $K_{\Omega_3} = K_{\Omega_1} * K_{\Omega_2}$. In the same way the symbol $K_{\Omega_1} * f$ will be used for the function g defined by the principal value integral

$$g(x) = \mathscr{P}\int_{E^n} \frac{\Omega_1(x-y)}{|x-y|^n} f(y)dy,$$

provided that this integral exists.

As will be seen presently, the preceding lemma need not be valid without the stated hypothesis on the parity of Ω_1 and Ω_2.

7 LEMMA. *Let K_Ω be a convolution kernel of Calderón-Zygmund type in n-dimensional Euclidean space E^n, and suppose that Ω is infinitely often differentiable for $x \neq 0$. Let f be a C^∞ function vanishing outside a bounded subset of E^n. Then, if $F(g)$ denotes the L_2-Fourier transform of a function g in $L_2(E^n)$, i.e.,*

$$F(g)(x) = \underset{R\to\infty}{\text{l.i.m.}}\, (2\pi)^{-n/2}\int_{|y|\leq R} g(y)e^{i(x,y)}\,dy,$$

we have $K_\Omega * f$ in $L_p(E^n)$ for $p > 1$, and

$$F(K_\Omega * f)(x) = F(f)(x)\cdot \mathscr{P}\int_{E^n} \frac{\Omega(y)}{|y|^n} e^{i(x,y)}\,dy.$$

In particular, the principal value integral $\mathscr{P}\int_{E^n}\Omega(y)|y|^{-n}e^{i(x,y)}\,dx$ exists.

PROOF. Suppose that $f(y)$ vanishes for $|y| \geqq R_0$. Then, if $|x| \geqq R_0+1$,

$$\begin{aligned}|(K_\Omega * f)(x)| &= \left|\mathscr{P}\int_{E^n} \frac{\Omega(y)}{|y|^n} f(x-y)dy\right| \\ &\leq M\frac{1}{(|x|-R_0)^n}\int_{E^n} |f(y)|dy,\end{aligned}$$

M denoting an upper bound for $|\Omega(y)|$. Thus $(K_\Omega * f)(x) = O(|x|^{-n})$ as $|x| \to \infty$. Since

$$(K_\Omega * f)(x) = \lim_{\varepsilon\to 0}\int_{|y|\geq\varepsilon} \frac{\Omega(y)}{|y|^n} f(x-y)dy,$$

it follows that $K_\Omega * f$ is measurable.

Let M_1 be an upper bound for $|f(y)|$ and for

$$\left\{\left|\frac{\partial}{\partial y_1} f(y)\right|^2 + \left|\frac{\partial}{\partial y_2} f(y)\right|^2 + \cdots + \left|\frac{\partial}{\partial y_n} f(y)\right|^2\right\}^{\frac{1}{2}}.$$

Let χ be the characteristic function of the sphere of radius $2(R_0+1)$. Now according to Definition 4(ii) $\int_S \Omega(\omega)\mu(d\omega) = 0$ and so

$$\mathscr{P}\int_{E^n} \frac{\Omega(y)}{|y|^n}\chi(y)dy = 0,$$

and we have for $|x| < R_0+1$,

$$[*] \qquad |(K_\Omega * f)(x)| \leq \mathscr{P}\int_{E^n} \frac{|\Omega(y)|}{|y|^n}|f(x-y)-f(x)\chi(y)|dy$$

$$\leq MM_1\omega_n\int_0^{2(R_0+1)} dr$$

$$= 2MM_1\omega_n(R_0+1),$$

where ω_n denotes as before the hypersurface of the unit sphere in E^n. Thus $(K_\Omega * f)$ is bounded. We remark for use below that an almost identical argument will show that

$$\left|\int_{|y|\geq\varepsilon} \frac{\Omega(y)}{|y|^n} f(x-y)dy\right|$$

is uniformly bounded for $\varepsilon > 0$. If the constants C_0, C_1, and $\alpha > 0$ are such that $|(K_\Omega * f)(x)| \leq C_0$, $x \in E^n$, and $|(K_\Omega * f)(x)| \leq C_1|x|^{-n}$, $|x| \geqq \alpha$, we have

$$\int_{E^n} |(K_\Omega * f)(x)|^p dx \leq \omega_n C_0^p\alpha^n + \int_{|x|\geq\alpha} C_1^p|x|^{-np}dx$$

$$= \omega_n C_0^p\alpha^n + C_1^p\int_\alpha^\infty r^{-n(p-1)-1}dr$$

$$= \omega_n C_0^p\alpha^n + C_1^p(n(p-1))^{-1}\alpha^{-n(p-1)} < \infty.$$

Thus $K_\Omega * f$ is in $L_p(E^n)$.

It follows from Plancherel's Theorem 8.9 and Theorem 8.22, that if $\{R_j\}$ is a sequence of real numbers approaching infinity, we have

$$
\begin{aligned}
F(K*f)(u) &= (2\pi)^{-n/2} \underset{j\to\infty}{\text{l.i.m.}} \int_{|x|\le R_j} e^{iux}\left\{\mathscr{P}\int_{E^n} \frac{\Omega(y)}{|y|^n} f(x-y)dy\right\} dx \\
(1) \qquad &= (2\pi)^{-\frac{n}{2}} \underset{j\to\infty}{\text{l.i.m.}} \int_{|x|\le R_j} e^{iux}\left\{\lim_{\varepsilon\to 0}\int_{|y|\ge\varepsilon} \frac{\Omega(y)}{|y|^n} f(x-y)dy\right\} dx,
\end{aligned}
$$

where here, and later, we write ux for the scalar product (u, x). Since, as remarked above, the integral in the inner braces of this last formula is uniformly bounded in ε, it follows from the Lebesgue dominated convergence theorem (III.6.16), and by Fubini's Theorem III.11.9 that equation (1) may be written as

$$
\begin{aligned}
F(K*f)(u) &= (2\pi)^{-n/2} \underset{j\to\infty}{\text{l.i.m.}} \lim_{\varepsilon\to 0}\int_{|x|\le R_j} e^{iux}\left\{\int_{|y|\ge\varepsilon} \frac{\Omega(y)}{|y|^n} f(x-y)dy\right\} dx \\
&= (2\pi)^{-n/2} \underset{j\to\infty}{\text{l.i.m.}} \lim_{\varepsilon\to 0}\int_{|y|\ge\varepsilon} \frac{\Omega(y)}{|y|^n}\left\{\int_{|x|\le R_j} e^{iux} f(x-y)dx\right\} dy \\
&= (2\pi)^{-n/2} \underset{j\to\infty}{\text{l.i.m.}}\, \mathscr{P}\int_{E^n} \frac{\Omega(y)}{|y|^n}\left\{\int_{|x|\le R_j} e^{iux} f(x-y)dx\right\} dy.
\end{aligned}
$$

Passing to a subsequence without loss of generality, we may assume (cf. III.6.3 and III.3.6) that for almost all u in E^n

$$
(2) \quad F(K*f)(u) = \underset{j\to\infty}{\text{l.i.m.}}\, (2\pi)^{-n/2} \mathscr{P}\int_{E^n} \frac{\Omega(y)}{|y|^n}\left\{\int_{|x|\le R_j} e^{iux} f(x-y)dx\right\} dy.
$$

If $f(x) = 0$ for $|x| \geqq R_0$, then

$$
\begin{aligned}
\int_{|x|\le R_j} e^{iux} f(x-y)dx &= \int_{E^n} e^{iux} f(x-y)dx \qquad |y| < R_j - R_0 \\
&= 0 \qquad\qquad\qquad\qquad\qquad |y| > R_j + R_0.
\end{aligned}
$$

Thus, if χ_j is the characteristic function of the set $(x \mid |x| \le R_j + R_0)$,

$$
\begin{aligned}
\Bigg| \mathscr{P}\int_{E^n} \frac{\Omega(y)}{|y|^n}&\left\{\int_{|x|\le R_j} e^{iux} f(x-y)dx\right\} dy \\
&- \mathscr{P}\int_{E^n} \frac{\Omega(y)}{|y|^n} \chi_j(y)\left\{\int_{E^n} e^{iux} f(x-y)dx\right\} dy \Bigg|
\end{aligned}
$$

$$
\begin{aligned}
&< \int_{R_j - R_0 \leq |y| \leq R_j + R_0} \frac{|\Omega(y)|}{|y|^n} \, dy \cdot \int_{E^n} |f(x)| dx \\
&\leq \max_{|x|=1} |\Omega(x)| \cdot \int_{E^n} |f(x)| dx \cdot \mu(S) \int_{R_j - R_0}^{R_j + R_0} \frac{dr}{r} \\
&\leq \frac{2R_0}{R_j - R_0} \mu(S) \max_{|x|=1} |\Omega(x)| \int_{E^n} |f(x)| dx,
\end{aligned}
$$

which evidently approaches zero as $j \to \infty$. Thus (2) gives

$$
\begin{aligned}
F(K * f)(u) &= (2\pi)^{-n/2} \lim_{j \to \infty} \mathscr{P} \int_{E^n} \frac{\Omega(y)}{|y|^n} \chi_j(y) \left\{ \int_{E^n} e^{iux} f(x-y) dx \right\} dy \\
&= \left\{ \lim_{j \to \infty} \mathscr{P} \int_{E^n} \frac{\Omega(y)}{|y|^n} \chi_j(y) e^{iuy} \, dy \right\} F(f)(u),
\end{aligned}
$$

provided only that the limit in the braces in this last equation exists. Thus, to complete the proof of the present lemma, it suffices to show that

$$
\theta(u) = \mathscr{P} \int_{E^n} \frac{\Omega(y)}{|y|^n} e^{iyu} \, dy = \lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \int_{\varepsilon \leq |y| \leq R} \frac{\Omega(y)}{|y|^n} e^{iyu} \, dy \tag{3}
$$

exists for each u. By Lemma 2, the integral $\theta(tu)$ exists if $\theta(u)$ exists and $t > 0$; and the integral $\theta(Vu)$ exists and equals

$$
\mathscr{P} \int_{E^n} \frac{\Omega(x)}{|x|^n} e^{i(x, Vu)} \, dx = \mathscr{P} \int_{E^n} \frac{\Omega(Vy)}{|y|^n} e^{i(y,u)} \, dy
$$

if $\mathscr{P} \int_{E^n} \Omega(Vy) |y|^{-n} e^{i(y,u)} \, dy$ exists and V is a rotation of E^n. Thus, to show that the proper value integral (3) exists generally, it is sufficient for us to consider the cases $u = 0$ and $u = [1, 0, \ldots, 0]$. If $u = 0$, we have evidently

$$
\mathscr{P} \int_{E^n} \frac{\Omega(x)}{|x|^n} \, dx = 0,
$$

since $\int_S \Omega(\omega) \mu(d\omega) = 0$. Hence, it suffices to show that the integral $\mathscr{P} \int_{E^n} e^{ix_1} \Omega(x) |x|^{-n} dx$ exists. Let χ_0 be the characteristic function of the interior of S. Then

$$\mathscr{P}\int_{E^n} e^{ix_1}\chi_0(x)\,\frac{\Omega(x)}{|x|^n}\,dx$$

exists by Lemma 3. Hence, it suffices to show the existence of the limit

$$(*)\qquad \lim_{R\to\infty}\int_{1\le|x|\le R} e^{ix_1}\frac{\Omega(x)}{|x|^n}\,dx = \lim_{R\to\infty}\int_S \Omega(\omega)\left\{\int_1^R \frac{e^{ir\omega_1}}{r}\,dr\right\}\mu(d\omega)$$

where ω_1 denotes the first component of the unit vector ω. Since $\lim_{R\to\infty}\int_1^R r^{-1}e^{ir\omega_1}\,dr$ exists for all ω such that $\omega_1 \neq 0$, i.e., for μ-almost all ω, the existence of the limit $(*)$ will follow by the Lebesgue dominated convergence theorem if we can find a μ-integrable function φ of ω such that

$$\left|\int_1^R \frac{e^{ir\omega_1}}{r}\,dr\right| \le \varphi(\omega), \qquad \omega\in S, \qquad R\ge 1.$$

Now, for $-1\le\omega_1\le 1$, $\omega_1\neq 0$, and $R\ge 1$,

$$\left|\int_1^R \frac{e^{ir\omega_1}}{r}\,dr\right| = \left|\int_{|\omega_1|}^{R|\omega_1|}\frac{e^{ir\operatorname{sgn}\omega_1}}{r}\,dr\right| \le -\log|\omega_1| + \left|\int_1^{R|\omega_1|}\frac{e^{ir\operatorname{sgn}\omega_1}}{r}\,dr\right|$$
$$\le \log|\omega_1| + K,$$

where K is a bound for $|\int_1^{R|\omega_1|}\exp(\pm ir)dr/r|$ for all $R\ge 1$; such a bound exists since the limits $\lim_{R\to\infty}\int_1^R\exp(\pm ir)dr/r$ exist. Thus, to complete the proof, it will suffice to show that the function $\varphi(\omega) = -\log|\omega_1|$ is μ-integrable on S. The Fubini theorem gives

$$\int_{\frac12\le|x|\le 1}\frac{-\log|x_1|}{|x|^{n-1}}\,dx = \int_{\frac12}^1 dr\left\{\int_S(-\log r - \log|\omega_1|)\mu(d\omega)\right\},$$

and so

$$\int_S \varphi(\omega)\mu(d\omega) = 2\int_{\frac12\le|x|\le 1}\frac{-\log|x_1|}{|x|^{n-1}}\,dx + 2\mu(S)\int_{\frac12}^1 \log(r)dr,$$

and it suffices to prove that the first integral on the right side of this equation is finite. This integral is readily seen to be finite since $|x|^{1-n}$ is bounded on the region indicated and

$$\int_{\frac12\le|x|\le 1} -\log|x_1|dx \le \int_{\max|x_i|\le 1} -\log|x_1|dx$$
$$= 2^{n-1}\int_{-1}^{+1} -\log|x_1|dx < \infty. \qquad \text{Q.E.D.}$$

The next theorem gives the inequality and the argument of M. Riesz.

8 THEOREM. *Let $1 < p < \infty$, let f be in $C^\infty(-\infty, +\infty)$, and vanish outside a bounded set. Then there exists a finite constant K_p such that the Hilbert proper value integral*

$$(Hf)(x) = \mathscr{P}\int_{-\infty}^{+\infty} \frac{f(y)}{x-y}\, dy$$

satisfies the inequality $|Hf|_p \leq K_p |f|_p$.

PROOF. We know by Lemma 7 that $(F(Hf))(y) = \pi i \operatorname{sgn}(y) (F(f))(y)$, $F(g)$ denoting the Fourier transform of g. By Plancherel's theorem (3.21), we know that $|Hf|_2 \leq \pi |f|_2$. We will first show that a finite constant K_p with the property specified in the statement of our theorem may be found for $p = 2n$, and will subsequently use this fact and auxiliary arguments (the Riesz convexity theorem, and an argument involving adjoints) to obtain the same result for other values of p. Let $J = \pi i I + H$. Then clearly (cf. Lemma 7)

$$\begin{aligned} F(Jf)(y) &= 0 & y < 0, \qquad & f \in L_2(-\infty, +\infty) \\ &= 2\pi i\, F(f)(y) & y \geqq 0. \qquad & \end{aligned}$$

Let f be in $C^\infty(-\infty, +\infty)$ and vanish outside a bounded set, and let $g = F(Jf)$. Since $F(f) \in L_1(-\infty, +\infty)$ and $F(f)$ is bounded, $F(Jf) \in L_1(-\infty, +\infty)$ and is bounded. If we put $g_1 = g$, $g_n = g * g_{n-1}$, it follows from an inductive argument that each function g_n vanishes for almost all $x < 0$. Moreover, by 3.21,

$$g_n = F((Jf)^n).$$

It may be shown, exactly as in the second paragraph of the proof of Lemma 7, that $(Hf)(x)$ is uniformly bounded, and that $|(Hf)(x)| = O(|x|^{-1})$ as $|x| \to \infty$. $(Jf)^n$ is in $L_1(-\infty, +\infty)$ for $n > 1$, so that by 3.21,

$$g_n(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \{(Jf)(y)\}^n e^{ixy}\, dy,$$

g_n is continuous, and thus $g_n(0) = 0$ for $n \geqq 2$. Putting $x = 0$, we consequently find that

$$\int_{-\infty}^{+\infty} (\pi i f(y) + (Hf)(y))^{2m}\, dy = 0, \qquad m \geqq 1. \tag{1}$$

Let f be real, so that Hf is real. Then it follows by taking the real part of (1) that

$$\sum_{k=0}^{m} \int_{-\infty}^{+\infty} (-1)^{m-k} \binom{2m}{2k} \{f(y)\}^{2(m-k)} \{(Hf)(y)\}^{2k}\, dy = 0.$$

By Hölder's inequality, this gives

$$|Hf|_{2m}^{2m} - \sum_{k=0}^{m-1} \binom{2m}{2k} |Hf|_{2m}^{2k} |f|_{2m}^{2(m-k)} \leqq 0.$$

Let $\alpha = |Hf|_{2m}/|f|_{2m}$. Then we have

$$\alpha^{2m} - \sum_{k=0}^{m-1} \binom{2m}{2k} \alpha^{2k} \leq 0.$$

Hence α is bounded by the largest real root r_m of the equation

$$t^{2m} - \sum_{k=0}^{m-1} \binom{2m}{2k} t^{2k} = 0.$$

Thus $|Hf|_{2m} \leqq r_m |f|_{2m}$. Since a complex function may be written as the sum of its real and imaginary parts, this proves the present theorem for the special case $p = 2m$.

It follows immediately that for $p = 2m$, H may be extended in a unique manner to a bounded mapping of $L_p(-\infty, +\infty)$ into itself. That the same conclusion is valid for all $p \geqq 2$ follows at once from the Riesz convexity theorem (VI.10.11).

Next we turn our attention to the range $1 < p \leqq 2$. Since, by Lemma 7, FHF^{-1} is the operation of multiplication by the function $\pi i \operatorname{sgn}(y)$ it follows at once that $(FHF^{-1})^* = -FHF^{-1}$. Since, by Plancherel's theorem (8.21) F is unitary it follows at once that $H^* = -H$ and hence

$$(Hf, g) = -(f, Hg) \qquad f, g \in L_2.$$

Thus, if $2 < p < \infty$ and $1/p + 1/p' = 1$, while f is in C^∞ and vanishes outside a bounded set, we have

$$\int_{-\infty}^{+\infty} (Hf)(x)\overline{g(x)}dx = -\int_{-\infty}^{+\infty} f(x)\overline{Hg(x)}dx \qquad g \in L_p(-\infty, +\infty),$$

by continuity. Thus

$$|Hf|_{p'} = \sup \left| \int_{-\infty}^{+\infty} Hf(x)\overline{g(x)}dx \right| \leqq K_p |f|_{p'},$$

by Theorem IV.8.1, the Hahn-Banach theorem (II.3.14) and Hölder's inequality, and the theorem is proved for all p, $1 < p < \infty$. Q.E.D.

Having proved the basic inequality of M. Riesz, we now proceed to prove the full inequality of Calderón and Zygmund. Our first step is to put the result of M. Riesz in an equivalent but technically more convenient form. This is done in the following lemma.

9 Lemma. *Let $1 < p < \infty$. There exists a finite constant Λ_p such that if the function $f \in C^\infty(-\infty, +\infty)$ vanishes outside a bounded set, then the function g defined by the equation*

$$g(x) = \int_{|y| \geq 1} \frac{1}{y} f(x-y)dy$$

satisfies the inequality $|g|_p \leqq \Lambda_p |f|_p$.

Proof. Let φ be an infinitely often differentiable non-negative function, vanishing for $|t| \geqq 1$, and satisfying the equation $\int_{-1}^{+1} \varphi(t)dt = 1$. Let

$$(Hf)(x) = \mathscr{P}\int_{-\infty}^{+\infty} \frac{1}{y} f(x-y)dy = \mathscr{P}\int_{-\infty}^{+\infty} \frac{f(y)}{x-y}\, dy.$$

Then, by Lemmas 7 and 3.1, $\varphi * Hf = H\varphi * f$.

Now it follows from an argument like that in the second paragraph of the proof of Lemma 7 that $H(\varphi)(y)$ is bounded and thus, since

$$\left| H(\varphi)(y) - \frac{1}{y} \right| \leqq \int_{y-1}^{y+1} \left| \frac{1}{u} - \frac{1}{y} \right| \varphi(y-u)du$$

$$\leqq \frac{1}{|y|\,\big||y|-1\big|} = O(|y|^{-2}), \qquad \text{as } |y| \to \infty,$$

it is seen that $H(\varphi) - \psi \in L_1$, where ψ is the function defined by the equations $\psi(y) = y^{-1}$, $|y| \geqq 1$; $\psi(y) = 0$, $|y| < 1$. Hence, by Theorem

9, Lemma 8.1, and the Riesz convexity theorem, we have

$$
\begin{aligned}
|y|_p = |\psi * f|_p &= |\varphi * H(f) - H(\varphi) * f + \psi * f|_p \\
&\leqq |\varphi * H(f)|_p + |(H(\varphi)-\psi) * f|_p \\
&\leqq \{K_p + |H(\varphi)-\psi|_1\}|f|_p, \qquad \text{Q.E.D.}
\end{aligned}
$$

It is now easy to obtain an inequality, exactly analogous to that stated in Lemma 9, for *odd* kernels of Calderón Zygmund type.

10 Lemma. *Let Ω be an odd function defined in E^n, with $\Omega(tx) = \Omega(x)$ for $t > 0$, and suppose that Ω is infinitely often differentiable in the neighborhood of the unit sphere S of E^n. Let the function f in $C^\infty(E^n)$ vanish outside a bounded set. Then, if $1 < p < \infty$ and Λ_p is the constant of Lemma 9, the $L_p(E^n)$-norm of the function g defined by the equation*

$$
g(x) = \int_{|u|\geq 1} \frac{\Omega(u)}{|u|^n} f(x-u)\,du
$$

is at most $2^{-1}\Lambda_p|f|_p \int_S |\Omega(\omega)|\mu(d\omega)$, μ being the measure of hypersurface on S.

Proof. By writing the integral in polar coordinates, it is seen that

$$
\begin{aligned}
g(x) &= \int_S \Omega(\omega)\mu(d\omega) \int_1^\infty f(x-\omega r)\frac{dr}{r} \\
&= \tfrac{1}{2}\int_S \Omega(\omega)\mu(d\omega) \int_{|t|\geq 1} \frac{f(x-\omega t)}{t}\,dt.
\end{aligned}
$$

By Lemma III.11.16(b) and Theorems III.11.17 and III.2.20(a) we have

$$
(1)\quad \left\{\int_B \left|\int_A F(a,b)\nu_1(da)\right|^p \nu_2(db)\right\}^{1/p} \leq \int_A \left\{\int_B |F(a,b)|^p \nu_2(db)\right\}^{1/p} v(\nu_1, db)
$$

for each function F which is $(\nu_1 \times \nu_2)$-measurable on the product space $(A, \nu_1) \times (B, \nu_2)$ of two measure spaces, the second of these measure spaces being positive. Hence

$$
|g|_p \leqq \tfrac{1}{2}\int_S |\Omega(\omega)|\mu(d\omega) \left\{\int_{E^n} \left|\int_{|t|\geq 1} \frac{f(x-\omega t)}{t}\,dt\right|^p dx\right\}^{1/p},
$$

and it is sufficient for us to prove that

$$\int_{E^n}\left|\int_{|t|\geq 1}\frac{f(x-\omega t)}{t}\,dt\right|^p dx \leq \Lambda_p^p|f|_p^p. \tag{2}$$

The outer integral in (2) being invariant under a rotation of coordinate systems, we may without loss of generality choose a coordinate system in E^n in which $\omega = [1, 0, \ldots, 0]$. Then what we have to show is that

$$\begin{aligned}\int_{E^n}\left|\int_{|t|\geq 1}\frac{f(x_1-t, x_2, \ldots, x_n)}{t}\,dt\right|^p dx_1\,dx_2\ldots dx_n \\ \leq \Lambda_p^p\int_{E^n}|f(x_1, \ldots, x_n)|^p\,dx_1\,dx_2\ldots dx_n.\end{aligned} \tag{3}$$

Now, it is an evident consequence of Lemma 9 that

$$\begin{aligned}\int_{-\infty}^{+\infty}\left|\int_{|t|\geq 1}\frac{f(x_1-t, x_2, \ldots, x_n)}{t}\,dt\right| dx_1 \\ \leq \Lambda_p^p\int_{-\infty}^{+\infty}|f(x_1, \ldots, x_n)|^p\,dx_1, \quad -\infty < x_j < \infty, \quad j = 2, \ldots, n.\end{aligned}$$

Upon integrating this inequality with respect to $x_2, \ldots, x_n$, the inequality (3) follows. Q.E.D.

Now we are able to establish the Calderón-Zygmund inequality for *odd* kernels.

11 THEOREM. *Let $1 < p < \infty$. Let Ω be an odd, measurable bounded function defined in E^n such that $\Omega(x) = \Omega(tx)$ for $t > 0$. Then, for f in $L_p(E^n)$, the limit*

$$(K_\Omega * f)(x) = \lim_{\varepsilon\to 0}\int_{|u|\geq\varepsilon}\frac{\Omega(u)}{|u|^n}f(x-u)du$$

exists in the mean of order p, and, if Λ_p is the constant of the preceding lemma,

$$|K_\Omega * f|_p \leq \Lambda_p\int|\Omega(\omega)|\mu(ds)|f|_p.$$

PROOF. We first wish to prove that the function g defined by the equation

$$g(x) = \int_{|u|\geq 1} \frac{\Omega(u)}{|u|^n} f(x-u)du \tag{1}$$

satisfies the inequality $|g|_p \leq I\Lambda_p|f|_p$, where $I = \int_S |\Omega(\omega)|\mu(d\omega)$. To do this, let $\{\Omega_m\}$ be a sequence of odd functions, each infinitely often differentiable in the neighborhood of the unit sphere, such that $\Omega_m(tx) = \Omega_m(x)$, $t > 0$, $m \geqq 1$, and such that

$$\lim_{m\to\infty} \int_S |\Omega_n(\omega) - \Omega(\omega)|\mu(d\omega) = 0.$$

First suppose that f is infinitely often differentiable and vanishes outside a bounded set. Let

$$g_m(x) = \int_{|u|\geq 1} \frac{\Omega_m(u)}{|u|^n} f(x-u)du.$$

By the preceding lemma, $\{g_m\}$ is a Cauchy sequence in $L_p(E^n)$, and consequently converges in the norm of $L_p(E^n)$ to a function φ in $L_p(E^n)$, which, by the preceding lemma, satisfies the inequality $|\varphi|_p \leq \Lambda_p I|f|_p$. Passing to a subsequence, we may assume that $g_m(x) \to \varphi(x)$ for almost all x. But it is clear that $g_m(x) \to g(x)$ for all x. Hence $\varphi = g$, proving that $|g|_p \leq \Lambda_p I|f|_p$ if f is in $C^\infty(I)$ and vanishes outside a bounded set.

Next let f be in $L_p(E^n)$, and let $\{f_m\}$ be a sequence of functions in $C^\infty(E^n)$, each vanishing outside a bounded set, and such that $|f-f_m|_p \to 0$. Put

$$h_m(x) = \int_{|u|\geq 1} \frac{\Omega(u)}{|u|^n} f_m(x-u)du.$$

By what has just been proved, $\{h_m\}$ is a Cauchy sequence in $L_p(E^n)$, and consequently converges in the norm of $L_p(E^n)$ to a function ψ in $L_p(E^n)$, which satisfies the inequality $|\psi|_p \leq \Lambda_p I|f|_p$. Passing to a subsequence, we may assume that $h_m(x) \to \psi(x)$ for almost all x. On the other hand, since

$$\begin{aligned}\int_{|x|\geq 1} \left|\frac{\Omega(x)}{|x|^n}\right|^q dx &\leqq \max_{x\in S} |\Omega(x)|^q \int_1^\infty r^{-n(q-1)-1}\, dr \\ &= \max_{x\in S} |\Omega(x)|^q \cdot (n(q-1))^{-1} < \infty\end{aligned}$$

for each $q > 1$. it is clear that $h_m(x) \to g(x)$ for each x. Thus the integral in (1) exists for each x in E^n and for each f in $L_p(E^n)$, and the function g defined by that integral satisfies the inequality $|g|_p \leq A_p I |f|_p$.

Next, for each $\varepsilon > 0$ define a mapping H_ε of $L_p(E^n)$ into itself as follows:

$$(H_\varepsilon f)(x) = \int_{|u| \geq \varepsilon} \frac{\Omega(u)}{|u|^n} f(x-u) du, \qquad f \in L_p(E^n).$$

Then clearly, putting $u = \varepsilon v$

$$\begin{aligned}(H_\varepsilon f)(x) &= \int_{|v| \geq 1} \frac{\Omega(v)}{\varepsilon^n |v|^n} f(x-\varepsilon v) \varepsilon^n dv \\ &= (T_\varepsilon^{-1} H_1 T_\varepsilon f)(x),\end{aligned}$$

where T_ε denotes the mapping of $L_p(E^n)$ into itself defined by the equation $(T_\varepsilon f)(x) = f(\varepsilon x)$. Since

$$|T_\varepsilon f|_p = \left\{\varepsilon^{-n} \int_{E^n} |f(v)|^p dv\right\}^{1/p} = \varepsilon^{-n/p} |f|_p, \qquad f \in L_p,$$

it follows immediately from what has been proved that $|H_\varepsilon| = |H_1| \leq A_p I$. We wish to show that $\lim_{\varepsilon \to 0} H_\varepsilon f$ exists strongly for each f in a set fundamental in $L_p(E^n)$, since then by Theorem II.3.6 the present theorem will be proved. Suppose that f is in $C^\infty(E^n)$ and vanishes outside the set $\{x \mid |x| \leq R\}$. Then, as was shown at the beginning of the proof of Lemma 7, the proper value integral

$$(K_\Omega * f)(x) = \lim_{\varepsilon \to 0} (H_\varepsilon f)(x) \tag{2}$$

exists and is bounded in x. Since $f(y) = 0$ for $|y| > R$ it follows that, for $|x| > R+1$ and $0 < \varepsilon < 1$, we have $(H_\varepsilon f)(x) = (K_\Omega * f)(x)$ and so it suffices to prove that

$$\lim_{\varepsilon \to 0} \int_{|x| \leq R+1} |(H_\varepsilon f)(x) - (K_\Omega * f)(x)|^p dx = 0.$$

By Corollary III.6.16, this will follow from (2) once it is shown that $(H_\varepsilon f)(x)$ is uniformly bounded. Let L be a bound for

$\{\sum_{i=1}^n |(\partial/\partial y_i)f(y)|^2\}^{\frac{1}{2}}$. Then using the mean value theorem of the calculus and Definition 4(i) and (ii) we have

$$\begin{aligned}|(H_\varepsilon f)(x)| &= \left|\int_{\varepsilon \leq r \leq 2(R+1)} \frac{\Omega(u)}{|u|^n} \{f(x-u)-f(x)\}\, du\right| \\ &\leq L\mu(S) \max_{|u|=1} |\Omega(u)| \int_0^{2(R+1)} dr \\ &= 2L\mu(S)(R+1) \max_{|u|=1} |\Omega(u)| < \infty,\end{aligned}$$

completing the proof. Q.E.D.

Next we introduce a convenient *odd* auxiliary function.

12 LEMMA. *There exists a non-zero constant c such that*

$$\mathscr{P}\int_{E^n} \frac{x_j}{|x|^{n+1}} e^{i(x,y)}\, dx = c\,\frac{y_j}{|y|}, \qquad j = 1, \ldots, n.$$

PROOF. If $n = 1$, we have calculated the integral and found $c = \pi i$ in our heuristic discussion of Hilbert's integral in the fourth paragraph of the present section. Thus, only the case $n > 1$ will be considered here. We need only consider the case $j = 1$. Let the integral on the left of the displayed formula be called $\varphi(y)$. Then, by Lemma 2,

(i) $\varphi(ty) = \varphi(y), \qquad t > 0,$

(ii) $\varphi(y) = -\varphi(-y),$

(iii) $\varphi(Uy) = \varphi(y)$ if U is a rotation in E^n leaving the point $[1, 0, \ldots, 0]$ fixed.

Any two points $u, v \in E^n$ such that $u_1 = v_1$ and $|u| = |v|$ can be mapped into each other by one of the rotations described in (iii). Hence, $\varphi(y)$ can be written for $y \neq 0$ and $i = 1$ as $\varphi(y) = \psi(y_1/|y|, |y|)$. By (i), ψ is independent of its second variable; by (ii), ψ is odd. Thus $\varphi(y) = \psi(y_1/|y|)$.

Let $\Phi(y)$ denote the integral

$$\mathscr{P}\int_{E^n} \frac{x_1 + ix_2}{|x|^{n+1}} e^{i(x,y)}\, dx.$$

Then clearly

$$\text{(iv)} \qquad \Phi(y) = \psi\left(\frac{y_1}{|y|}\right) + i\psi\left(\frac{y_2}{|y|}\right).$$

Let V be a rotation in the plane of y_1, y_2 (i.e., a rotation in E^n which keeps all points in the plane of $y_3, \ldots, y_n$) fixed. Then V maps $y_1 + iy_2$ into $e^{i\theta}(y_1 + iy_2)$ and $x_1 + ix_2$ into $e^{i\theta}(x_1 + ix_2)$. By Lemma 2

(v) $\Phi(Vy) = e^{i\theta}\Phi(y)$.

Let $\eta(y_1 + iy_2) = \Phi([y_1, y_2, 0, \ldots])$. Then by (iv) and (v)

(vi) $\eta(e^{i\theta}z) = e^{i\theta}\eta(z), \qquad z \neq 0.$

(vii) $\eta(tz) \quad = \eta(z), \qquad t > 0, \qquad z \neq 0.$

By (vi), $\eta(re^{i\theta}) = e^{i\theta}\eta(r)$, by (iv) $\eta(r)$ is real, and by (vii) $\eta(r) = c$ is independent of r. Thus $\eta(re^{i\theta}) = ce^{i\theta}$. and hence

$$\begin{aligned}\psi\left(\frac{y_1}{\sqrt{y_1^2 + y_2^2}}\right) = \varphi(y_1, y_2, 0, \ldots, 0) &= \mathscr{R}\Phi(y_1, y_2, 0, \ldots, 0) \\ &= \mathscr{R}\eta(y_1 + iy_2) \\ &= \mathscr{R}\eta(e^{i\theta}) \\ = c\cos\theta = c\left(\frac{y_1}{\sqrt{y_1^2 + y_2^2}}\right)&\end{aligned}$$

so that $\psi(x) = x$, and the present lemma will be proved once we show that $c \neq 0$.

If $c = 0$, then, by Lemma 8, $K_\Omega * f = 0$ for any f in $C^\infty(E^n)$ which vanishes outside a bounded set; here we have placed

$$\Omega(x) = \frac{x_1}{|x|}.$$

Let $f \neq 0$ be chosen to vanish if $x_1 < 0$ and to be non-negative. Then for $y_1 > 0$,

$$(K_\Omega * f)(y) = \mathscr{P}\int_{E^n} \frac{x_1}{|x|^n} f(x-y)dx = \int_{x_1>0} \frac{x_1}{|x|^n} f(x-y)dy > 0,$$

contradicting the fact that $K_\Omega * f = 0$. Q.E.D.

13 LEMMA. *Let ψ be in $L_2(E^n)$, f in $L_1(E^n)$, and let Ω be as in Theorem* 11. *Then*

$$K_\Omega * (\psi * f) = (K_\Omega * \psi) * f. \tag{1}$$

PROOF. By Theorem 11 and Lemma 3.1, both sides of this equation vary continuously in the topology of $L_2(E^n)$ as ψ and f vary continuously in the topology of $L_2(E^n)$ and $L_1(E^n)$ respectively. Hence, a continuity argument shows that it is sufficient to prove (1) if f and ψ both belong to $C^\infty(E^n)$ and vanish outside bounded sets. Let $\hat{f}$ and $\hat{\psi}$ be the Fourier transforms of f and ψ respectively, and let

$$\hat{K}(y) = \mathscr{P}\int_{E^n} \frac{\Omega(u)}{|u|^n} e^{iyu}\, du.$$

Then by Lemma 7 and 3.15 the Fourier transforms of the right and left hand sides of (1) are both $\hat{K}\hat{f}\hat{\psi}$, so that the present lemma follows from Plancherel's theorem (3.9 and 3.22). Q.E.D.

14 LEMMA. *Let Ω_1, Ω_2, and Ω_3 be related as in Lemma 5 and Definition 6. Let*

$$\psi_2(x) = \frac{\Omega_2(x)}{|x|^n}, \quad |x| \geqq 1; \qquad \psi_3(x) = \frac{\Omega_3(x)}{|x|^n}, \quad |x| \geqq 2$$
$$= 0, \quad |x| < 1; \qquad = 0, \quad |x| < 2.$$

Let S be the unit sphere in E^n, and let μ be the measure of hypersurface on S. Let $\varepsilon > 0$. Then there exists a constant K_ε depending only on ε and Ω_1 such that

$$\text{(i)} \quad \int_S |\Omega_3(\omega)|\mu(d\omega) \leqq K_\varepsilon \left\{\int_S |\Omega_2(\omega)|^{1+\varepsilon}\mu(d\omega)\right\}^{1/(1+\varepsilon)}$$

$$\text{(ii)} \quad \int_{E^n} |\psi_3(x) - (K_{\Omega_1} * \psi_2)(x)|dx \leqq K_\varepsilon \left\{\int_S |\Omega_2(\omega)|^{1+\varepsilon}\mu(d\omega)\right\}^{1/(1+\varepsilon)}.$$

Moreover, Ω_3 is odd and bounded.

PROOF. For $|x| \geqq 2$,

$$\begin{aligned}(K_{\Omega_1} * \psi_2)(x) & \\ &= \mathscr{P}\int_{E^n} \frac{\Omega_2(u)}{|u|^n}\frac{\Omega_1(x-u)}{|x-u|}\, du - \mathscr{P}\int_{E^n} \chi(u)\frac{\Omega_2(u)}{|u|^n}\frac{\Omega_1(x-u)}{|u-x|^n}\, du \\ &= \psi_3(x) - \mathscr{P}\int_{E^n} \chi(u)\frac{\Omega_2(u)}{|u|^n}\frac{\Omega_1(x-u)}{|x-u|^n}\, du,\end{aligned} \tag{1}$$

χ denoting the characteristic function of the set $\{x \mid |x| \leq 1\}$. The function $|u|^{-n}\Omega_1(u)$ is infinitely often differentiable for $|u| \geq 1$, and has a derivative with respect to u_i equal to

$$nu_i\,|u|^{-n-2}\Omega_1(u) + \left(\frac{\partial}{\partial u_i}\,\Omega_1\right)(u)\,|u|^{-n}.$$

Since $\Omega_1(tu) = \Omega_1(u)$ for $t > 0$, we have

$$t\left(\frac{\partial}{\partial u_i}\,\Omega_1\right)(tu) = \left(\frac{\partial}{\partial u_i}\,\Omega_1\right)(u), \quad i = 1, \ldots, n; \quad t > 0, \quad |u| > 0.$$

Thus

$$\begin{aligned}\frac{\partial}{\partial u_i}\,\{|u|^{-n}\Omega_1(u)\} &= -nu_i|u|^{-n-2}\Omega_1(u) + \left(\frac{\partial}{\partial u_i}\,\Omega\right)\left(\frac{u}{|u|}\right)|u|^{-n-1}\\ &= O(|u|^{-n-1}).\end{aligned}$$

It follows from the mean value theorem of calculus that there exists a finite constant L (depending only on Ω_1) such that

$$\left|\frac{1}{|u|}\left\{\frac{\Omega_1(x-u)}{|x-u|^n} - \frac{\Omega_1(x)}{|x|^n}\right\}\right| \leq L|x|^{-n-1}$$

for $|u| \leq 1$, $|x| \geq 2$. Thus the second integral in (1) is bounded by

$$\begin{aligned}&\left|\int_{|u|\leq 1} \frac{\Omega_2(u)}{|u|^n}\left\{\frac{\Omega_1(x-u)}{|x-u|^n} - \frac{\Omega_1(x)}{|x|^n}\right\} du\right|\\ &\qquad\leq L|x|^{-n-1}\int_S |\Omega_2(\omega)|\mu(d\omega)\cdot\int_0^1 dr\\ &\qquad\leq (\mu(S))^{\varepsilon/(1+\varepsilon)}\,L|x|^{-n-1}\left(\int_S |\Omega_2(\omega)|^{1+\varepsilon}\,(d\omega)\right)^{1/(1+\varepsilon)}.\end{aligned}$$

Hence

$$(2)\quad \int_{|u|\geq 2} |\psi_3(u) - (K_{\Omega_1} * \psi_2)(u)|du \leq K_\varepsilon\left\{\int_S |\Omega_2(\omega)|^{1+\varepsilon}\mu(d\omega)\right\}^{1/(1+\varepsilon)},$$

where

$$K_\varepsilon = (\mu(S))^\delta L\int_2^\infty r^{-2}\,dr = \tfrac{1}{2}L(\mu(S))^\delta$$

and $\delta = \varepsilon/(1+\varepsilon)$, so that K_ε clearly depends only on ε and Ω_1. (In the remainder of the present proof, we shall continue to use the

symbol K_ε for a constant depending only on ε and Ω_1). We may conclude in the same way that there exists another constant K_ε such that

$$(3)\quad \int_{|u|\geq 2} |\psi_3(u)-(K_{\Omega_1}*\psi_2)(u)|^{1+\varepsilon}du < K_\varepsilon \int_S |\Omega_2(\omega)|^{1+\varepsilon}\mu(d\omega).$$

Since by Theorem 11 there exists a constant K_ε such that

$$(4)\quad \begin{aligned} |K_{\Omega_1}*\psi_2|_{1+\varepsilon} &\leq K_\varepsilon |\psi_2|_{1+\varepsilon} \\ &\quad K_\varepsilon \left\{\int_1^\infty r^{-n\varepsilon-1}dr\right\}^{1/(1+\varepsilon)} \left\{\int_S |\Omega_2(\omega)|^{1+\varepsilon}\mu(d\omega)\right\}^{1/(1+\varepsilon)} \\ &= K_\varepsilon \left\{\int_S |\Omega_2(\omega)|^{1+\varepsilon}\mu(d\omega)\right\}^{1/(1+\varepsilon)}, \end{aligned}$$

it follows from (3) that there exists a constant K_ε such that

$$\begin{aligned} |\psi_3|_{1+\varepsilon} &= \left\{\int_S |\Omega_3(\omega)|^{1+\varepsilon}\mu(d\omega)\right\}^{1/(1+\varepsilon)} \left\{\int_2^\infty r^{-n\varepsilon-1}dr\right\}^{1/(1+\varepsilon)} \\ &\leq K_\varepsilon \left\{\int_S |\Omega_2(\omega)|^{1+\varepsilon}\mu(d\omega)\right\}^{1/(1+\varepsilon)}; \end{aligned}$$

thus

$$\begin{aligned} \int_S |\Omega_3(\omega)|\mu(d\omega) &\leq \mu(S)^{\varepsilon/(1+\varepsilon)} \left\{\int_S |\Omega_3(\omega)|^{1+\varepsilon}\mu(d\omega)\right\}^{1/(1+\varepsilon)} \\ &\leq \mu(S)^{\varepsilon/(1+\varepsilon)} K_\varepsilon \left\{\int_2^\infty r^{-n\varepsilon-1}dr\right\}^{-1/(1+\varepsilon)} \left\{\int_S |\Omega_2(\omega)|^{1+\varepsilon}\mu(d\omega)\right\}^{1/(1+\varepsilon)} \end{aligned}$$

proving (i).

By (4) and Hölder's inequality we have

$$\begin{aligned} \int_{|x|\leq 2} |(K_{\Omega_1}*\psi)(x)|dx &\leq [2^n\mu(S)]^{\varepsilon/(1+\varepsilon)} \left\{\int_{|x|\leq 2} |(K_\Omega*\psi)(x)|^{1+\varepsilon}dx\right\}^{1/(1+\varepsilon)} \\ &\leq [2^n\mu(S)]^{\varepsilon/(1+\varepsilon)} K_\varepsilon \left\{\int_S |\Omega_2(\omega)|^{1+\varepsilon}\mu(d\omega)\right\}^{1/(1+\varepsilon)}; \end{aligned}$$

so that (ii) follows from this and (2). That Ω_3 is odd follows from Lemma 5. Finally, to show that Ω_3 is bounded, note that since the second integral on the right of (1) has been shown to be bounded for $|x| \geq 2$, we have only to show that $K_{\Omega_1}*\psi_2$ is bounded for $|x| \geq 2$. If we put $\psi_1(x) = \Omega_1(x)|x|^{-n}$, $|x| \geq 1$, and $\psi_1(x) = 0$, $|x| \leq 1$, we have clearly

$$(5) \qquad (K_{\Omega_1} * \psi_2)(x) = (\psi_1 * \psi_2)(x) + \mathscr{P}\int_{E_n} \chi(u)\, \frac{\Omega_1(u)}{|u|^n}\, \frac{\Omega_2(x-u)}{|x-u|^n}\, du.$$

The second integral on the right of (5) may be shown to be bounded exactly as the second integral on the right of (1) was shown to be bounded. Since $|(\psi_1 * \psi_2)(x)| \leqq |\psi_1|_2 |\psi_2|_2$, the present lemma is fully proved. Q.E.D.

15 LEMMA. *Let $1 < p < \infty$, and let Ω be an even function defined in E^n, infinitely often differentiable on the surface of the unit sphere, and satisfying the equation $\Omega(tx) = \Omega(x)$ for $t > 0$. Let f be defined in E^n, infinitely often differentiable, and let f vanish outside a bounded set. Let $\varepsilon > 0$, and*

$$I_\varepsilon = \left\{\int_S |\Omega(\omega)|^{1+\varepsilon} \mu(d\omega)\right\}^{1/(1+\varepsilon)},$$

S being the unit sphere in E^n, and μ being the measure of hypersurface on S. Then there exists a finite constant $\Lambda_{\varepsilon,p}$ depending only on p and ε, such that the $L_p(E^n)$ norm of the function g defined by the equation

$$g(x) = \int_{|u| \geqq 1} \frac{\Omega(u)}{|u|^n} f(x-u)du$$

is at most $\Lambda_{\varepsilon,p} I_\varepsilon |f|_p$.

PROOF. Let $\Omega_i(x) = c^{-1} x_i |x|^{-1}$, where c is the constant of Lemma 12. Then, by Lemma 12 and Lemma 7, $F(K_{\Omega_i} * f)(y) = y_i |y|^{-1} F(f)(y)$ for each f in $C^\infty(E^n)$ which vanishes outside a bounded set; here $F(g)$ denotes the Fourier transform of g. By Plancherel's theorem (8.9) and by Theorem 11, both sides of this equation are continuous linear operators in $L_2(E^n)$, so that we have

$$F(K_{\Omega_i} * h)(y) = \frac{y_i}{|y|} F(h)(y), \qquad h \in L_2(E^n).$$

Consequently,

$$\sum_{i=1}^{n} K_{\Omega_i} * (K_{\Omega_i} * h) = h, \qquad h \in L_2(E^n).$$

Let $\psi(u) = \Omega(u)|u|^{-n}$, $|u| \geqq 1$; $\psi(u) = 0$, $|u| < 1$. By Lemma 14 and Theorem 11,

$$|g|_p = \Big|\sum_{i=1}^{n} K_{\Omega_i} * (K_{\Omega_i} * \psi) * f\Big|$$
$$\leq nL \max_{1\leq i\leq n} |(K_{\Omega_i} * \psi) * f|_p,$$

where $L = \int_S |y_1|\mu(dy)$. By Lemma 14, $K_{\Omega_i} * \psi$ may be written as

$$K_{\Omega_i} * \psi = \varphi_i + h_i,$$

where $\varphi_i(u) = \Omega_3^{(i)}(u)|u|^{-n}$, $|u| \geq 1$, $\varphi_i(u) = 0$, $|u| < 1$; $\Omega_3^{(i)}(u) = \Omega_3^{(i)}(tu)$, $t > 0$; $\Omega_3^{(i)}$ is odd and bounded;

$$\int_S |\Omega_3^{(i)}(\omega)|\mu(d\omega) \leq K_\varepsilon I_\varepsilon;$$

and

$$\int_{E^n} |h_i(x)|dx \leq K_\varepsilon I_\varepsilon.$$

Then, by Lemma 3.1 and Lemma 10,

$$|(K_{\Omega_i} * \psi) * f|_p \leq 2K_\varepsilon I_\varepsilon |f|_p. \qquad \text{Q.E.D.}$$

Now we can prove the general result of Calderón and Zygmund.

16 THEOREM. (Calderón-Zygmund) *Let* $1 < p < \infty$, *and* $\varepsilon > 0$. *Let* Ω *be a bounded measurable function defined in* E^n *with* $\Omega(x) = \Omega(tx)$ *for* $t > 0$ *and with*

$$\int_S \Omega(\omega)\mu(d\omega) = 0,$$

S *being the unit sphere in* E^n, *and* μ *being the measure of hypersurface on* S. *Then, for every* f *in* $L_p(E^n)$, *the limit*

$$(K_\Omega * f)(x) = \lim_{\varepsilon\to 0} \int_{|u|\geq\varepsilon} \frac{\Omega(u)}{|u|^n} f(x-u)du$$

exists in the mean order p, *and there exists a finite constant* $\Lambda_{\varepsilon,p}$ *depending only on* p *and* ε, *such that*

$$|K_\Omega * f|_p \leq \Lambda_{\varepsilon,p} I_\varepsilon |f|_p,$$

where

$$I_\varepsilon = \left\{\int_S |\Omega(\omega)|^{1+\varepsilon} \mu(d\omega)\right\}^{1/(1+\varepsilon)} < \infty.$$

PROOF. Let $\omega_n = \mu(S)$. Then, by Hölder's inequality, $\int_S |\Omega(\omega)|\mu(ds) \leq \omega_n^{\varepsilon/(1+\varepsilon)} I_\varepsilon$. Hence, by Theorem 11, the present theorem holds if Ω is odd. Since Ω may be written as the sum of its odd and even parts, it is sufficient for us to consider the case in which Ω is even. But in this case the theorem follows from Lemma 15 in just the same way that Theorem 11 followed from Lemma 10. Q.E.D.

8. Exercises

A. *Exercises on Fourier Integrals*

1 Let $\mathfrak{L}_1^+$ be the subspace of $L_1(-\infty, +\infty)$ consisting of all functions which vanish for t negative. Show that $\mathfrak{L}_1^+$ forms a closed subalgebra of the convolution algebra $L_1(-\infty, +\infty)$. Show that if m is a multiplicative linear functional on the algebra $\mathfrak{L}_1^+$, there exists a complex number z with $\mathscr{I}z > 0$ and such that

$$m(f) = \int_0^\infty e^{ixz} f(x)dx, \qquad f \in \mathfrak{L}_1^+.$$

2 Let f_1 and f_2 be in $L_2(-\infty, +\infty)$, and let $f(x) = f_1(x)f_2(x)$. If F, F_1, F_2 are the Fourier transforms of f, f_1, f_2 respectively, show that

$$F(x) = \int_{-\infty}^{+\infty} F_1(x-y)F_2(y)dy.$$

3 Generalize Exercise 2 to an arbitrary locally compact Abelian topological group.

4 Let $p \geq 1$, let f be in $L_1(-\infty, +\infty)$, and let g be in $L_p(-\infty, +\infty)$. Show that the convolution

$$(f*g)(x) = \int_{-\infty}^{+\infty} f(x-y)g(y)dy$$

exists for almost all x, is in $L_p(-\infty, +\infty)$, and that

$$|f*g|_p \leq |f|_1 \, |g|_p.$$

5 If f and g are in $L_1(-\infty, \infty) \cap L_2(-\infty, \infty)$, and if

$$g(x) = \int_a^x f(y)dy,$$

show that the Fourier transforms F and G of f and g respectively are related by the equation

$$it\ F(t) = G(t).$$

6 Let $1 \leqq p \leqq 2$, and let f be in $L_p(-\infty, +\infty)$. Show that

$$F(t) = \lim_{A\to\infty} \int_{-A}^{+A} e^{itx} f(x) dx$$

exists in the norm of $L_q(-\infty, +\infty)$, where $p^{-1}+q^{-1} = 1$. (Hint: Cf. VI.11.43.)

7 Show, with the hypotheses and notation of Exercise 6, that

$$\lim_{A\to\infty} \frac{1}{2\pi} \int_{-A}^{+A} F(t) e^{-itx} dt = f(x)$$

in the topology of $L_p(-\infty, +\infty)$. (Hint: Cf. IV.4.19.)

8 Show, with the hypotheses and notation of Exercise 6, that if b is in $L_p(-\infty, +\infty)$, then

$$\int_{-\infty}^{+\infty} |b(t)|^{2-p} |F(t)|^p dt < \infty.$$

9 Let λ be a real function of a real variable such that $\lambda(\cdot)F(\cdot)$ is the Fourier transform of a function in $L_1(-\infty, +\infty)$ whenever F is the Fourier transform of a function in $L_1(-\infty, +\infty)$. Show that for $1 \leqq p \leqq 2$, $\lambda(\cdot)F(\cdot)$ is the Fourier transform of a function in $L_p(-\infty, +\infty)$ whenever F is the Fourier transform of a function in $L_p(-\infty, +\infty)$, the Fourier transforms being defined as in Exercise 6.

10 Let λ be a function defined on $(-\infty, +\infty)$ which is of finite total variation. Show that if $1 < p \leqq 2$, and if F is the Fourier transform of a function in $L_p(-\infty, +\infty)$, then so is $\lambda(\cdot)F(\cdot)$. Fourier transforms are to be defined as in Exercise 6. (Hint: Use the inequality of M. Riesz.)

11 Let λ_n be a sequence defined for $-\infty < n < +\infty$ which is of finite total variation. Show that if $\sum_{n=-\infty}^{+\infty} a_n e^{inx}$ is the Fourier series of a function in $L_p(0, 2\pi)$, then so is $\sum_{n=-\infty}^{+\infty} a_n \lambda_n e^{inx}$.

12 Let f be in $L_1(-\infty, +\infty)$, and let F be its Fourier transform. Show that

$$f(x) = \lim_{A\to\infty} \frac{1}{2\pi} \int_{-A}^{+A} F(t) e^{-itx} dt$$

if f is of bounded variation in the neighborhood of x. (Hint: Cf. IV.14.17.)

13 Show that the conclusion of Exercise 12 remains valid if the condition that f is in $L_1(-\infty, +\infty)$ is replaced by the requirement that f be in $L_2(-\infty, +\infty)$.

14 Show that there exists a continuous function f in $L_1(-\infty, +\infty) \cap L_2(-\infty, +\infty)$ such that the limit in Exercise 12 fails to exist for $x = 0$.

15 Show that there exists a function f in $L_1(-\infty, +\infty)$ for which the family of functions

$$f_A(x) = \frac{1}{2\pi}\int_{-A}^{+A} F(t)e^{-itx}\,dt,$$

F denoting the Fourier transform of f, fails to satisfy the inequality

$$\sup_{A>0}\int |f_A(x)|\,dx < \infty.$$

16 Show that not every continuous function, defined for $-\infty < t < \infty$ and approaching zero as t approaches $+\infty$ or $-\infty$, is the Fourier transform of a function f in $L_1(-\infty, +\infty)$.

17 Find a function f in $L_1(-\infty, +\infty)$ which is the indefinite integral of another function in $L_1(-\infty, +\infty)$ such that the Fourier transform F of f satisfies the condition

$$\int_{-\infty}^{+\infty} |F(t)|\,dt = \infty.$$

Show that if f is the integral of a function in $L_2(-\infty, +\infty)$, this is impossible.

18 Let f be in $L_1(-\infty, +\infty)$ and let F be its Fourier transform. Then

$$f(x) = \lim_{A\to\infty}\frac{1}{2\pi}\int_{-\infty}^{+\infty} F(t)e^{-itx}\alpha\left(\frac{t}{A}\right)dt$$

provided that the function α is bounded, continuously differentiable at the origin, and that both α and its Fourier transform belong to $L_1(-\infty, +\infty)$. The limit is in the topology of $L_1(-\infty, +\infty)$. Moreover, under the additional hypothesis that the Fourier trans-

form of α is bounded above by an even, monotone decreasing function in L_1, we have

$$f(x) = \lim_{A\to\infty} \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(t)e^{-itx}\alpha\left(\frac{t}{A}\right) dt,$$

(i) for almost all x, and

(ii) for each x at which f is continuous.

(Hint: Cf. VIII.9.5.)

19 Let f be in $L_1(-\infty, +\infty)$, and suppose that the Fourier transform F of f also belongs to $L_1(-\infty, +\infty)$. Show that

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ixt} F(t)dt$$

almost everywhere. (Hint: Use Exercise 18.)

20 Let $1 \leqq p \leqq 2$, let E^n denote real n-dimensional Euclidean space, and let f be in $L_p(E^n)$, the measure being Lebesgue measure. Show that

$$F(t_1, \ldots, t_n) = \lim_{A\to\infty} \int_{-A}^{+A} \cdots \int_{-A}^{+A} e^{i(t_1x_1+\cdots+t_nx_n)} f(x_1, \ldots, x_n)dx_1 \ldots dx_n$$

exists in the norm of $L_q(E^n)$, where $p^{-1}+q^{-1} = 1$. Show also that

$$f(x_1, \ldots, x_n) = \lim_{A\to\infty} (2\pi)^{-n} \int_{-A}^{+A} \cdots \int_{-A}^{+A} e^{-i(t_1x_1+\cdots+t_nx_n)} F(t_1, \ldots, t_n)dt_1 \ldots dt_n$$

in the topology of $L_p(E^n)$. (Hint: Cf. Exercises 6 and 7.)

B. *Exercises on Inequalities and Singular Integrals*

21 (Hardy and Littlewood) Let the function f in $L_1(0, 2\pi)$ have

$$c_n(f) = \int_0^{2\pi} f(x)e^{-inx}dx = 0, \qquad n < 0.$$

Show that

$$\sum_{n=0}^{\infty} \frac{|c_n(f)|}{n} \leqq K \int_0^{2\pi} |f(x)|dx,$$

where K is a certain absolute constant. (Hint: Show, using Exercise

IV.14.88. that there exist two functions f_1, f_2 in $L_2(0, 2\pi)$ such that $c_n(f_2) = 0, n < 0. i = 1. 2.$ and such that $f = f_1 f_2$. Then use Exercise 8.)

22 (Conjugate Trigonometric series) Let f be in $L_p(0, 2\pi)$ where $1 < p < \infty$. Let

$$c_n(f) = \int_0^{2\pi} f(x)e^{-inx}dx, \qquad -\infty < n < +\infty.$$

Show that the series

$$[*] \qquad \sum_{n=-\infty}^{+\infty} \operatorname{sgn}(n)c_n(f)e^{inx} = \lim_{p\to\infty} \sum_{n=-p}^{p} \operatorname{sgn}(n)c_n(f)e^{inx}$$

converges in the mean of order p, to a function which is given hy the singular integral

$$I = \frac{1}{2\pi i}\int_0^{2\pi} \frac{1}{\tan \frac{1}{2}(y-\pi)} f(x-y)dy.$$

Here. the integral is to be understood as the limit

$$[\dagger] \qquad I = \lim_{\varepsilon\to 0} \frac{1}{2\pi i}\left\{\int_0^{\pi-\varepsilon} + \int_{\pi+\varepsilon}^{2\pi}\right\} \frac{1}{\tan \frac{1}{2}(y-\pi)} f(x-y)dy,$$

which exists in the mean of order p. (Hint: Use the method of Exercise IV.14.19 to establish the mean convergence of the series and Theorem 7.8 to establish the mean convergence of the integral.)

23 (Calderõn-Zygmund) Let $1 < p < \infty$, and let Ω be a function defined in E^n and having the properties

(i) $\Omega(tx) = \Omega(x)$, $\qquad t > 0$,

(ii) $\int_S \Omega(\omega)\mu(d\omega) = 0$,

S being the unit sphere in E^n. and μ the measure of hypersurface on S,

(iii) $\Omega(t)$ is a continuously differentiable function of t for $t \neq 0$. Show that, for f in $L_p(E^n)$, the singular integral $\lim_{\varepsilon\to 0}(J^\varepsilon f)(y)$. where

$$(J^\varepsilon f)(y) = \int_{|x|>\varepsilon} \frac{\Omega(x)}{|x|^n} f(y-x)dx,$$

exists for almost all y. (Hint: Let φ be a function in $C^\infty(E^n)$ which is non-negative, vanishes outside a bounded set, and has integral equal to one. Put $\varphi_\varepsilon(x) = \varepsilon^n\varphi(\varepsilon x)$ for each function φ. Show that there exists a function ρ in $L_1(E^n)$ such that

$$J^{\varepsilon} f - \varphi_{\varepsilon} * (K_{\Omega} * f) = \rho_{\varepsilon} * f.$$

Then use a suitable generalization of Exercise VIII.9.6.)

24 Let $1 < p < \infty$; then the integral [†] of Exercise 22 converges almost everywhere for each f in L_p.

C. *Exercises on Eigenvalue Location*

25 (A. Brauer) Let $A = (a_{jk})$ be an $n \times n$ matrix, regarded as a linear transformation in E^n. For each k, $1 \leq k \leq n$, let C_k be the circle with center a_{kk} and radius

$$r_k = \min \left(\sum_{j \neq k} |a_{jk}|, \quad \sum_{j \neq k} |a_{kj}| \right).$$

Show that each eigenvalue of A lies in some circle C_k. (Hint: Let $[x_1, \ldots, x_n]$ be the eigenvector belonging to an eigenvalue λ. Show that if $|x_k| = \max_{1 \leq i \leq n} |x_i|$ then $|\lambda - a_{kk}| \leq \sum_{j \neq k} |a_{jk}|$.)

26 (A. Brauer) Suppose that all the circles C_k of Exercise 1 are disjoint. Show that each contains exactly one simple eigenvalue of A. If in addition, it is assumed that all the elements of A are real, it follows that all the eigenvalues of A are real. (Hint: Let D be the matrix with the same diagonal elements as A, and with zero off-diagonal elements. Consider $\alpha A + (1-\alpha) D$, $0 \leq \alpha \leq 1$.)

27 Using the notation of Exercise 25 show that

$$|\det(A)| \geq \prod_{k=1}^{n} (|a_{kk}| - r_k)$$

if all the factors in the product are positive.

28 (Perron) Let A be as in Exercise 25. Let $c_1, \ldots, c_n$ be an arbitrary set of positive numbers. Then each eigenvalue λ of A satisfies the inequality

$$|\lambda| \leq \max_{1 \leq j \leq n} c_j^{-1} \sum_{k=1}^{n} |a_{jk}| c_k.$$

29 For each n and each $m \leq n$, let E_m^n be the space of all skew-symmetric functions $a(i_1, i_2, \ldots, i_m)$ of m indices running between 1 and n. (That is, $a(i_1, i_2, \ldots, i_m) = -a(j_1, j_2, \ldots, j_m)$ if the ordered sequences $i_1, \ldots, i_m$ and $j_1, \ldots, j_m$ differ by a single interchange.) Put $(a, b) = \sum_{i_1, \ldots, i_m} a(i_1, \ldots, i_m) \overline{b(i_1, \ldots, i_m)}$ thereby

making E_m^n into a Hilbert space. If A is a matrix as in Exercise 25, let $A^{(m)}$ be the transformation in E_m^n defined by the equation

$$(A^{(m)}b)(i_1, \ldots, i_m) = \sum_{j_1, \ldots, j_m} a_{i_1 j_1} \ldots a_{i_m j_m} b(j_1, \ldots, j_m).$$

Show that $(AB)^{(m)} = A^{(m)}B^{(m)}$, and that $(A^*)^{(m)} = (A^{(m)})^*$. Show that if $\lambda_1, \ldots, \lambda_n$ are eigenvalues of A (each eigenvalue λ being repeated a number of times equal to the dimension of the range of $E(\lambda; A)$), then the eigenvalues of $A^{(m)}$ are

$$\lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_m}$$

$i_1, i_2, \ldots, i_m$ being an arbitrary sequence of integers such that $1 \leqq i_1 < i_2 \ldots < i_m \leqq n$.

30 (H. Weyl) Let A and $\{\lambda_i\}$ be as in Exercise 29. Let B be the unique positive square root of AA^*. Let $\{\mu_i\}$ be the eigenvalues of B. Suppose that $\{\lambda_i\}$ and $\{\mu_i\}$ are arranged in order of decreasing absolute value. Show that

$$|\lambda_1| \leqq \mu_1, \ |\lambda_1 \lambda_2| \leqq \mu_1 \mu_2, \ |\lambda_1 \lambda_2 \lambda_3| \leqq \mu_1 \mu_2 \mu_3, \text{ etc.}$$

[*Remark*: This result is the best possible (Horn).]

31 (H. Weyl) Let $\{\lambda_i\}$ and $\{\mu_i\}$ be as in Exercise 30. Show that the inequalities of Exercise 30 imply that

$$\sum_{i=1}^{j} |\lambda_i|^p \leqq \sum_{i=1}^{j} \mu_i^p, \qquad p \geqq 1, \qquad 1 \leqq j \leqq n.$$

(Hint: $k_1 + k_2 k_1^{-1} + k_3 k_2^{-1} + \ldots$ is an increasing function of all its variables as long as $k_1^2 \geq k_2$, $k_2^2 \geq k_1 k_3$, $k_3^2 \geq k_2 k_4$, etc.)

32 (H. Weyl) Let $\{\lambda_i\}$ and $\{\mu_i\}$ be as in Exercise 30. Show that

$$\sum_{i \neq j} |\lambda_i \lambda_j|^p \leqq \sum_{i \neq j} (\mu_i \mu_j)^p, \qquad p \geqq 1,$$

$$\sum_{i \neq j \neq k} |\lambda_i \lambda_j \lambda_k|^p \leqq \sum_{i \neq j \neq k} (\mu_i \mu_j \mu_k)^p \qquad p \geqq 1,$$

etc.

33 Let A be an operator in an n-dimensional Hilbert space, and let $B = \frac{1}{2}(A + A^*)$. Let μ_- and μ_+ be respectively the least and the greatest eigenvalues of B, and let λ be an eigenvalue of A. Then

$$\mu_- \leqq \mathscr{R}(\lambda) \leqq \mu_+ .$$

34 (Bendixon) Let A be as in Exercise 25, and suppose also that the matrix elements of A are real. Let $C = (A-A^*)$, and let g be the maximum of the absolute values of the matrix elements of C. Then

$$|\mathscr{I}\lambda| \leqq g\left(\frac{n(n-1)}{2}\right)^{\frac{1}{2}}.$$

(Hint: Use Exercise 33 and the case $n = 2$ of Exercise 31.)

35 (Pick) Let C be a real, $n\times n$, skew symmetric matrix, with elements (a_{ij}), and regard A as a mapping of n-dimensional Hilbert space into itself. Show that

$$|C| \leqq g \cot \frac{\pi}{2n},$$

where g is the maximum of the absolute values of the elements of the matrix C. Show consequently that the inequality of Exercise 34 may be improved to give the inequality

$$|\lambda| \leqq g \cot \frac{\pi}{2n} \leqq g\left(\frac{n(n-1)}{2}\right)^{\frac{1}{2}}.$$

(Hint: By arranging the components x_i of a vector x in an order such that $\mathscr{I}(x_i\bar{x}_j - x_j\bar{x}_i) \geqq 0$ if $i < j$, show that

$$|(Ax, x)| \leqq -ig\sum_{i<j}(x_i\bar{x}_j - x_j\bar{x}_i) \leqq g|x| \cot \frac{\pi}{2n}.)$$

36 (Parker) Let A be as in Exercise 26. Show that if λ is an eigenvalue of A,

$$|\lambda| \leqq \tfrac{1}{2} \max_k \sum_{i=1}^{n} \{|a_{ik}|+|a_{ki}|\}.$$

37 (Frobenius) Let A be a matrix whose elements are all positive. Show that

(i) The eigenvalue λ largest in modulus is simple, and in fact real and positive.

(ii) The eigenvector x belonging to the eigenvalue λ has positive components.

(iii) Any eigenvector of A having positive components is proportional to x.

(iv) The eigenvalue λ is the largest number λ_0 for which there exists a vector y with positive components for which each component of Ay is at least as large as the corresponding component of $\lambda_0 y$. (Hint: Use the Brouwer fixed point theorem. Better, consider the iterates A^n.)

38 Let $p \geqq 1$, let (S, Σ, μ) be a positive measure-space, and let A be a compact operator in $L_p(S, \Sigma, \mu)$. Suppose that A maps positive functions into positive functions. Show that A has an eigenvalue λ which is positive, which is at least as large as the modulus of any other eigenvalue of A, and to which there corresponds a non-negative eigenfunction.

39 Let A and B be $n \times n$ matrices and let $\{\lambda_i\}$ be an enumeration of the eigenvalues of AB. Show that

$$\sum_{i=1}^{n} |\lambda_i| \leqq \|A\| \, \|B\|.$$

(Hint: Put AB in subdiagonal form. $\|A\|$ is the Hilbert-Schmidt norm of A.)

40 (Lalesco) Let A and B be operators of Hilbert-Schmidt class. Let $\{\lambda_i\}$ be an enumeration of the eigenvalues of AB, repeated according to multiplicity. Show that

$$\sum_{i=1}^{\infty} |\lambda_i| \leqq \|A\| \, \|B\|.$$

D. *Exercises on the Fredholm Theory of Hilbert-Schmidt Operators*

In the following set of exercises, A is a given Hilbert-Schmidt operator. $\{\lambda_i\}$ is the family of non-zero eigenvalues of A, repeated according to their multiplicities, $\delta(\lambda)$ is the function defined by the convergent infinite product

$$\delta(\lambda) = \prod_{i=1}^{\infty} (1-\lambda\lambda_i)e^{\lambda\lambda_i},$$

and $\Delta(\lambda)$ is the analytic operator-valued function

$$R(\lambda^{-1}; A)\delta(\lambda).$$

(Cf. Theorem 6.26.)

41 (Smithies) Show that

(a) if $f(z, \lambda) = z^{-1}\log(1-\lambda z)$,

$$\delta(\lambda) = \exp\{\mathrm{tr}(f(A, \lambda), A)\}, \qquad \lambda \neq \lambda_i^{-1},$$

(b) $\delta(\lambda) = \exp\left\{-\sum_{n=2}^{\infty} \frac{\lambda^n}{n}\sigma_n\right\}$,

where

$$\sigma_n = \sum_{j=1}^{\infty} \lambda_j^n, \qquad n \geqq 2.$$

(c) $\delta'(\lambda) = (-\sum_{n=2}^{\infty} \lambda^{n-1}\sigma_n)\delta(\lambda)$.

(d) $\delta(\lambda) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \det(P_n)$,

where $P_0 = I$, and where for $n \geqq 1$, P_n is the $n \times n$ matrix

$$P_n = \begin{pmatrix} 0 & n-1 & 0 & \cdots & \cdots & 0 \\ \sigma_2 & 0 & n-2 & 0 & \cdots & \\ \sigma_3 & \sigma_2 & 0 & \cdot & & \\ \cdot & \cdot & \cdot & & & \\ & & & \cdot & & 0 \\ & & \cdot & & \cdot & 1 \\ \sigma_n & \cdots & & \cdots & \sigma_3 \quad \sigma_2 & 0 \end{pmatrix}.$$

42 (Smithies) Show that

(a) $(I-\lambda A)\Delta(\lambda) = \lambda\delta(\lambda)$.

(b) If $\delta(\lambda) = \sum_{n=0}^{\infty}\delta_n\lambda_n$ and $\Delta(\lambda) = \sum_{n=0}^{\infty}\Delta_n\lambda^n$, then $\Delta_0 = 0$, and $\Delta_n = \delta_{n-1}I + A\delta_{n-2} + A^2\delta_{n-3} + \dots + A^{n-1}\delta_0$.

(c) Δ_n may be written as a symbolic $n \times n$ determinant

$$\Delta_n = \frac{(-1)^n}{n!} \det\left(\begin{array}{c|c} I & n-1 \quad 0 \dots 0 \\ \hline A & \\ \vdots & P_{n-1} \\ A^{n-1} & \end{array}\right), \qquad n \geqq 2,$$

where P_{n-1} is the $(n-1)\times(n-1)$ determinant of (d) of the preceding exercise.

43 (Smithies) The function $\Delta(\lambda)-\lambda\delta(\lambda)I$ is an analytic function of λ, even if we regard it as having values in the space HS of operators of Hilbert-Schmidt class. Consequently, the series

$$\Delta(\lambda)-\lambda\delta(\lambda)I = \sum_{n=2}^{\infty} \Delta_n \lambda^n$$

of the preceding exercise converges in the Hilbert-Schmidt norm.

44 Let (S, Σ, μ) be a positive measure space. Then an operator A in the Hilbert space $L_2(S, \Sigma, \mu)$ is of Hilbert-Schmidt class if and only if there exists a $\mu\times\mu$ measurable function $A(\cdot, \cdot)$ on $S\times S$ such that

$$\text{(i)} \qquad \left\{\int_S\int_S |A(s, t)|^2 \mu(ds)\mu(dt)\right\}^{\frac{1}{2}} < \infty$$

and such that

$$\text{(ii)} \quad (Af)(s) = \int_S A(s, t)f(t)\mu(dt), \qquad f \in L_2(S, \Sigma, \mu),$$

for μ-almost all s, i.e., if and only if A can be represented as an integral operator with a kernel satisfying (i). If such a kernel $A(\cdot, \cdot)$ exists, then it is unique, and $\|A\|$ is exactly equal to the finite quantity (i). Conversely, if $A(\cdot, \cdot)$ is a $\mu\times\mu$ measurable function defined on $S\times S$, and satisfying (i), then (ii) defines an operator A in $L_2(S, \Sigma, \mu)$ which is of Hilbert-Schmidt class.

45 Suppose that the Hilbert space H in Exercise 43 is the space $L_2(S, \Sigma, \mu)$ of Exercise 44. Let $\Delta_n(s, t)$ be the kernel of Exercise 44 which represents, in the sense of Exercise 44, the operator Δ_n of Exercise 42. Then the power series

$$\Delta(s, t; \lambda) = \sum^{\infty} \lambda^n \Delta_n(s, t)$$

converges for $\mu\times\mu$-almost all $[s, t]$, and $\Delta(s, t; \lambda)$ is the kernel which represents the operator $\Delta(\lambda)-\lambda\delta(\lambda)I$ in the sense of Exercise 44.

46 (Hilbert) Suppose that the hypotheses of the preceding exercise are satisfied.

(a) Show that the constants δ_n and the operators Δ_n of Exercise 42 are uniquely determined by the equations $\delta_0 = 1$, $\delta_1 = 0$, $\Delta_0 = 0$, $\Delta_1 = \delta_0 I$, and by the recursion relations

$$\Delta_n = \delta_{n-1} I + A\Delta_{n-1}$$

and

$$\delta_n = \frac{(-1)^n}{n!} \operatorname{tr}(\Delta_n - \delta_{n-1} I, A).$$

(b) Let $A(\cdot, \cdot)$ be the kernel representing the operator A in the sense of Exercise 44. Let $\hat{\delta}_n$ be the sequence of constants determined by the formulae $\hat{\delta}_0 = 1$,

$$\hat{\delta}_n = \frac{(-1)^n}{n!} \int_S \cdots \int_S B_n(s_1, \ldots, s_n)\mu(ds) \ldots \mu(ds_n). \qquad n \geqq 1,$$

where $B(s_1, \ldots, s_n)$ is the determinant of the $n \times n$ matrix whose general element, i.e., element in the ith row and jth column, is $A(s_i, s_j)$ if $i \neq j$, and zero if $i = j$. Let $\hat{\Delta}_n(s, t)$ be the function defined by the formulae

$$\hat{\Delta}_n(s, t) = \frac{(-1)^{n-1}}{(n-1)!} \int_S \cdots \int_S \tilde{B}_n(s, t; s_1, \ldots, s_n)\mu(ds_1) \ldots \mu(ds_n),$$
$$n \geqq 2,$$

where $\tilde{B}_n(s, t; s_1, \ldots, s_n)$ is the determinant of the $(n+1)\times(n+1)$ matrix whose general element α_{ij} is given by the equations

$$\begin{aligned}
&\alpha_{1,1} = A(s, t); \alpha_{1,j} = A(s, s_{j-1}), && n+1 \geqq j > 1,\\
&\alpha_{j,1} = A(s_{j-1}, t), && n+1 \geqq j > 1,\\
&\alpha_{i,j} = 0, && n+1 \geqq i = j > 1,\\
&\alpha_{i,j} = A(s_{i-1}, s_{j-1}), && n+1 \geqq i \neq j > 1.
\end{aligned}$$

Show that $\hat{\Delta}_n(s, t)$ satisfies (i) of Exercise 44, and hence represents an operator $\hat{\Delta}_n$ of Hilbert-Schmidt type. Show finally that if $\Delta_n(s, t)$ are the functions of the preceding exercise, then $\hat{\Delta}_n(s, t) = \Delta_n(s, t)$ for $\mu\times\mu$-almost all $[s, t]$, and

$$\hat{\delta}_n = \delta_n, \qquad n \geqq 2.$$

47 Let a number σ_1 be chosen arbitrarily, and let $d(\lambda) = \delta(\lambda)$ $\exp(-\lambda\sigma_1)$. Show that

$$d'(\lambda) = (-\sum_{n=1}^{\infty} \lambda^{n-1}\sigma_n)d(\lambda),$$

and that

$$d(\lambda) = \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{n!} \det(R_n),$$

where $R_0 = 1$, and where, for $n \geq 1$, R_n is the $n \times n$ matrix

$$R_n = \begin{pmatrix} \sigma_1 & n-1 & 0 & \cdots & 0 \\ \sigma_2 & \sigma_1 & n-2 & & 0 \\ \sigma_3 & \sigma_2 & \ddots & \ddots & \\ \vdots & & \ddots & \ddots & 1 \\ \sigma_n & & & \sigma_2 & \sigma_1 \end{pmatrix}.$$

Show that the operator-valued function $D(\lambda) = d(\lambda)\big(I - (A/\lambda)\big)^{-1}$ is analytic in λ, and that $D(\lambda) = \sum_{n=0}^{\infty} D_n \lambda^n$, where $D_0 = 0$, $D_1 = d_0 I$, and where for $n \geq 2$, D_n is the operator defined by the symbolic determinant

$$\frac{(-1)^n}{n!} \det \left(\begin{array}{c|c} I & n-1 \quad 0 \ldots 0 \\ \hline A & \\ \vdots & R_{n-1} \\ A^{n-1} & \end{array} \right).$$

48 (Fredholm Determinant Series) Let the hypotheses of the preceding exercise be satisfied, and suppose that the Hilbert space of that exercise is $L_2(S, \Sigma, \mu)$, where (S, Σ, μ) is a positive measure space. Let $A(\cdot, \cdot)$ be a kernel representing the operator A in the sense of Exercise 44. Suppose that $A(\cdot, \cdot)$ is chosen in such a way that the formula $f(s) = A(s, s)$ defines a μ-measurable, μ-integrable function. Suppose that the number σ_1 of the preceding exercise is chosen to be

$$\sigma_1 = \int_S A(s, s)\mu(ds).$$

(a) Then the numbers $d_n = \det(R_n)$ of the prereding exercise are given by the formulae $d_0 = 1$,

$$d_n = \int_S \cdots \int_S C(s_1, \ldots, s_n)\mu(ds_1) \ldots \mu(ds_n), \qquad n \geq 1,$$

where $C(s_1, \ldots, s_n)$ is the determinant of the $n \times n$ matrix whose general element is $A(s_i, s_j)$.

(b) The formula

$$D_n(s, t) = \int_S \cdots \int_S C'(s, t; s_1, \ldots, s_n)\mu(ds_1) \ldots \mu(ds_n), \qquad n \geqq 2,$$

where $C'(s, t; s_1, \ldots, s_n)$ is the determinant of the $(n+1) \times (n+1)$ matrix whose elements are given by the formulae

$$\alpha_{11} = A(s, t);$$

$$\alpha_{1,j} = A(s, s_{j-1}), \quad n+1 \geqq j > 1; \quad \alpha_{j,1} = A(s_{j-1}, t), \quad n+1 \geqq j > 1;$$
$$\alpha_{i,j} = A(s_{i-1}, s_{j-1}), \qquad n+1 \geqq i, j > 1;$$

determines a kernel satisfying (i) of Exercise 44, which represents the operator $D_n - d_{n-1}I$ of the preceding exercise. Moreover, the series

$$D(s, t; \lambda) = \sum_{n=2}^{\infty} \frac{(-\lambda)^n}{(n-1)!} D_n(s, t)$$

converges for all λ, for $\mu \times \mu$-almost all $[s, t]$, and $D(s, t; \lambda)$ is the kernel which represents the operator $D(\lambda) - \lambda d(\lambda)I$ of the preceding exercise, in the sense of Exercise 44.

Show, finally, that by choosing $A(s, s) = 0$ for all s in S, we obtain the result of Exercise 46 as a special case of the present result. (Hint: Generalize the method of Exercise 46.)

49 The operator A of Hilbert-Schmidt class is said to be of *trace class* if $\sum_{i=1}^{\infty} |\lambda_i| < \infty$. The trace $\operatorname{tr}(A)$ of an operator of trace class is defined to be $\operatorname{tr}(A) = \sum_{i=1}^{\infty} \lambda_i$. Prove the following statements.

(a) If the operator A of Exercise 47 is of trace class, and we take $\alpha_1 = \operatorname{tr}(A)$ in that exercise, then $d(\lambda) = \prod_{i=1}^{\infty}(1-\lambda_i\lambda)$, the infinite product converging absolutely and uniformly for λ in any bounded set.

(b) The product A of two operators A_1 and A_2 of Hilbert-Schmidt class is of trace class; moreover, $\operatorname{tr}(A) = \operatorname{tr}(A_1, A_2)$.

(c) Let the operators A, A_1, and A_2 in (b) be operators in $L_2(S, \Sigma, \mu)$, where (S, Σ, μ) is a positive measure space, and let $A_1(\cdot, \cdot)$, $A_2(\cdot, \cdot)$ be kernels representing A_1, A_2 respectively in the sense of Exercise 44. Then the kernel $A(\cdot, \cdot)$ defined by

$$A(s, t) = \int_S A_1(s, r)A_2(r, t)\mu(dr)$$

for each s and t for which the integral on the right exists, represents the operator A in the sense of Exercise 44. Moreover,

$$\operatorname{tr}(A) = \int_S A(s, s)\mu(ds).$$

(d) If $(AA^*)^{\frac{1}{2}}$ is of trace class, A is of trace class. (Hint: For (d), use Weyl's inequality, Exercise 30.)

E. *Miscellaneous Exercises*

50 (Halberg) Let (S, Σ, μ) be a σ-finite measure space. Let T_p be a 1-parameter family of bounded operators defined in a subinterval I of the parameter interval $1 \leq p \leq \infty$, each operator T_p acting in the space $L_p(S, \Sigma, \mu)$. Suppose that for p_1, p_2 in I, T_{p_1} and T_{p_2} always agree on the intersection of $L_{p_1}(S, \Sigma, \mu)$ and $L_{p_2}(S, \Sigma, \mu)$. Prove that $\log |\sigma(T_p)|$ is a convex function of p.

51 Let the hypotheses of Exercise 50 be satisfied. Show that

$$\sigma(T_{p_2}) \subseteq \sigma(T_{p_1}) \cap \sigma(T_{p_3}) \qquad \text{if} \qquad p_1 \leqq p_2 \leqq p_3; \qquad p_1, p_2, p_3 \in I.$$

52 Let the hypotheses of Exercise 50 be satisfied. Show that if p_1 and p_2 are in I, then any component of $\sigma(T_{p_1})$ intersects $\sigma(T_{p_2})$.

53 Let the hypotheses of Exercise 50 be satisfied, and suppose in addition that the number 2 is in I and that T_2 is Hermitian. Show that $\sigma(T_2) \subset \sigma(T_p)$ for every p in I.

54 Let the hypotheses of Exercise 50 be satisfied, and suppose in addition that (S, Σ, μ) is finite. Let p, q be in I, $p < q$, and $\lambda \notin \sigma(T_p)$. Then $\lambda \notin \sigma(T_q)$ if and only if $(\lambda I - T_p)(L_p - L_q) \subseteq L_p - L_q$.

55 Let the hypotheses of Exercise 50 be satisfied, and suppose in addition that S is the set of integers, and that each point in S has measure 1. Let p, q be in I, $p > q$, and $\lambda \notin \sigma(T_p)$. Then $\lambda \notin \sigma(T_q)$ if and only if $(\lambda I - T_p)(L_p - L_q) \subset L_p - L_q$.

56 (Schmidt) Let the hypotheses of Exercise 44 be satisfied, and suppose that the operator A is Hermitian. Let $\{\varphi_i\}$ be an enumeration of the normalized eigenfunctions of A, and $\{\mu_i\}$ an enumeration of the corresponding eigenvalues. Show that

$$A(s, t) = \sum_i \mu_i \varphi_i(s)\varphi_i(t),$$

the series converging in the topology of $L_2((S, \Sigma, \mu) \times (S, \Sigma, \mu))$.

57 Let S be a compact space, and (S, Σ, μ) a finite regular measure space. Let $K(s, t)$ be a continuous function of the variable

$[s, t]$ in $S \times S$, and suppose that

$$K(s, t) = \overline{K(t, s)},$$

so that

$$(Kf)(s) = \int K(s, t)f(t)\mu(dt)$$

defines a compact operator in $L_2(S, \Sigma, \mu)$. Let $\{\varphi_i\}$ be an enumeration of the eigenfunctions of K, and $\{\mu_i\}$ an enumeration of the corresponding eigenvalues. Show that if $g = Kf$ for some f in $L_2(S, \Sigma, \mu)$, then g is continuous and the eigenfunction expansion of g converges uniformly and unconditionally.

58 (Mercer) Let the hypotheses of the preceding exercise be satisfied, and suppose that the operator K is non-negative. Show that

$$K(s, t) = \sum_i \mu_i \varphi_i(s)\overline{\varphi_i(t)},$$

the series converging uniformly. (Hint: Show that $K(t, t) = \sum_i \mu_i |\varphi_i(t)|^2$, and hence prove that the latter series converges uniformly. Use Exercises 57 and 56.)

59 Let φ_n be an orthonormal set of functions in the Hilbert space $L_2(S, \Sigma, \mu)$ with $|\varphi_n(s)| \leqq M < \infty$ for s in S and $n = 1, 2, \ldots$. For each f in L_1, put $c_n = \int_S f(s)\overline{\varphi_n(s)}\mu(ds)$. Show that for each p with $1 < p \leqq 2$, there exists a finite constant K_p such that

$$\left(\sum_{n=1}^{\infty} |c_n|^p n^{p-2}\right)^{1/p} \leqq K_p \left\{\int_0^1 |f(s)|^p \mu(ds)\right\}^{1/p}.$$

(Hint: Use the interpolation theorem of Marcinkiewicz stated in the section of notes and comments concluding the present chapter.)

9. The Classes C_p of Compact Operators. Generalized Carleman Inequalities

If T is a compact operator in Hilbert space, the non-negative self adjoint operator T^*T is compact (Corollary VI.5.5); thus, by Corollary X.3.5, Corollary VI.5.5, and Corollary X.2.8, $A = (T^*T)^{1/2}$ is also a compact non-negative self adjoint operator. The eigenvalues

$\mu_1, \mu_2, \ldots$ of A, arranged in decreasing order and repeated according to multiplicity, form, by Theorem VII.4.5, a sequence of numbers approaching zero. These numbers are called the *characteristic numbers* of the operator T; we write $\mu_n(T)$ for the nth characteristic number of T.

In terms of these characteristic numbers, we may define various norms for and classes of compact operators.

1 Definition. (a) $|T|_p = \{\sum_{n=1}^{\infty}(\mu_n(T))^p\}^{1/p}$, $\infty > p > 0$;

(b) $|T|_\infty = \sup_{1\leq n<\infty}|\mu_n(T)| = |\mu_1(T)| = |T|$;

(c) C_p is the set of all compact operators T such that $|T|_p$ is finite.

The final equality in (b) follows from Theorem X.2.1 and Lemma IX.8.2. The basic properties of the characteristic numbers $\mu_n(T)$ are stated in the following lemma and corollaries.

2 Lemma. *The characteristic numbers $\mu_n(T)$ of a compact operator are given by the following formula:*

$$\mu_{n+1}(T) = \min_{\varphi_1,\ldots,\varphi_n} \max_{\substack{|\varphi|=1\\(\varphi,\varphi_1)=\cdots=(\varphi,\varphi_n)=0}} |T\varphi|, \qquad n \geqq 0.$$

Proof. This formula may be written

$$(\mu_n(T))^2 = \min_{\varphi_1,\ldots,\varphi_n} \max_{\substack{|\varphi|=1\\(\varphi,\varphi_1)=\cdots=(\varphi,\varphi_n)=0}} |T\varphi|^2.$$

Since $|T\varphi|^2 = (T\varphi, T\varphi) = (T^*T\varphi, \varphi)$, we see our lemma to be a special case of the "minimax formula" for the eigenvalues of a compact operator, given as Theorem X.4.3. Q.E.D.

It will be convenient in what follows to acopt the formula of Lemma 2 as a definition of $\mu_n(T)$ in case T is not compact. Note that $|T| = \mu_1(T)$ quite generally by this definition.

3 Corollary. *The characteristic numbers of a compact or non-compact operator satisfy the inequalities*

$$\mu_{n+m+1}(T_1+T_2) \leqq \mu_{n+1}(T_1)+\mu_{m+1}(T_2)$$

$$\mu_{n+m+1}(T_1T_2) \leqq \mu_{n+1}(T_1)\mu_{m+1}(T_2).$$

PROOF. We observe that

$$\begin{aligned}\min_{\varphi_1,\ldots,\varphi_{n+m}} \max_{\substack{|\varphi|=1\\(\varphi,\varphi_1)=\ldots=(\varphi,\varphi_{n+m})=0}} |(T_1+T_2)\varphi| &\leq \min_{\varphi_1,\ldots,\varphi_{n+m}} \max_{\substack{|\varphi|=1\\(\varphi,\varphi_1)=\ldots=(\varphi,\varphi_{n+m})=0}} (|T_1\varphi|+|T_2\varphi|)\\ &\leq \min_{\varphi_1,\ldots,\varphi_n} \max_{\substack{|\varphi|=1\\(\varphi,\varphi_1)=\ldots=(\varphi,\varphi_n)=0}} |T_1\varphi|\\ &\quad+ \min_{\varphi_{n+1},\ldots,\varphi_{n+m}} \max_{\substack{|\varphi|=1\\(\varphi,\varphi_{n+1})=\ldots=(\varphi,\varphi_{n+m})=0}} |T_2\varphi|,\end{aligned}$$

proving the first assertion.

Similarly,

$$\begin{aligned}&\min_{\varphi_1,\ldots,\varphi_{n+m}} \max_{\substack{|\varphi|=1\\(\varphi,\varphi_1)=\ldots=(\varphi,\varphi_{n+m})=0}} |T_1T_2\varphi|\\ &\leq \min_{\varphi_1,\ldots,\varphi_{n+m}} \max_{\substack{|\varphi|=1\\(\varphi,T_2^*\varphi_1)=\ldots=(\varphi,T_2^*\varphi_n)=0\\(\varphi,\varphi_{n+1})=\ldots=(\varphi,\varphi_{n+m})=0}} |T_1T_2\varphi|\\ &= \min_{\varphi_1,\ldots,\varphi_{n+1}} \max_{\substack{(T_2\varphi,\varphi_1)=\ldots=(T_2\varphi,\varphi_n)=0\\(\varphi,\varphi_{n+1})=\ldots=(\varphi,\varphi_{n+m})=0}} \left(\frac{|T_1(T_2\varphi)|}{|T_2\varphi|}\right)\left(\frac{|T_2\varphi|}{|\varphi|}\right)\\ &\leq \Big\{\min_{\varphi_1,\ldots,\varphi_n} \max_{\substack{|\psi|=1\\(\psi,\varphi_1)=\ldots=(\psi,\varphi_n)=0}} |T_1\psi|\Big\}\Big\{\min_{\varphi_{n+1},\ldots,\varphi_{n+m}} \max_{\substack{|\varphi|=1\\(\varphi,\varphi_{n+1})=\ldots=(\varphi,\varphi_{n+m})=0}} |T_2\varphi|\Big\},\end{aligned}$$

proving the second assertion. Q.E.D.

4 COROLLARY. (a) $|\mu_n(T_1)-\mu_n(T_2)| \leq |T_1-T_2|$.

(b) $\mu_n(TA) \leq \mu_n(T)|A|$; $\mu_n(AT) \leq |A|\mu_n(T)$.

(c) $\mu_n(TU) = \mu_n(T)$ *if* $1 = |U|$ *and* $1 = |U^{-1}|$, *and, in particular, if* U *is unitary.*

PROOF. Assertions (a) and (b) are special cases of the statements of Corollary 3; assertion (c) follows at once from (b). Q.E.D.

Statement (a) of the preceding corollary enables us to prove various results by approximation of general compact operators with compact operators having finite-dimensional range. This will be a main technique of the present section. The following lemma gives a

useful auxiliary statement for the application of this process, showing how the eigenvalues of an operator behave in such a process of approximation.

5 LEMMA. *Let T_n, T be compact operators, and let $T_n \to T$ in the uniform operator topology. Let $\lambda_m(T)$ be an enumeration of the non-zero eigenvalues of T, each repeated according to its multiplicity. Then there exist enumerations $\lambda_m(T_n)$ of the non-zero eigenvalues of T_n, with repetitions according to multiplicity, such that*

$$\lim_{n\to\infty} \lambda_m(T_n) = \lambda_m(T), \qquad m \geqq 1,$$

the limit being uniform in m.

PROOF. Choose a decreasing sequence ε_k of numbers approaching zero such that the periphery of the disc C_{ε_k} with radius ε_k centered at the origin lies wholly in the resolvent set of the compact operator T. Find a decreasing sequence of numbers $\delta_k < \varepsilon_k$ such that the circles of radius δ_k centered at those points of $\sigma(T)$ outside C_{ε_k} are non-overlapping. Then, by Lemmas VII.6.4, VII.6.5, and VII.6.7, there exists an increasing sequence n_k of integers such that for $n \geqq n_k$ each point in $\sigma(T_n)$ lies either in C_{ε_k} or within a distance δ_k of a point in $\sigma(T)$. Moreover, if the points of $\sigma(T_n)$ are repeated according to multiplicity, then the number of points in $\sigma(T_n)$ lying within δ_k of a point $\lambda \in \sigma(T) - C_{\varepsilon_k}$ is precisely equal to the multiplicity of λ.

For $n_k \leqq n < n_{k+1}$, enumerate the points of $\sigma(T_n)$ as follows:

(i) Arrange the points $\bar{\lambda}_1, \ldots, \bar{\lambda}_l$ in $\sigma(T) - C_{\varepsilon_k}$ in the order in which they occur in the sequence $\lambda_m(T)$. Enumerate first all the points in $\sigma(T_n)$ lying within δ_k of $\bar{\lambda}_1$, next all the points in $\sigma(T_n)$ lying within δ_k of $\bar{\lambda}_2$, and finally all the points lying within δ_k of $\bar{\lambda}_l$. In each case, repetition should be made according to multiplicity.

(ii) Enumerate the points of $\sigma(T_n) \cap C_{\varepsilon}$, repeated according to multiplicity, in any convenient way.

Now choose any positive ε and M. It is plain that we may find k so large that $\varepsilon_k < \varepsilon$ and $\lambda_1(T), \ldots, \lambda_M(T)$ all lie outside C_{ε_k}, while $|\lambda_j(T)| + \delta_k < \varepsilon$ for $j > M$. Let $n > n_k$. By the above construction, $|\lambda_j(T_n) - \lambda_j(T)| < \delta_k$ for $j \leqq M$, while both $|\lambda_j(T_n)|$ and $|\lambda_j(T)|$ are bounded by ε for $j > M$. Thus the lemma is proved. Q.E.D.

6 Lemma. *Let T be a compact operator, and $\lambda_n(T)$ an enumeration of its eigenvalues, repeated according to multiplicity, and in decreasing order of absolute values. (If there are only a finite number N of non-zero eigenvalues, we write $\lambda_n(T) = 0$, $n > N$). Then, for each positive integer m*

(a) $|\lambda_1(T) \ldots \lambda_m(T)| \leqq |\mu_1(T) \ldots \mu_m(T)|$;

(b) $\sum_{i=1}^{m} |\lambda_i(T)|^p \leqq \sum_{i=1}^{m} |\mu_i(T)|^p$;

(c) $\mu_m(T) = \mu_m(T^*)$.

Proof. We group these three loosely related statements together because of the similarity of their proofs. By Lemma 5 and Corollary 4, and the elementary fact that any compact operator may be approximated in norm by a sequence of operators T_n with finite-dimensional range, it is enough to prove the lemma in the special case that T has finite-dimensional domain and range.

Note that if T has finite-dimensional range, $T = ET$, where E is the orthogonal projection on the range of T. Thus $T^* = T^*E^*$, so that T^* also has finite-dimensional range. This remark will be used implicitly from time to time in what follows.

Let $\mathfrak{S}$ be a finite-dimensional space including both the range of T and the range of T^*; suppose that the dimension of $\mathfrak{S}$ is d. Then, plainly, $\mathfrak{S}$ is invariant under T and T^*, and, since $(T\mathfrak{S}^\perp, x) = (\mathfrak{S}^\perp, T^*x) = 0$ for all x, we have $T\mathfrak{S}^\perp = 0$ and similarly $T^*\mathfrak{S}^\perp = 0$. Thus, it is easily seen that

$$\begin{aligned}
&\lambda_n(T) = \lambda_n(T|\mathfrak{S}), \quad 1 \leqq n \leqq d; \quad \lambda_n(T) = 0, \quad n > d;\\
(1)\quad &\mu_n(T) = \mu_n(T|\mathfrak{S}), \quad 1 \leqq n \leqq d; \quad \mu_n(T) = 0, \quad n > d;\\
&\mu_n(T^*) = \mu_n((T|\mathfrak{S})^*), \quad 1 \leqq n \leqq d; \quad \mu_n(T^*) = 0, \quad n > d.
\end{aligned}$$

Consequently, it is enough to prove (a), (b), and (c) for operators in *finite-dimensional* Hilbert space.

In this case, (a) and (b) are known inequalities of Weyl, given in Section 8 as Exercises 8.31 and 8.32. To prove (c), note that any finite-dimensional operator can be approximated arbitrarily closely by non-singular operators; thus, without loss of generality, we may assume T non-singular. Let $U = (T^*T)^{1/2}T^{-1}$. Then plainly U is non-

singular, and $U^*U = (T^*)^{-1}T^*TT^{-1} = I$, so that U is unitary. We have $UT = (T^*T)^{1/2}$; thus $UTT^*U^{-1} = T^*T$, so that $U(TT^*)^{1/2}U^{-1} = (T^*T)^{1/2}$. Since unitarily-equivalent operators have the same eigenvalues, (c) follows. Q.E.D.

7 COROLLARY. *If $T \in C_p$, $\infty > p > 0$, then the series $\sum_{i=1}^{\infty}(\lambda_i(T))^p$ converges absolutely and*

$$\sum_{i=1}^{\infty} |\lambda_i(T)|^p \leqq \sum_{i=1}^{\infty} |\mu_i(T)|^p.$$

PROOF. This is clear from (b) of the preceding lemma. Q.E.D.

8 COROLLARY. *If $T \in C_1$, $\infty > p > 0$, then the series* $\text{tr}(T) = \sum_{i=1}^{\infty}\lambda_i(T)$ *converges absolutely and*

$$|\text{tr}(T)| \leqq |T|_1.$$

Remark. The expression $\text{tr}(T)$ is called the *trace* of T.

The following lemma states some useful elementary properties of the spaces C_p. The norm inequalities stated in the lemma are inexact for the range $1 \leqq p < \infty$ and will be improved a little later in our discussion.

9 LEMMA. (a) *We have $C_p \subseteq C_{p'}$ if $p \leqq p'$; $|T|_p$ decreases as p increases.*

(b) *If T_1, T_2 are in C_p, then T_1+T_2 is in C_p and*

$$|T_1+T_2|_p \leqq 2^{1/p}|T_1|_p+2^{1/p}|T_2|_p, \qquad p \geqq 1,$$
$$|T_1+T_2|_p^p \leqq 2|T_1|_p^p+2|T_2|_p^p, \qquad 0 < p \leqq 1.$$

(c) *If T_1 is in C_{r_1} and T_2 is in C_{r_2}, then T_1T_2 is in C_r, where* $1/r_1+1/r_2 = 1/r$. *Moreover*

$$|T_1T_2|_r \leqq 2^{1/r}|T_1|_{r_1}|T_2|_{r_2}, \qquad 0 < r < \infty.$$

(d) *If T is in C_r and A is bounded, then AT and TA are in C_r; moreover,*

$$|AT|_r \leqq |A|\,|T|_r; \qquad |TA|_r \leqq |T|_r|A|.$$

(e) *C_2 is the Hilbert-Schmidt class of operators, and $|T|_2 = \|T\|$ for T in C_2.*

Proof. Part (d) follows at once from (b) of Corollary 4. Since, if $\{\varphi_i\}$ is the complete set of eigenvectors of T^*T, we have

$$|T|_2^2 = \sum_{n=1}^{\infty} (\mu_n(T))^2 = \sum_{n=1}^{\infty} |T\varphi_n|^2 = \|T\|^2.$$

This proves (e).

Statement (a) is evident from Definition 1.

To prove (b), we put $T = T_1+T_2$ and note that, by Corollary 3,

$$\mu_{2n+1}(T_1+T_2) \leqq \mu_{n+1}(T_1)+\mu_{n+1}(T_2)$$
$$\mu_{2n+2}(T_1+T_2) \leqq \mu_{n+1}(T_1)+\mu_{n+2}(T_2).$$

First let $p \geqq 1$. Then by Minkowski's inequality,

$$\left(\sum_{n=0}^{\infty} |\mu_{2n+1}(T)|^p\right)^{1/p} \leqq \left\{\sum_{n=0}^{\infty} |\mu_{n+1}(T_1)|^p\right\}^{1/p} + \left\{\sum_{n=0}^{\infty} |\mu_{n+1}(T_2)|^p\right\}^{1/p}$$
$$\leqq |T_1|_p + |T_2|_p.$$

In the same way

$$\left\{\sum_{n=0}^{\infty} |\mu_{2n+2}(T)|^p\right\}^{1/p} \leqq |T_1|_p + |T_2|_p.$$

Thus the first assertion of (b) follows on addition. The assertion of (b), for the range $0 < p \leqq 1$, follows in just the same way on replacing Minkowski's inequality by the elementary inequality $|x|^p+|y|^p \geqq |x+y|^p$, valid in this range of p.

Similarly, using Corollary 3, and the Hölder inequality

$$\left\{\sum |\alpha_i\beta_i|^r\right\}^{1/r} \leqq \left\{\sum |\alpha_i|^{r_1}\right\}^{1/r_1} \left\{\sum |\beta_i|^{r_2}\right\}^{1/r_2},$$

valid if $r_1^{-1}+r_2^{-1} = r^{-1}$ and $0 < r_1, r_2, r < \infty$, we obtain (c). Q.E.D.

The slightly unorthodox "triangle inequality" given in (b) of the preceding lemma does not prevent us from using our "norms" to define a topology for C_p. A set $U \subseteq C_p$ is to be called *open* if for every T in U there is an $\varepsilon > 0$ such that $\{T' \mid |T'-T| < \varepsilon\} \subseteq U$. It follows at once from the preceding lemma that $T \to T^*$ is a continuous mapping of C_p into itself, that $T \to T$ is a continuous mapping of C_p into $C_{p'}$ if $p' > p$, that $[T_1, T_2] \to T_1+T_2$ is a continuous mapping of $C_p \times C_p$ into C_p, and that $[T_1, T_2] \to T_1T_2$ is a continuous mapping of $C_p \times C_q$ into C_r, where $r^{-1} = p^{-1}+q^{-1}$. What is missing of the

ordinary properties of a metric space is the assertion that $|T|_p$ is a continuous function on C_p. Later, when we improve Lemma 9(b) by removing the superfluous constant 2, even this will follow.

The space C_p has the completeness property expressed in the following lemma.

10 LEMMA. *If $T_n \in C_p$ is a sequence of operators such that $|T_n - T_m|_p \to 0$ as $m, n \to \infty$, there exists a compact operator T such that $T_n \to T$ (in the topology of C_p) as $n \to \infty$.*

PROOF. By Lemma 9(a) and the fact that the family of compact operators is closed in the uniform topology of operators, (Corollary VI.5.5), there exists a compact operator T such that $T_n \to T$ in the uniform topology. Thus, by Corollary 4(a), $\lim_{m\to\infty} \mu_k(T_n - T_m) = \mu_k(T_n - T)$. It follows that

$$\left\{\sum_{k=1}^{N} |\mu_k(T_n - T)|^p\right\}^{1/p} \leq \limsup_{m\to\infty} \left\{\sum_{k=1}^{\infty} |\mu_k(T_n - T_m)|^p\right\}^{1/p}$$
$$= \limsup_{m\to\infty} |T_n - T_m|_p.$$

Therefore, letting $N \to \infty$, we find

$$|T_n - T|_p \leq \limsup_{m\to\infty} |T_n - T_m|_p,$$

so that

$$\lim_{n\to\infty} |T_n - T|_p \leq \lim_{m,n\to\infty} |T_n - T_m|_p = 0.$$

Q.E.D.

The following easy lemma will be useful in the sequel.

11 LEMMA. *Let T be compact. Then there exists a sequence T_n of compact operators having finite-dimensional ranges, such that*

(a) *$T_n \to T$ in the uniform topology as $n \to \infty$;*

(b) *$|T_n - T|_p \to 0$ as $n \to \infty$ if $T \in C_p$;*

(c) *$|T_n|_p \to |T|_p$ as $n \to \infty$ if $T \in C_p$.*

PROOF. Let $\varphi_1, \varphi_2, \ldots$ be an orthonormal basis for Hilbert space consisting of eigenvectors of T^*T corresponding to the eigenvalues $(\mu_1(T))^2, (\mu_2(T))^2, \ldots$ of this operator. Let E_n be the projection on the space spanned by $\varphi_1, \ldots, \varphi_n$ and let $E_n' = I - E_n$,

Let $T_n = TE_n$ and let $T'_n = TE'_n$. Then it is clear that T_n has finite-dimensional range and $T = T_n + T'_n$.

Since $E'_n \to 0$ strongly, $|E'_n x| \to 0$ uniformly for x in any compact set. Thus, $E'_n T^* x \to 0$ uniformly for x in any bounded set and $|E'_n T^*| = |T'_n| \to 0$. This proves (a).

Since $T_n^* T_n = E_n T^* T E_n = T^* T E_n$, it is plain that

$$\mu_m(T_n) = \mu_m(T), \quad m \leqq n, \qquad \mu_m(T_n) = 0, \quad m > n.$$

Thus (c) is obvious. Since it is plain by an exactly similar argument that

$$\mu_m(T'_n) = \mu_{m+n}(T), \qquad m \geqq 1,$$

(b) is equally obvious. Q.E.D.

We shall ultimately need to demonstrate the continuity (or, quite equivalently, the additivity) of the function $\operatorname{tr}(T)$ of Corollary 8. The proof, unfortunately, involves the same delicate point that gave us difficulty on the road to Theorems XI.6.24 and XI.6.25. We shall surmount this obstacle by a complex-variable argument of much the same sort.

12 Lemma. *If T_1 and T_2 are operators with finite-dimensional range, then* $\operatorname{tr}(T_1+T_2) = \operatorname{tr}(T_1)+\operatorname{tr}(T_2)$. *Hence, for two such operators,* $|\operatorname{tr}(T_1-T_2)| \leqq |T_1-T_2|_1$.

Proof. Let $\mathfrak{S}$ be a finite-dimensional subspace of Hilbert space including the ranges of T_1, T_2, T_1^* and T_2^*. Put $T_3 = T_1+T_2$. It follows, by arguments such as those given in the third paragraph of the proof of Lemma 6 that $\operatorname{tr}(T_1) = \operatorname{tr}(T_1|\mathfrak{S})$; $\operatorname{tr}(T_2) = \operatorname{tr}(T_2|\mathfrak{S})$; $\operatorname{tr}(T_3) = \operatorname{tr}(T_3|\mathfrak{S})$. Thus the first assertion of our lemma follows from the corresponding finite-dimensional assertion (cf. Definition XI.6.8, Lemma XI.6.10). The second assertion follows from the first, from the elementary equation $\operatorname{tr}(\alpha T) = \alpha \operatorname{tr}(T)$, and from Corollary 8. Q.E.D.

Lemma 12 makes it clear that $\operatorname{tr}(T)$ has a unique extension by continuity from the dense subset of finite-dimensional operators in C_1 to the whole of C_1. We call this extension $\widetilde{\operatorname{tr}}(T)$. The next main landmark of our argument will be the proof that $\widetilde{\operatorname{tr}}(T)$ and $\operatorname{tr}(T)$ are identical. We first state some easily proved properties of $\widetilde{\operatorname{tr}}(T)$ in the following lemma.

13 LEMMA. (a) *The function* $\widetilde{\mathrm{tr}}(T)$ *is linear and continuous on the space* C_1.

(b) *We have*

$$\widetilde{\mathrm{tr}}(T) = \sum_{i=1}^{\infty} (T\varphi_i, \varphi_i),$$

where $\{\varphi_i\}$ *is any orthonormal basis and the series converges absolutely.*

PROOF. Statement (a) follows immediately from the preceding lemma and from the definition of $\widetilde{\mathrm{tr}}(T)$. To prove the absolute convergence of the series in (b), let $\psi_1, \psi_2, \ldots$ be an orthonormal basis for Hilbert space consisting of a sequence of eigenvectors of $(TT^*)^{1/2}$ corresponding to the eigenvalues $\mu_1(T), \mu_2(T), \ldots$ of this latter operator, so that $|T^*\psi_j| = \mu_j(T)$. Then

$$\begin{aligned}
\sum_{i=1}^{\infty} |(T\varphi_i, \varphi_i)| &\leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |(T\varphi_i, \psi_j)||(\psi_j, \varphi_i)| \\
(*) \qquad &\leq \sum_{j=1}^{\infty} \left\{ \sum_{i=1}^{\infty} |(T^*\psi_j, \varphi_i)|^2 \right\}^{1/2} \left\{ \sum_{i=1}^{\infty} |(\psi_j, \varphi_i)|^2 \right\}^{1/2} \\
&= \sum_{j=1}^{\infty} \mu_j(T) = |T|_1.
\end{aligned}$$

This demonstrates the convergence of the expression on the right of (b); since that expression is linear, the inequality $(*)$ also implies its continuous dependence on T. Thus, using Lemma 11, (b) will follow generally if we establish it for an operator T with finite-dimensional range. Such an operator is the sum of a finite collection of operators with one-dimensional range. Thus, without loss of generality, we may assume T to have a one-dimensional range, i.e., we may assume that T has the form $x \to (x, v)u$. It is easy to compute that $\mathrm{tr}(T) = (u, v)$ for such an operator; thus the formula of (b) reduces to the formula

$$(u, v) = \sum_{i=1}^{\infty} (u, \varphi_i)(\varphi_i, u)$$

which we know to be an elementary consequence of the completeness of the orthonormal basis $\{\varphi_i\}$. Q.E.D.

The next lemma will allow us to sharpen the norm inequalities of Lemma 9, insofar as these pertain to the spaces C_p with $\infty \geq p \geq 1$.

14 LEMMA. (a) *Let* $\infty \geqq p \geqq 1$ *and let* $q^{-1}+p^{-1} = 1$. *Let* C^0 *denote the set of non-zero operators with finite-dimensional ranges. Then, if* A *is in* C_p,

$$|A|_p = \sup_{B\in C^0} \frac{|\widetilde{\mathrm{tr}}(AB)|}{|B|_q}. \tag{1}$$

(b) *Let* p *and* q *be as above, let* A *be in* C_p *and* B *in* C_q. *Then* AB *and* BA *are in* C_1,

$$\widetilde{\mathrm{tr}}(AB) = \widetilde{\mathrm{tr}}(BA), \tag{2}$$

and $|\widetilde{\mathrm{tr}}(AB)| \leqq |A|_p|B|_q$.

(c) *Let* p, q, A, *and* B *be as above. Then*

$$|AB|_1 \leqq |A|_p|B|_q.$$

(d) *Let* p *be as above and* A, A_1 *be in* C_p. *Then*

$$|A+A_1|_p \leqq |A|_p+|A_1|_p.$$

PROOF. It is clear from formula (1) that (d) will follow from (a). Similarly, if (a) and (b) are proved, we have

$$\begin{aligned}|AA_1|_1 = \sup_{B\in C^0} \frac{|\mathrm{tr}(AA_1B)|}{|B|} &\leqq \sup_{B\in C^0} \frac{|A|_p|A_1B|_q}{|B|}\\ &\leqq |A|_p|A_1|_q,\end{aligned}$$

by Lemma 9(d). Thus, only (a) and (b) need be proved.

Suppose that (a) is known in the special case in which A has a finite-dimensional range. For general A, we may use Lemma 11 to find a sequence $A_n \to A$ in the uniform topology and in the topology of C_p, such that A_n has finite-dimensional range. If $B \in C^0 \subseteq C_1$, it follows from the continuity of the mapping $[A, B] \to AB$ of $C_\infty \times C_1 \to C_1$ and the continuity of $\widetilde{\mathrm{tr}}$ on C_1 that

$$\begin{aligned}|\widetilde{\mathrm{tr}}(AB)| = \lim_{n\to\infty} |\widetilde{\mathrm{tr}}(A_nB)| &\leqq \lim_{n\to\infty} |A_n|_p|B|_q\\ &\leqq |A|_p|B|_q.\end{aligned}$$

Conversely, there exists a sequence $\{B_n\}$ in C^0 such that $|B_n|_q = 1$ and $|\widetilde{\mathrm{tr}}(A_nB_n)| \geqq |A_n|_p - 1/n$. Thus, since $AB_n - A_nB_n$ belongs to C^0

it follows from Lemma 12 and Lemma 9(c) that

$$|\widetilde{\text{tr}}(AB_n - A_nB_n)| \leqq |(A-A_n)B_n|_1 \leqq 2|A-A_n|_p$$

so that

$$|\widetilde{\text{tr}}(AB_n)| \geqq |A_n|_p - 2|A-A_n|_p - \frac{1}{n} \to |A|_p.$$

Hence the general validity of A would follow from the validity of A for operators with a finite-dimensional range. By elementary arguments such as those employed in the third paragraph of the proof of Lemma 6, which we leave to the reader to elaborate in detail, we may conclude that to establish (a) in general it is sufficient to consider the case in which the Hilbert space is finite-dimensional.

The argument in this special case is as follows. Since both sides of (1) are continuous in T and since every finite matrix may be approximated arbitrarily closely by non-singular matrices, it is sufficient to consider the case in which T is non-singular. Then $A = (TT^*)^{1/2}$ is also non-singular and if $U = A^{-1}T$, $UU^* = A^{-1}A^2A^{-1} = I$, then U is unitary, and $T = AU$. Let $B_0 = U^{-1}A^{p-1}$. Then $TB_0 = A^p$, so that $\text{tr}(TB_0) = \sum|\mu_i(T)|^p$. On the other hand $B_0B_0^* = U^{-1}A^{2(p-2)}U$, so that $\mu_i(B_0) = \mu_i(T)^{p-1}$, and

$$|B_0|_q = \{\sum|\mu_i(T)|^{pq-q}\}^{1/q} = \{\sum|\mu_i(T)|^p\}^{1-1/p}.$$

This shows that

$$\frac{|\text{tr}(TB_0)|}{|B_0|_q} = |T|_p,$$

and hence that the right side of formula (1) is not less than the left side of formula (1).

To prove the converse it suffices to show that

$$|\text{tr}(TB)| \leqq |T|_p|B|_q; \tag{3}$$

as above we may see that it is sufficient to prove this equation when both T and B are non-singular. Since a non-singular matrix has been shown above to have the form $T = AU$, where U is unitary and A is positive definite and Hermitian, and since by the spectral theorem A may be written as $A = VDV^*$, where D is a positive diagonal

matrix with the same eigenvalues as A, the statement (3) follows from the inequality

$$|\mathrm{tr}(VD_1V'D_2V'')| \leq |D_1|_p|D_2|_q.$$

Using the identity $\mathrm{tr}(SS') = \mathrm{tr}(S'S)$, it is enough to establish

$$|\mathrm{tr}(D_1 U D_2 \tilde{U})| \leq |D_1|_p|D_2|_q, \tag{4}$$

for each pair $U, \tilde{U}$ of unitary matrices, and each pair D_1, D_2 of positive diagonal matrices. The inequality (4) may be written as

$$\Big|\sum_{i,j} u_{ij}\tilde{u}_{ji}(d_1)_i(d_2)_j\Big| \leq \Big(\sum (d_1)_i^p\Big)^{1/p}\Big(\sum (d_2)_i^q\Big)^{1/q}.$$

This last inequality follows by the Riesz convexity theorem (VI.10.7) from its special cases $p = 1, q = \infty$, and $q = 1, p = \infty$, which are implied by the two evident inequalities

$$\sum_j |u_{ij}\tilde{u}_{ji}| \leq 1; \qquad \sum_i |u_{ij}\tilde{u}_{ji}| \leq 1$$

for the pair of unitary matrices $U, \tilde{U}$. Thus (1) is proved for the case $p \neq \infty$. The easy extension of this proof to the case $p = \infty$ is left to the reader.

Since the linear functional $\widetilde{\mathrm{tr}}$ is continuous on C_1, and since, as we have observed in the paragraph following Lemma 9, $[A, B] \to AB$ is a continuous mapping of $C_p \times C_q$ into C_1, it follows from Lemma 11 that to prove (b) in general, we have only to prove (b) in the special case in which A and B have finite-dimensional ranges. But then the inequality in (b) is plainly a special case of (a), while the identity (3) follows readily from the corresponding identity for operators in finite-dimensional spaces. This completes the proof of the present lemma. Q.E.D.

It follows immediately from the preceding lemma, from Lemma 9 (d), from the fact that $|A| \leq |A|_p$, and from Lemma 10 that for $p \geq 1$ the family C_p of operators, with the norm $|\cdot|_p$, is a complete B-algebra; if the Hilbert space is infinite-dimensional, this B-algebra has no unit. Thus, the argument used to prove Theorem 6.7 may with the slightest of adaptations be used to prove the following lemma.

15 Lemma. *If $p \geqq 1$, if $T \in C_p$, and if f is a single valued analytic function on the spectrum of T which vanishes at zero, then $f(T) \in C_p$ and the mapping $T \to f(T)$ of C_p into itself is continuous. Furthermore, if $\{f_n\}$ is a sequence of such functions having as domain a common neighborhood N of the spectrum of T and if $f_n(\lambda) \to f(\lambda)$ uniformly for λ in N then $f_n(T) \to f(T)$ in the topology of C_p.*

The modifications necessary to carry over the proof of Theorem 6.7 from the special case $p = 2$ to the general case $p \geqq 1$ are sufficiently slight that they can cause the reader no anxiety. We therefore omit to set them out.

Let $T \in C_1$. By Lemma 15, the mapping $T \to \log(I+zT)$ is defined whenever $-z^{-1} \notin \sigma(T)$ and depends continuously on T. Thus, the function $\det(1+zT) = \exp\left(\tilde{\text{tr}}(\log(I+zT))\right)$ is defined if $-z^{-1}$ lies in the resolvent set of T, and depends continuously on T. Since, if $-z^{-1} \notin \sigma(T)$,

$$\lim_{h\to 0} \frac{\log(1+(z+h)\zeta)-\log(1+z\zeta)}{h}$$

converges uniformly for ζ in a neighborhood of $\sigma(T)$, it follows by Lemma 15 that

$$\lim_{h\to 0} \frac{\log(I+(z+h)T)-\log(I+zT)}{h}$$

exists in the topology of C_1. Thus $\tilde{\text{tr}}(\log(I+zT))$ is an analytic function of z, defined for all z such that $-z^{-1} \notin \sigma(T)$. The following lemma states an important inequality for this function.

16 Lemma. *Let $T \in C_1$. Then*

(a) $\quad |\det(I+zT)| \leqq \prod_{n=1}^{\infty} (1+|z|\mu_n(T))$;

(b) *the function* $\det(1+zT)$ *is analytic in z for all z and has only removable singularities at the points z such that $-z^{-1} \in \sigma(T)$.*

Proof. The left side of formula (a) is continuous in T. In the course of proving Lemma 11, we showed how to construct a sequence of operators T_n with finite-dimensional range such that $|T_n - T|_1 \to 0$ and $\mu_m(T_n) = \mu_m(T)$ if $m \leqq n$, $\mu_m(T) = 0$ if $m > n$. Thus the inequality (a) will follow in general once we establish its validity

for operators T with finite-dimensional range. Arguing as previously along the lines of the third paragraph of the proof of Lemma 6, we may even say that T may be considered without loss of generality to be an operator in a Hilbert space of dimension $d < \infty$.

The formula $e^{\operatorname{tr} A} = \det(A)$ is valid for finite-dimensional matrices, $\det(A)$ denoting the determinant of A. Since the determinant of A is the product of its eigenvalues, it follows by Lemma 6(a) that we have

$$\begin{aligned} |\det(I+zT)| &= \prod_{n=1}^{d} |\lambda_n(I+zT)| \\ &\leq \prod_{n=1}^{d} \mu_n(I+zT) \\ &\leq \prod_{n=1}^{d} (1+|z|\mu_n(T)); \end{aligned}$$

which completes the proof of (a).

The determinant $\det(I+zT_n)$ is an analytic (and even a polynomial) function of z, if T_n operates in finite-dimensional space, and hence more generally if T_n has a finite-dimensional range. Thus, since a bounded convergent sequence of analytic functions converges to an analytic function, it follows that $\det(I+zT)$ is analytic if $-z^{-1} \notin \sigma(T)$. Since by (a) $\det(I+zT)$ is bounded, the singularities are removable and (b) is proved. Q.E.D.

Remark. Since, by the maximum modulus principle, a bounded sequence of analytic functions which converges at all but an isolated set of points converges everywhere, it follows that $\det(I+T)$ is a continuous function of $T \in C_1$ for all T, irrespective of whether $-1 \in \sigma(T)$ or not. It is equally evident that the preceding argument suffices to prove the following slightly more general lemma.

17 LEMMA. *Let T, $B \in C_1$. Then the function*

$$\det(I+T+zB)$$

is analytic in the complex variable z.

Now we are able to prove that $\operatorname{tr}(T) = \widetilde{\operatorname{tr}}(T)$. The following lemma takes a first step by demonstrating this equality in case T is quasi-nilpotent.

18 LEMMA. *If $N \in C_1$ is quasi nilpotent, then $\widetilde{\mathrm{tr}}(N) = 0$.*

PROOF. Since $\sigma(N) = \{0\}$, $-z^{-1} \notin \sigma(N)$ is true for all z. Thus $L(z) = \widetilde{\mathrm{tr}}(\log (I+zN))$ is an entire function of z. Choose k so large that $\sum_{n=k+1}^{\infty} \mu_n(N) < \varepsilon$. Then, from Lemma 16(a), it follows that

$$|\exp (L(z))| \leq \prod_{n=1}^{k} (1+|z|\mu_n(N)) \exp (|z| \sum_{n=k+1}^{\infty} \mu_n(N))$$
$$\leq \prod_{n=1}^{k}(1+|z|\mu_n(N)) \exp (\varepsilon|z|);$$

we have used the inequality $1+\alpha \leq e^{\alpha}$ for $\alpha > 0$. Consequently, $\mathscr{R}L(z) \leq 2\varepsilon|z|$ for sufficiently large $|z|$. Lemma 6.22 then implies that $|L(z)| < 8\varepsilon|z|$ for sufficiently large $|z|$, so that

$$\lim_{|z|\to\infty} \frac{|L(z)|}{|z|} = 0.$$

Thus $L(z)/z$ is analytic and vanishes at $z = \infty$. The Laurent series of $L(z)$ at $z = \infty$ is consequently

$$L(z) = a + \frac{b}{z} + \cdots,$$

so that L is bounded at infinity. By Liouville's theorem, it follows that $L(z)$ is constant. Since $L(0) = \widetilde{\mathrm{tr}}(\log I) = \mathrm{tr}(0) = 0$, we have $L = 0$. Since, by Lemma 15,

$$\log (I+zN) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}N^k z^k}{k},$$

we have

$$0 = L(z) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}z^k\, \widetilde{\mathrm{tr}}(N^k)}{k}.$$

Thus $\widetilde{\mathrm{tr}}(N^k) = 0$, $k \geq 1$. Q.E.D.

Next, let T be a general operator in C_1. Let $E_i = E(\lambda_i(T); T)$ be the finite-dimensional projection corresponding to the non-zero isolated point $\lambda_i(T)$ of $\sigma(T)$. Let

$$\mathfrak{H}_n = \sum_{i=1}^{n} E(\lambda_i(T); T)\mathfrak{H},$$

$\mathfrak{H}_\infty$ be the closure of $\bigcup_n \mathfrak{H}_n$, and $\mathfrak{H}^\perp$ be the orthocomplement of $\mathfrak{H}_\infty$.

Let $\{\varphi_n\}$ be an orthonormal basis for $\mathfrak{H}_\infty$ chosen so that $\{\varphi_1, \ldots, \varphi_{n_1}\}$ is a basis for $\mathfrak{H}_1$, $\{\varphi_1, \ldots, \varphi_{n_2}\}$ is a basis for $\mathfrak{H}_2$, etc. Let $\{\psi_n\}$ be an orthonormal basis for $\mathfrak{H}^\perp$. Then it clearly follows from Lemma 13(b) that

$$(*) \qquad \widetilde{\mathrm{tr}}(T) = \sum_{i=1}^{\infty} (T\varphi_i, \varphi_i) + \sum_{n=1}^{\infty} (T\psi_i, \psi_i).$$

On the other hand,

$$\begin{aligned}\sum_{i=1}^{\infty} (T\varphi_i, \varphi_i) = \lim_{j\to\infty} \sum_{i=1}^{n_j} (T\varphi_i, \varphi_i) &= \lim_{j\to\infty} \mathrm{tr}(T|\mathfrak{H}_j) \\ &= \lim_{j\to\infty} \sum_{i=1}^{n_j} \lambda_i(T) \\ &= \mathrm{tr}(T),\end{aligned}$$

by Lemma 6.10 and the definition of $\mathrm{tr}(T)$. Thus, in order to show that $\widetilde{\mathrm{tr}}(T) = \mathrm{tr}(T)$, we have only to show that the second sum on the right side of formula $(*)$ is zero.

Now, since $\mathfrak{H}_\infty$ is invariant under T, $\mathfrak{H}^\perp = (\mathfrak{H}_\infty)^\perp$ is invariant under T^*. We have

$$\sum_{n=1}^{\infty} (T\psi_i, \psi_i) = \sum_{n=1}^{\infty} \overline{(T^*\psi_i, \psi_i)} = \widetilde{\mathrm{tr}}(T^*|\mathfrak{H}^\perp),$$

by Lemma 13(b). If the compact operator $T^*|\mathfrak{H}^\perp$ were not quasi-nilpotent, then by Theorem VII.4.5 there would exist a non-zero complex number μ and a non-zero $x \in \mathfrak{H}^\perp$ such that $T^*x = \mu x$. Thus, by Theorem VII.4.5 again, $E(\mu; T^*)\mathfrak{H}^\perp \neq 0$. From the paragraph following Definition VII.3.17, from Lemma VI.2.10, and from Definition VII.3.9, it is seen that

$$E(\mu; T^*) = E(\bar{\mu}; T)^*.$$

Hence, we would have $E(\bar{\mu}; T)^*\mathfrak{H}^\perp \neq 0$. i.e., $(\mathfrak{H}^\perp, E(\bar{\mu}; T)\mathfrak{H}) \neq 0$. But, since $E(\bar{\mu}; T)\mathfrak{H} \subseteq \mathfrak{H}_\infty = (\mathfrak{H}^\perp)^\perp$, this is impossible. It follows that $T^*|\mathfrak{H}^\perp$ is quasi-nilpotent, and hence from the preceding lemma, that $\widetilde{\mathrm{tr}}(T^*|\mathfrak{H}) = 0$. This completes the proof that $\widetilde{\mathrm{tr}}(T) = \mathrm{tr}(T)$ for all $T \in C_1$. We formulate this important result as a theorem.

19 THEOREM. *The functional* $\mathrm{tr}(T)$ *on* C_1 *is continuous and*

linear. We have $\mathrm{tr}(T) = \widetilde{\mathrm{tr}}(T)$, *where* $\widetilde{\mathrm{tr}}(T)$ *is the expression of Lemma* 13(b).

We now pause to sharpen another of the inequalities of Lemma 9.

20 Lemma. *Let* $A_1 \in C_{r_1}$, $A_2 \in C_{r_2}$, $A_3 \in C_{r_3}$ *where* $r_1^{-1}+r_2^{-1}+r_3^{-1} = 1$. *Then*

(a) $|\mathrm{tr}(A_1A_2A_3)| \le |A_1|_{r_1}|A_2|_{r_2}|A_3|_{r_3}$;

(b) *if* $r^{-1} = r_1^{-1}+r_2^{-1}$, $r_1, r_2, r_3 \ge 1$, *then*

$$|A_1A_2|_r \le |A_1|_{r_1}\,|A_2|_{r_2}.$$

Proof. By the continuity of $\mathrm{tr}(T)$ for $T \in C_1$, the continuity of the product TS which was noted in the paragraph following Lemma 9, the continuity of the norm function which follows from the triangle inequality of Lemma 14(d), and by Lemma 11, it follows that we may without loss of generality assume all operators to have finite-dimensional ranges. By arguments such as those of the third paragraph of Lemma 6, we may then assume our Hilbert space to have finite dimension d. Lemma 14(a) shows that (b) follows immediately from (a). Thus, we have only to prove the trilinear inequality (a) for operators in a d-dimensional Hilbert space.

Arguing as in the paragraphs of the proof of Lemma 14 following formula (3) of that proof, where we proved a bilinear inequality quite similar to our present trilinear inequality, we see that it is sufficient to prove the inequality

$$(1) \qquad |\mathrm{tr}(D_1U_1D_2U_2D_3U_3)| \le |D_1|_{r_1}|D_2|_{r_2}|D_3|_{r_3},$$

U_i being unitary and D_i positive and diagonal. The Riesz convexity theorem (Lemma VI.10.7) will imply (1) once we establish the special case $r_1 = 1, r_2 = \infty, r_3 = \infty$ of (1) and the two symmetrical special cases $r_1 = \infty, r_2 = 1, r_3 = \infty$ and $r_1 = \infty, r_2 = \infty, r_3 = 1$. But

$$|\mathrm{tr}(D_1U_1D_2U_2D_3U_3)| \le |D_1|_1|D_2|_\infty|D_3|_\infty$$

follows immediately from Lemma 9(d) and Corollary 8. Q.E.D.

We may now proceed rapidly to the main goal of the present section: the derivation of inequalities for the resolvent of an operator in C_p generalizing the Carleman inequality, which inequality has been

given as Theorem 6.27. We first define an appropriate family of generalized determinants.

21 Definition. Let $0 \leqq p \leqq k$, where k is an integer not less than 1, and let T be in C_p. Let $\lambda_i = \lambda_i(T)$, $i \geqq 1$. Then

$$\det_k(I+T) = \prod_{i=1}^{\infty} \left\{(1+\lambda_i)\exp\left(-\lambda_i + \frac{\lambda_i^2}{2} - \dots + \frac{(-1)^{k-1}}{k-1}\lambda_i^{k-1}\right)\right\}.$$

22 Lemma. *Let p and k be as in Definition 21.*

(a) *The product defining* $\det_k(I+T)$ *converges absolutely.*

(b) $\det_k(I+T)$ *is a continuous function of T on C_p.*

(c) *If $T' \in C_p$, $\det_k(I+T+zT')$ is an entire function of the complex variable z, whose derivatives all depend continuously on T and T'.*

(d) *If $k \geqq p \geqq k-1$ there exists a finite constant Γ, depending only on k and p, such that*

$$\det_k(I+T) \leq \exp(\Gamma|T|_p^p).$$

(e) *If $k \geq 2$ and $p < k-1$, we have*

$$\det_k(I+T) = \exp\left(\frac{(-1)^{k-1}}{k-1}\operatorname{tr}(T^{k-1})\right)\det_{k-1}(I+T).$$

(f) *For $k = 1$,* $\det_1(I+T)$ *coincides with the function* $\det(I+T)$ *of Lemma* 16.

Proof. We have

$$\left|\log(1+\lambda)-\lambda+\frac{\lambda^2}{2}-\dots+\frac{(-1)^{k-1}}{k-1}\lambda^{k-1}\right| = O(|\lambda|^k) \qquad (\text{as}\,|\lambda|\to 0)$$

$$= O(|\lambda|^{k-1}) \quad (\text{as}\,|\lambda|\to\infty).$$

Thus, using the elementary inequality $|e^z-1| \leq |z|e^{|z|}$, we find that there exists a sufficiently large constant Γ so that

$$(1) \qquad \left|(1+\lambda)\exp\left(-\lambda+\dots+\frac{(-1)^{k-1}}{k-1}\lambda^{k-1}\right)-1\right|$$
$$\leq \Gamma\frac{|\lambda|^k}{1+|\lambda|}\exp\left(\Gamma\frac{|\lambda|^k}{1+|\lambda|}\right),$$

both in the vicinity of $\lambda = 0$ and in the vicinity of $\lambda = \infty$. Since the function on the left is bounded in bounded regions not containing $\lambda = 0$ and the function on the right is non-vanishing, we may, by increasing Γ, assume that the above inequality holds for all λ.

Similarly, for each $k \geq p \geq k-1$, we find a constant Γ depending only on p such that

$$(2) \qquad \left|(1+\lambda)\exp\left(-\lambda+\ldots+\frac{(-1)^{k-1}}{k-1}\lambda^{k-1}\right)\right| \leq \exp(\Gamma|\lambda|^p).$$

The absolute convergence of the product of Definition 21 will consequently follow from the convergence of the series

$$(3) \qquad \sum_{n=0}^{\infty} \frac{|\lambda_n|^k}{1+|\lambda_n|} \exp\left(\Gamma \frac{|\lambda_n|^k}{1+|\lambda_n|}\right).$$

Since, if $\sum_{n=0}^{\infty}|\lambda_n|^k < \infty$, all but a finite number of the terms $|\lambda_n|$ are bounded by 1, it follows at once that the series (3) will converge if $\sum|\lambda_n|^k$ converges. Hence, by Corollary 7, the product of Definition 21 converges uniformly if $T \in C_p$, $p \leq k$. This proves (a).

The inequality in (d) follows similarly from (2) if $k \geq p \geq k-1$.

It is clear from Definition 21 that if $\sum_{i=0}^{\infty}|\lambda_i(T)|^k < \infty$, then by what has already been proved the products defining $\det_k(I+T)$ and $\det_{k+1}(I+T)$ converge, and

$$\det_k(I+T)\cdot\exp\left\{\frac{(-1)^k}{k}\sum_{i=1}^{\infty}\lambda_i^k\right\} = \det_{k+1}(I+T).$$

Since, from the spectral mapping theorem (VII.3.19), $\sum_{i=1}^{\infty}\lambda_i^k = \operatorname{tr}(T^k)$, (e) follows at once.

Now we turn to the proof of (b). If $|\lambda| \leq 1$, then the inequality (1) may be written, perhaps with an increase of the constant Γ, as

$$(4) \qquad \left|(1+\lambda)\exp\left(-\lambda+\ldots+\frac{(-1)^{k-1}}{k-1}\lambda^{k-1}\right) - 1\right| \leq \Gamma|\lambda|^k.$$

If we note the identity

$$a_1\ldots a_n - 1 = (a_1-1)a_2\ldots a_n + (a_2-1)a_3\ldots a_n + (a_n-1)a_1\ldots a_{n-1}$$

and the elementary inequality $|1+x| \leq e^{|x|}$, it follows that

$$\left|\prod_{n=1}^{\infty}(1+\alpha_n)-1\right| \leq \left(\sum_{i=1}^{\infty}|\alpha_i|\right)\exp\left(\sum_{i=1}^{\infty}|\alpha_i|\right).$$

Thus, if all the constants $|\lambda_i|$ are less than 1, we have

$$\left|\prod_{i=1}^{\infty}\left\{(1+\lambda_i)\exp\left(-\lambda_i+\cdots+\frac{(-1)^k}{k-1}\lambda_i^{k-1}\right)\right\}-1\right|$$
$$\leq \left\{\sum_{i=1}^{\infty}|\lambda_i|^k\right\}\exp\left\{\sum_{i=1}^{\infty}|\lambda_i|^k\right\}.$$

If $\lambda_i(m)$ is a parameter family of sequences such that

(i) $\lambda_i(m)\to\lambda_i$ as $m\to\infty$ uniformly in i,

(ii) $\lim_{r\to\infty}\sum_{i=r}^{\infty}|\lambda_i(m)|^k=0$ uniformly in m,

it follows that

(iii) $\sum_{i=1}^{\infty}|\lambda_i(m)|^k$ is bounded uniformly in m,

(iv) $\lambda_i(m)\to 0$ as $i\to\infty$ uniformly in m.

Thus, by the above inequality,

$$\lim_{r\to\infty}\prod_{i=r+1}^{\infty}\left\{(1+\lambda_i(m))\exp\left(-\lambda_i(m)+\cdots+\frac{(-1)^{k-1}}{k-1}\lambda_i^{k-1}(m)\right)\right\}\to 1$$

uniformly in m, so that

$$\prod_{i=1}^{r}\left\{(1+\lambda_i(m))\exp\left(-\lambda_i(m)+\cdots+\frac{(-1)^{k-1}}{k-1}\lambda_i^{k-1}(m)\right)\right\}$$
$$\to\prod_{i=1}^{\infty}\left\{(1+\lambda_i(m))\exp\left(-\lambda_i(m)+\cdots+\frac{(-1)^{k-1}}{k-1}\lambda_i^{k-1}(m)\right)\right\}$$

uniformly in m. On the other hand, by (i),

$$\lim_{m\to\infty}\prod_{i=1}^{r}\left\{(1+\lambda_i(m))\exp\left(-\lambda_i(m)+\cdots+\frac{(-1)^{k-1}}{k-1}\lambda_i^{k-1}(m)\right)\right\}$$
$$=\prod_{i=1}^{r}\left\{(1+\lambda_i)\exp\left(-\lambda_i+\cdots+\frac{(-1)^{k-1}}{k-1}\lambda_i^{k-1}\right)\right\}.$$

Thus, since the order of two limits may be interchanged if one of them is uniform, we find that (i) and (ii) imply

$$\begin{aligned}\lim_{m\to\infty}\prod_{i=1}^{\infty}&\left\{(1+\lambda_i(m))\exp\left(-\lambda_i(m)+\cdots+\frac{(-1)^{k-1}}{k-1}\lambda_i^{k-1}(m)\right)\right\}\\ &=\prod_{i=1}^{\infty}\left\{(1+\lambda_i(m))\exp\left(-\lambda_i+\cdots+\frac{(-1)^{k}}{k-1}\lambda_i^{k-1}\right)\right\}.\end{aligned}$$

Since

$$\sum_{i=r}^{\infty}|\lambda_i(m)|^k \leqq \max_{i>r}|\lambda_i(m)|^{\varepsilon}\sum_{i=r}^{\infty}|\lambda_i(m)|^{k-\varepsilon},$$

statements (i) and (ii) are implied by the following statements (i') and (ii'):

(i') $\lambda_i(m) \to \lambda_i$ as $m \to \infty$, uniformly in i.

(ii') $\sum_{i=1}^{\infty}|\lambda_i(m)|^{k-\varepsilon}$ is uniformly bounded.

Thus, using Definition 21, Lemma 5, and Corollary 7 it follows that

$$\det_k(I+T_m) \to \det_k(I+T)$$

if $T_m, T \in C_{k-\varepsilon}$, and $T_m \to T$ in the topology of $C_{k-\varepsilon}$. This proves (b) if $p < k$.

To handle the limiting case $p = k$, note that if $T_m \to T$ in the topology of C_k, it follows by what has already been proved that

$$\det_{k+1}(I+T_m) \to \det_{k+1}(I+T).$$

But, by (e),

$$\det_{k+1}(I+T_m) = \exp\left(\frac{(-1)^k}{k}\operatorname{tr}(T_m^k)\right)\det_k(I+T_m)$$

$$\det_{k+1}(I+T) = \exp\left(\frac{(-1)^k}{k}\operatorname{tr}(T^k)\right)\det_k(I+T).$$

Since, by the observations made in the paragraph following Lemma 9, $T_m^k \to T^k$ in the topology of C_1, and since by Theorem 19 this implies $\operatorname{tr}(T_m^k) \to \operatorname{tr}(T^k)$, (b) follows also if $p = k$.

Since a bounded, convergent sequence of analytic functions converges uniformly with all its derivatives, it follows readily from (b) and from Lemma 11 that (c) will follow in general if it is established in the special case in which both T and T' have finite-dimensional ranges. The arguments given in the third paragraph of the proof of Lemma 6 reduce this case to the case in which our Hilbert

space is finite-dimensional. But, for a matrix T in d-dimensional Hilbert space, since the determinant (in the ordinary sense) of T is the product of the eigenvalues of T and the trace of T is the sum of the eigenvalues of T, then

$$\prod_{i=1}^{d}\left\{(1+\lambda_i(T))\exp\left(-\lambda_i(T)+\dots+\frac{(-1)^{k-1}}{k-1}\lambda_i(T)^{k-1}\right)\right\}$$
$$=\det(I+T)\exp\left(-\mathrm{tr}(T)+\dots+\frac{(-1)^{k-1}}{k-1}\mathrm{tr}(T^{k})\right).$$

This expression clearly depends analytically on the entries of the finite matrix T; thus (e) is proved.

We saw in the course of proving Lemma 16 that the function $\det(I+T)$ of that lemma is continuous in T (cf. the remark which follows Lemma 16). Thus, by (b) and by Lemma 11, to prove (f) in general we have only to prove (f) in the special case in which T has finite-dimensional range. The arguments of the third paragraph of Lemma 6 suffice to reduce this case to the case in which our Hilbert space is finite-dimensional. But in this latter case, both $\det_1(I+T)$ and the function $\det(I+T)$ of Lemma 16 have been shown to coincide with the determinant, in the ordinary sense, of the matrix $I+T$. This was established for $\det_1(I+T)$ just above, and for the function of Lemma 16 in the semi-final paragraph of the proof of Lemma 16. This proves (f), and completes the proof of the present lemma. Q.E.D.

The next lemma gives a derivative formula which is the key to the subsequent course of our argument.

23 Lemma. *Let k be an integer not less than* 1, *and* $T, B \in C_k$. *Then, if* $(-1) \notin \sigma(T)$,

$$(*)\quad \frac{d}{dz}\det_k(I+T+zB)\Big|_{z=0}$$
$$=\det_k(I+T)\,\mathrm{tr}\left[\left\{(I+T)^{-1}-I+\dots+(-1)^{k-1}T^{k-2}\right\}B\right].$$

Proof. Let us first remark that, since the function $g(\zeta) = (1+\zeta)^{-1}-1+\dots+(-1)^{k-1}\zeta^{k-2}$ has a $(k-1)$-fold zero at $\zeta = 0$ and is analytic on the spectrum of T, we may write $g(\zeta) = h(\zeta)\zeta^{k-2}$, where $h(\zeta)$ vanishes at zero. Consequently,

$$g(T) = (I+T)^{-1} - I + \ldots + (-1)^{k-1}T^{k-2} = T^{k-2}h(T).$$

By Lemma 15 and the paragraph of discussion following Lemma 9, it follows that if $T_n \to T$ and $B_n \to B$ in the topology of C_k, $g(T_n) \to g(T)$ in the topology of $C_{1+1/k}$, while $g(T_n)B_n \to g(T)B$ in the topology of C_1. Thus, by Theorem 19, the trace on the left of formula (*) is defined and continuous in T and B. By (c) of the preceding lemma and by Lemma 11, it follows that to prove the general validity of formula (*), we have only to consider the special case in which T and B have finite-dimensional range. Arguments like those of the third paragraph of Lemma 6 suffice to reduce this case to the case in which our Hilbert space is of finite dimension n. In this latter case, as we have seen in the final paragraph of the proof of the preceding lemma,

$$\det_k(I+T+zB)$$
$$= \det(I+T+zB)\exp\left\{\operatorname{tr}\left[-(T+zB)+\ldots+\frac{(-1)^{k-1}}{k-1}(T+zB)^{k-1}\right]\right\}.$$

Since

$$\frac{d}{dz}(T+zB)^j\Big|_{z=0} = \sum_{i=0}^{j-1} T^i B T^{j-1-i},$$

we have

$$\frac{d}{dz}\operatorname{tr}\big((T+zB)^j\big)\Big|_{z=0} = j\operatorname{tr}(T^{j-1}B).$$

Thus

$$[\dagger]\quad \frac{d}{dz}\exp\left\{\operatorname{tr}\left[-(T+zB)+\ldots+\frac{(-1)^{k-1}}{k-1}(T+zB)^{k-1}\right]\right\}\Bigg|_{z=0}$$
$$= \exp\left\{\operatorname{tr}\left(-T+\ldots+\frac{(-1)^{k-1}}{k-1}T^{k-1}\right)\right\}\operatorname{tr}[(-I+\ldots+(-1)^{k-1}T^{k-2}$$

On the other hand, if a_{ij} and b_{ij} are the matrices of a pair of linear transformations A and B in n-dimensional space, then since a determinant is linear in each of its rows, we have

$$\frac{d}{dz}\det(A+zB)\big|_{z=0} = \begin{vmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & & & \\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n} \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{vmatrix}.$$

Therefore, by Lagrange's expansion formula and Cramer's formula for matrix inverses, we have

$$\begin{aligned}\frac{d}{dz}\det(A+zB)\big|_{z=0} &= \sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}\gamma_{ji} \\ &= \det(A)\operatorname{tr}(A^{-1}B),\end{aligned}$$

where γ_{ij} denotes the cofactor of the element a_{ij} of the matrix A. Substituting $A = I+T$ and using [†], we obtain formula (*) at once. Q.E.D.

Remark. Statement (c) of Lemma 22 shows that the expression on the left of (*) is continuous in T and B. Thus, the expression on the right of (*) may be defined by continuity for all T, whether or not $(-1)\in\sigma(T)$. This corresponds to the fact that since $\det_k(I+\mu T)$ has an n-fold zero at any point μ such that $\lambda = -\mu^{-1}\in\sigma(T)$ and λ is of multiplicity n, while $(I+\mu T)^{-1}$ has at most an n-fold pole there (cf. VII.3.20, VII.3.18), the expression

$$\det_k(I+T)(I+T)^{-1}-I+\ldots+(-1)^{k-1}T^{k-2}$$

may be defined by continuity, irrespective of whether $(-1)\in\sigma(T)$ or not. This remark will be used freely and implicitly in what follows.

It is now entirely trivial to derive a generalized Carleman inequality.

24 THEOREM. *Let k be an integer not less than 1, and $1\le p\le k$. Then there exists a constant Γ depending only on p, such that*

$$|\det_k(I+T)\{(I+T)^{-1}-I+\ldots+(-1)^kT^{k-2}\}|_q \le \exp(\Gamma(|T|_p^p+1)),$$

where $p^{-1}+q^{-1}=1$.

25 COROLLARY. *If k and p are as above, there exists a constant Γ depending only on p, such that*

$$|\det_k(I+T)(I+T)^{-1}| \le \exp(\Gamma|T|_p^p).$$

PROOF OF THEOREM 24. The formula of Lemma 23 and the derivative formula

$$f'(0) = -\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{f(\zeta)}{\zeta^2}\,d\zeta$$

of complex function theory imply that

$$\begin{aligned}\det_k(I+T)\operatorname{tr}[\{(I+T)^{-1}-I+\ldots+(-1)^{k-1}T^{k-2}\}B]\\ = -\frac{1}{2\pi i}\int_{|\zeta|=1}\zeta^{-2}\det_k(I+T+\zeta B)d\zeta.\end{aligned}$$

Thus, by Lemma 22(d), there exist constants Γ and Γ' depending only on p such that

$$\begin{aligned}|\operatorname{tr}[\det_k(I+T)\{(I+T)^{-1}-I+\ldots+(-1)^{k-1}T^{k-2})B]|\\ \leq \exp\{\Gamma'(|T|+|B|)_p^p\} \leq \exp(\Gamma(|T|_p^p+1))\end{aligned}$$

if $B \in C_p$ and $|B|_p \leq 1$. Theorem 24 follows immediately from Lemma 14(a). Q.E.D.

PROOF OF COROLLARY 25. Since there exists a constant Γ depending only on p such that

$$|T^j| \leq |T^j|_p \leqq \exp(\Gamma(|T|_p^p+1)), \qquad j = 1,\ldots,k-2,$$

Corollary 25 follows immediately from Theorem 24 and Lemma 9. Q.E.D.

For $0 < p \leqq 1$, we cannot argue in exactly this way, but a modification of our method of proof will establish the corresponding results for this range without difficulty.

➔ 26 THEOREM. *Let $0 < p \leqq 1$. Then there exists a constant Γ depending only on p, such that*

$$|\det(I+T)(I+T)^{-1}| \leqq \exp(\Gamma|T|_p^p).$$

PROOF. From the analytic dependence of $(I+\mu T)^{-1}$ on μ it follows at once that it is sufficient to prove our theorem under the assumption that $(-1) \notin \sigma(T)$. Then, if $T_n \to T$ in the topology of C_p, $T_n \to T$ uniformly by Lemma 9(a) and Definition 1(b), so that the expression

$$|\det(I+T_n)(I+T_n)^{-1}|$$

approaches the expression on the left of the inequality of our theorem. By Lemmas 11 and 9(b) it is then sufficient to prove our theorem in the special case in which T has a finite-dimensional range. Arguments like those of the third paragraph of the proof of Lemma 6 then show that it is sufficient to prove our theorem for the case of a finite-dimensional Hilbert space. We shall, in fact, show that for a non-singular operator S in a d-dimensional Hilbert space,

$$|(\det S)S^{-1}| \leq \prod_{i=1}^{d-1} \mu_i(S). \tag{1}$$

Since $\mu_i(I+T) \leq 1+\mu_i(T)$ by Corollary 3, and since there plainly exists a constant Γ depending only on p such that

$$1+|\mu| \leq e^{(\Gamma|\mu|^p)},$$

our theorem will follow.

Since we have shown in the third paragraph of the proof of Lemma 14 that $S = UMU'$, when U and U' are unitary and M is a diagonal matrix with the entries $\mu_1(S), \ldots, \mu_n(S)$, in proving (1) we may assume without loss of generality that $S = M$. But then $(\det M)M^{-1}$ is also a diagonal matrix, whose nth diagonal element is

$$\prod_{\substack{i=1\\ i\neq n}}^{d} \mu_i(S).$$

Since the $\mu_i(S)$ are arranged in decreasing order, the largest of these diagonal elements is plainly the expression on the right of (1), which proves our theorem. Q.E.D.

Theorem 26 and Theorem 24 yield the following corollary for the special case of a quasi-nilpotent operator.

27 COROLLARY. *Let N be a quasi-nilpotent operator and $N \in C_p$, where $0 < p < \infty$. Then the resolvent $R(\lambda; N)$ has a bound*

$$|R(\lambda; N)| \leq e^{(\Gamma|\lambda|^{-p})}$$

in the neighborhood of $\lambda = 0$, Γ being some finite constant depending on p and N.

The argument of Theorem 6.29 may now be carried over without difficulty to compact operators whose imaginary part is of

the class C_p. This argument was based upon the inequality of Corollary 6.28, of which the preceding corollary is a direct generalization, and on the Phragmén-Lindelöf theorem, Lemma 6.33. The generalized form of this theorem may be stated as follows:

28 LEMMA. (Phragmén-Lindelöf) *Let g be a function of the complex variable z defined and analytic in the interior of the angular sector σ bounded by a non-intersecting pair of differentiable Jordan arcs γ_1 and γ_2 and forming an angle of opening less than π/p at the origin. Suppose that g is also analytic in a neighborhood of each of the half open arcs $\gamma_1-\{0\}$ and $\gamma_2-\{0\}$, that g is also bounded on each of these half-open arcs, and that $|g(z)| = O(\exp |z|^{-p})$ as $z \to 0$, z remaining in the interior of σ. Then $|g(z)| = O(1)$ as $z \to \infty$, z remaining in the interior of σ.*

PROOF. Apply Lemma 6.33 to $h(z) = g(z^{2/p})$. Q.E.D.

With Corollary 27 and 28 in hand, we may use the argument of Theorem 6.29 to obtain the following theorem.

29 THEOREM. *Let $0 < p < \infty$ and let $T \in C_p$. Let $\gamma_1, \ldots, \gamma_s$ be a family of non-overlapping differentiable arcs in the complex plane starting at the origin. Suppose that each of the s regions into which the plane is divided by these arcs is contained in a sector of angular opening less than π/p. Let $N > 0$ be an integer and let the resolvent of T satisfy the inequality*

$$|R(\lambda; T)| = O(|\lambda|^{-N})$$

as $\lambda \to 0$ along any of the arcs γ_i. Then the subspace $\text{sp}(T)$ contains the subspace $T^N\mathfrak{H}$.

The very minor adaptations of the proof of Theorem 6.29 needed to yield the proof of the above theorem are left to the reader. We note that the minimum necessary number s of arcs is $[2p]+1$.

30 COROLLARY. *Suppose that as λ tends to zero along the arcs γ_i of the preceding theorem the resolvent of T satisfies the inequality $|R(\lambda; T)| = O(|\lambda|^{-1})$. Then the subspace $\text{sp}(T)$ coincides with the entire Hilbert space $\mathfrak{H}$.*

31 COROLLARY. *Let T be a densely defined unbounded operator in Hilbert space $\mathfrak{H}$ and $0 < p < \infty$. Suppose that for some λ_0 in the*

resolvent set of T the resolvent $R(\lambda_0; T)$ is in the class C_p. Let $\gamma_1, \ldots, \gamma_s$ be non-overlapping differentiable arcs having a limiting direction at infinity and suppose that no adjacent pair of arcs forms an angle as great as π/p at infinity. Suppose that the resolvent $R(\lambda; T)$ satisfies an inequality $|R(\lambda; T)| = O(|\lambda|^N)$ as $\lambda \to \infty$ along each such γ_i. Then the subspace $\mathrm{sp}(T)$ *coincides with the entire Hilbert space* $\mathfrak{H}$.

The proof of Corollary 30 is identical with that of Corollary 6.30 word for word, while the proof of Corollary 31 differs from that of Corollary 6.31 only in slight details, the elaboration of which we leave to the reader.

In order to apply the preceding theorems, it is necessary to have criteria guaranteeing that a given operator lies in a class C_p. Following are a number of simple conditions of this sort.

32 Lemma. *Let $2 \leq p < \infty$. Let φ_i be an orthonormal set and T a bounded operator. If*

$$\left\{\sum_{i=1}^{\infty} |T\varphi_i|^p\right\}^{1/p} < \infty,$$

then $T \in C_p$.

Proof. Let $\gamma_i = |T\varphi_i|$. Let the operator B be defined by $B\varphi_i = (T\varphi_i)\gamma_i^{(p/2)-1}$. Then plainly

$$\sum_{i=1}^{\infty} |B\varphi_i|^2 = \sum_{i=1}^{\infty} (\gamma_i^{p/2})^2 < \infty,$$

so that, by Definition 6.1, B belongs to the Hilbert-Schmidt class C_2. If we let $A\varphi_i = \gamma_i^{1-p/2}\varphi_i$, then A is plainly self adjoint and A belongs to the class C_r, where $r(1-p/2) = p$, i.e., $r = p(1-p/2)^{-1}$. Thus, by Lemma 9, $T = BA$ belongs to the class C_s, where $s^{-1} = \frac{1}{2} + (1-p/2)p^{-1} = p^{-1}$; i.e., T belongs to C_p. Q.E.D.

If A is the operator in $L_2[0, 2\pi]$ mapping e^{inx} into $(in)^{-1}$ for $n \neq 0$ and mapping $1 = e^{i0x}$ into $-x$, then A has a two-dimensional range and

$$\int_0^x e^{iny}\,dy = \frac{1}{in}e^{inx} - A(e^{inx}), \qquad n \neq 0,$$

$$= 0 - A(e^{inx}), \qquad n = 0.$$

It is plain from this that the operation of integration belongs to each

class $C_{1+\varepsilon}$ (but not to the class C_1). Thus, for instance, if $K(x, y)$ is a Hilbert-Schmidt kernel such that

$$\int_0^{2\pi}\int_0^{2\pi}\left|\frac{\partial}{\partial y}K(x, y)\right|^2 dydx < \infty,$$

we may write

$$\int_0^{2\pi} K(x, y)f(y)dy$$
$$= \int_0^{2\pi}\left\{\left(\frac{\partial}{\partial y}K(x, y)\right)\int_0^y f(t)dt\right\}dy + K(x, 2\pi)\int_0^{2\pi} f(y)dy,$$

so that, from Lemma 9, $K \in C_r$, where $r^{-1} = 2^{-1} + (1+\varepsilon)^{-1}$; i.e., $K \in C_{(2/3)+\varepsilon}$ for each ε. Similarly, if

$$\int_0^{2\pi}\int_0^{2\pi}\left|\left(\frac{\partial}{\partial y}\right)^s K(x, y)\right|^2 dxdy < \infty, \qquad 0 \leqq s \leqq k,$$

then K belongs to the class $C_{(2/2k+1)+\varepsilon}$. If

$$\sup_{0\leqq y\leqq 2\pi}\int_0^{2\pi}\left\{\left|\left(\frac{\partial}{\partial y}\right)^s K(x, y)\right| + \left|\left(\frac{\partial}{\partial y}\right)^s K(y, x)\right|\right\} dx < \infty, \qquad 0 \leqq s \leqq k,$$

we see in the same way that K belongs to the class $C_{(1/k)+\varepsilon}$.

Similar results may be derived by direct estimation of the characteristic numbers $\mu_n(T)$. Suppose, for instance, that the kernel K satisfies the Hölder condition

$$h^{-\alpha}\left\{\int_0^1 |K(x, y+h)-K(x, y)|^2 dx\right\}^{1/2} \leqq \Gamma, \tag{1}$$
$$0 \leqq x, y \leqq 1, \quad h > 0.$$

Then, if f is a function in $L_2(0, 1)$ satisfying the n linear conditions

$$\int_{j/n}^{(j+1)/n} f(x)dx = 0, \qquad j = 0, \ldots, n-1. \tag{2}$$

we may write

$$(Kf)(x) = \int_0^1 K(x, y)f(y)dy \tag{3}$$
$$= \int_0^1 \{K(x, y)-K_n(x, y)\}f(y)dy,$$

where

$$K_n(x, y) = K\left(x, \frac{j}{n}\right), \qquad \frac{j}{n} \leq y < \frac{j+1}{n} \qquad 0 \leqq j < n-1.$$

If μ_n is a decreasing sequence of numbers, then plainly

$$n\mu_n^2 \leq \sum_{j=1}^{\infty} \mu_n^2;$$

thus $\mu_n(K) \leq n^{-1/2}|K|_2$. Consequently, if f is subject not only to the n linear conditions (2) but also to n additional linear conditions adapted to the kernel $K-K_n$, we may be sure that

$$\begin{aligned} |Kf| &\leq n^{-1/2}\left\{\int_0^1\int_0^1 |K(x, y)-K_n(x, y)|^2 dxdy\right\}^{1/2} |f| \\ &\leqq \Gamma\left\{\int_0^{1/n} t^{2\alpha} dt\right\}^{1/2} |f| = \Gamma n^{-1/2-\alpha}|f|, \end{aligned}$$

i.e.,

$$\mu_{2n}(K) \leq \Gamma n^{-1/2-\alpha}. \tag{4}$$

Thus $K \in C_p$ if $p(\frac{1}{2}+\alpha) > 1$, i.e., $p > 2/(1+2\alpha)$.

Numerous results of this sort are to be found in the work of Hille-Tamarkin [1].

We may derive a simple result of this sort for the infinite interval as follows. Since the mapping $I_A : f(x) \to \int_0^x f(y)dy$ in the space $L_2[0, A]$ is related after division by A to the formally identical mapping I_1 in $L_2[0, 1]$, by the unitary transformation $f(x) \to f(Ax)A^{1/2}$, it follows from our earlier calculations that there exists a constant M such that $\mu_n(I_A) \leq MAn^{-1}$. Thus, if K is a kernel on $[0, \infty)\times[0, \infty)$ such that

$$\int_0^{\infty}\int_0^{\infty}\left\{|K(x, y)|^2 + \left|\frac{\partial}{\partial y} K(x, y)\right|^2\right\} dxdy < \infty,$$

we have $K = K_B + K_B' I_B$, where K_B and K_B' are the integral operators with kernels

$$K_B(x, y) = K(x, B),\ y \leq B; \qquad K_B(x, y) = K(x, y),\ y > B.$$

$$K_B'(x, y) = 0,\ y > B; \qquad K_B'(x, y) = -\frac{\partial}{\partial y} K(x, y),\ y \leq B.$$

Thus, by Corollary 3 and by the estimate for the characteristic numbers of a Hilbert-Schmidt operator used above,

$$\mu_{3n}(K) \leqq MBn^{-3/2} + n^{-1/2}\left\{\int_0^\infty \int_0^\infty |K_B(x, y)|^2 dx dy\right\}^{1/2}.$$

Let us now suppose that

$$\int_0^\infty \int_0^\infty |K_B(x, y)|^2 dx dy = O(B^{-\alpha}).$$

Then the above inequality may be written as

$$\mu_{3n}(K) \leqq M'n^{-1/2}[Bn^{-1} + B^{-\alpha}],$$

in terms of a certain finite constant M'. Since B may be chosen at our discretion, it is seen that

$$\mu_{3n}(K) \leqq M''n^{-(1/2+(\alpha/\alpha+1))}.$$

Thus K is in C_p if $p > (2\alpha+2)/(3\alpha+1)$.

If the variables x and y of the kernel $K(x, y)$ lie in a bounded region of d-dimensional space, we may easily adapt the argument based on formula (3). It is necessary in this case to divide our region into n^d subregions. Thus the result corresponding to (4) is

$$\mu_{2n^d}(K) \leqq \Gamma n^{-d/2} n^{-\alpha},$$

if K satisfies the Hölder condition (1). This may be written

$$\mu_n(K) \leqq \Gamma' n^{-(1/2+(\alpha/d))}.$$

Similarly, if K has derivatives of the first s orders with respect to y which are square-integrable in both variables, then

$$\mu_n(K) \leqq \Gamma'' n^{-(1/2+(s/d))}.$$

We have $K \in C_p$ for $p > 2d/(d+\alpha)$ in the first case and $K \in C_p$ for $p > 2d/(d+s)$ in the second case.

10. Subdiagonalization of Compact Operators

Any Hermitian operator in a finite-dimensional space can be reduced to diagonal form by a unitary transformation. This statement is, of course, false if we omit the word "Hermitian" as the theorem

on reduction to Jordan canonical form shows. It is even false that every finite matrix can be reduced to diagonal form by a non-singular transformation. On the other hand, as the proof of Lemma 6.21 makes plain, every finite matrix can be reduced to *subdiagonal* form by a unitary transformation. In the present section, we shall discuss the analogue of this result for arbitrary compact operators in Hilbert space. Our analysis will yield a number of interesting inequalities which permit a useful extension of the result of the preceding section. Throughout the present section, we assume for simplicity of statement that Hilbert space is separable.

Subdiagonal representations of an operator are connected with the study of its invariant subspaces. Thus, the key to the situation that we wish to analyze is the following general, interesting, and important theorem of Aronszajn and Smith.

→ 1 THEOREM. *Let T be a compact operator in a B-space $\mathfrak{X}$ of dimension greater than 1. Then $\mathfrak{X}$ has a proper non-zero closed subspace $\mathfrak{Y}$ such that $T\mathfrak{Y} \subseteq \mathfrak{Y}$.*

PROOF. If $\mathfrak{X}$ is finite-dimensional, the result follows trivially from the existence of an eigenvector. Hence we may assume that $\mathfrak{X}$ is infinite-dimensional and has no eigenvectors. Pick a vector $\bar{x}$ with $|\bar{x}| = 1$. Since the closed subspace $\mathfrak{X}_0$ spanned by the vectors $T^i\bar{x}$, $i \geqq 1$, is invariant, we may suppose that $\mathfrak{X}_0 = \mathfrak{X}$. If the vector $T^n\bar{x}$ was linearly dependent on the set $A = \{T^{n-1}\bar{x}, \ldots, \bar{x}\}$ of vectors, then $T^{n+1}\bar{x}$ would also be dependent on A, and, inductively, all $T^{n+i}\bar{x}$ would be dependent on A, so that $\mathfrak{X}$ would be finite-dimensional. Summarizing, then, we may assume without loss of generality that

(i) the vectors $\bar{x}, T\bar{x}, \ldots$ are linearly independent;
(ii) the closure of the linear span of the vectors $\bar{x}, T\bar{x}, \ldots$ is $\mathfrak{X}$.

For each integer k, let $\mathfrak{X}^{(k)}$ denote the $(k+1)$-dimensional space spanned by $\bar{x}, T\bar{x}, \ldots, T^k\bar{x}$. For each closed subspace $\mathfrak{Y}$ of $\mathfrak{X}$ and each $x \in \mathfrak{X}$, let $\rho(x, \mathfrak{Y})$ denote

$$\inf_{y \in \mathfrak{Y}} |x-y|.$$

Then it is readily verified that we have

(iii) $\rho(x+\tilde{x}, \mathfrak{Y}) \leq \rho(x, \mathfrak{Y})+\rho(\tilde{x}, \mathfrak{Y})$

(iv) $\rho(\alpha x, \mathfrak{Y}) = |\alpha|\rho(x, \mathfrak{Y})$

(v) $\rho(x, \mathfrak{Y}) \leq |x|$.

By (ii), we have

(vi) $\lim_{k\to\infty} \rho(x, \mathfrak{X}^{(k)}) \to 0$, $x \in \mathfrak{X}$.

For each sequence $\mathfrak{Y}_k$ of subspaces of $\mathfrak{X}$, put

$$\underline{\lim}\, \mathfrak{Y}_k = \{x \mid \lim_{k\to\infty} \rho(x, \mathfrak{Y}_k) = 0\}.$$

It follows readily from (iii). (iv) and (v) that $\underline{\lim}\, \mathfrak{Y}_k$ is a closed subspace of $\mathfrak{X}$.

For each subspace $\mathfrak{Y}$ of $\mathfrak{X}$ and each $x \in \mathfrak{X}$, let $\mathfrak{Y}(x)$ denote some vector $y \in \mathfrak{Y}$ such that $|\mathfrak{Y}(x)-x| \leq 2\rho(x, \mathfrak{Y})$. Define the linear transformation T_k of the space $\mathfrak{X}^{(k)}$ into itself by putting

$$\begin{aligned} T_k(\alpha_0\bar{x}+\ldots+\alpha_k T^k\bar{x}) \\ &= \alpha_0 T\bar{x}+\ldots+\alpha_{k-1}T^k\bar{x}+\alpha_k\mathfrak{X}^k(T^{k+1}\bar{x}). \end{aligned}$$

It is plain that

$$\begin{aligned} |(T-T_k)(\alpha_0\bar{x}+\ldots+\alpha_k T^k\bar{x})| \\ &\leq 2|\alpha_k|\rho(T^{k+1}\bar{x}, \mathfrak{X}^{(k)}) = 2\rho(\alpha_k T^{k+1}\bar{x}, \mathfrak{X}^{(k)}). \end{aligned}$$

Thus, since $\rho(x+y, \mathfrak{Y}) = \rho(x, \mathfrak{Y})$ for any $y \in \mathfrak{Y}$, we have

(vii) $|(T-T_k)x| \leq 2\rho(x, \mathfrak{X}^{(k)})$, $\qquad x \in \mathfrak{X}^{(k)}$.

Since $\mathfrak{X}^{(k)}$ is finite-dimensional, Lemma 6.21 shows that $\mathfrak{X}^{(k)}$ has an increasing chain

$$\{0\} = \mathfrak{X}^{(k,0)} \subseteq \mathfrak{X}^{(k,1)} \subseteq \ldots \subseteq \mathfrak{X}^{(k,k)} = \mathfrak{X}^{(k)}$$

of subspaces such that $\mathfrak{X}^{(k,j)}$ is j-dimensional and $T_k\mathfrak{X}^{(k,j)} \subseteq \mathfrak{X}^{(k,j)}$.

We now prove that if k_i, j_i are any pair of sequences such that $k_i \to \infty$, and $0 \leq j_i \leq k_i$, then $\mathfrak{Z} = \underline{\lim}\, \mathfrak{X}^{(k_i, j_i)}$ is an invariant subspace. Suppose indeed that $z \in \mathfrak{Z}$, so that there exists a sequence x_n of vectors in $\mathfrak{X}^{(k_n, j_n)}$ such that $x_n \to z$ as $n \to \infty$. Put $Tz = u$, then $Tx_n \to u$. Moreover, $T_n x_n \in \mathfrak{X}^{(k_n, j_n)}$, while

$$\begin{aligned}|u-T_nx_n| &\leq |u-Tx_n|+|(T-T_n)x_n| \\ &\leq |u-Tx_n|+2\rho(Tx_n, \mathfrak{X}^{(n)}) \\ &\leq |u-Tx_n|+2\rho(u-Tx_n, \mathfrak{X}^{(n)}) \\ &\qquad\qquad +2\rho(u, \mathfrak{X}^{(n)}) \\ &\leq 3|u-Tx_n|+2\rho(u, \mathfrak{X}^{(n)}) \to 0\end{aligned}$$

by (iii)–(vii) above, proving our assertion.

Note that we have actually proved the following somewhat stronger statement:

(viii) If $\mathfrak{Z} = \underline{\lim}\, \mathfrak{X}^{(k_n, j_n)}$ and $u = \lim Tx_n$, where $x_n \in \mathfrak{X}^{(k_n, j_n)}$, then $u \in \mathfrak{Z}$.

It only remains to construct sequences k_i, j_i for which the invariant space $\mathfrak{Z}$ differs both from $\{0\}$ and from $\mathfrak{X}$. This we do as follows.

Choose some $\alpha > 0$ so small that $\alpha < 1$ and $|T\bar{x}| \geq \alpha|T|$. Since $\bar{x} \in \mathfrak{X}^{(k)} = \mathfrak{X}^{(k,k)}$ for all k, the numbers $\rho(\bar{x}, \mathfrak{X}^{(k,0)}), \ldots, \rho(\bar{x}, \mathfrak{X}^{(k,k)})$ decrease from 1 to 0. Thus, for each k there exists a unique $j(k)$ such that

(ix) $\rho(\bar{x}, \mathfrak{X}^{(k,j(k))}) \geq \alpha > \rho(\bar{x}, \mathfrak{X}^{(k,j(k)+1)})$.

Choose a sequence $x_n \in \mathfrak{X}^{(n,j(n)+1)}$ such that $|x_n-\bar{x}| \leq \alpha$. Then there exists a subsequence x_{n_i} such that Tx_{n_i} converges strongly to an element y (since T is compact). Put $\mathfrak{Z}_i = \mathfrak{X}^{(k_i, j(k_i))}$, $\mathfrak{Z}_i' = \mathfrak{X}^{(k_i, j(k_i)+1)}$, and let $\mathfrak{Z} = \underline{\lim}\, \mathfrak{Z}_i$, $\mathfrak{Z}' = \underline{\lim}\, \mathfrak{Z}_i'$. Then, since $\mathfrak{Z}_i \subseteq \mathfrak{Z}_i'$ for all i, it follows easily that $\mathfrak{Z} \subseteq \mathfrak{Z}'$. Plainly, by (ix), $y \in \mathfrak{Z}'$, so that since $|y-T\bar{x}| = \lim |T(x_{n_i}-\bar{x})| \leq \alpha|T|$ while $|T\bar{x}| > \alpha|T|$, we have $y \neq 0$ and thus $\mathfrak{Z}' \neq \{0\}$. On the other hand, since by (ix) there is no point of any of the spaces within a distance α of $\bar{x}$, there can be no point of $\mathfrak{Z}$ within a distance α of $\bar{x}$, so that $\mathfrak{Z} \neq \mathfrak{X}$.

If there is no proper invariant subspace of $\mathfrak{X}$, it must follow that $\mathfrak{Z} = \{0\}$, $\mathfrak{Z}' = \mathfrak{X}$. By (ix) and the compactness of T, this implies that

(x) $Tx_i \to 0$ if x_i is a bounded sequence in $\mathfrak{Z}_i$.

On the other hand, since $\dim(\mathfrak{Z}_i'/\mathfrak{Z}_i) = 1$, there exists a vector $u_i \in \mathfrak{Z}_i'$ such that $\mathfrak{Z}_i + (u_i) = \mathfrak{Z}_i'$. Since z_i is finite-dimensional, we may even suppose that u_i has been chosen so that $|u_i| = 1$ and

$$\inf_{z \in \mathfrak{Z}_i} |z-u_i| = |u_i| = 1.$$

Then plainly

$$\text{(xi)} \quad |z_i + \alpha u_i| \geqq |\alpha| \text{ and } 2|z_i + \alpha u_i| \geqq |z_i|, \ z_i \in \mathfrak{Z}_i.$$

Since $\mathfrak{Z}' = \mathfrak{X}$, we may find sequences z_i, $\tilde{z}_i \in \mathfrak{Z}_i$ and α_i, $\tilde{\alpha}_i$ such that $z_i + \alpha_i u_i \to \bar{x}$ and $\tilde{z}_i + \tilde{\alpha}_i u_i \to T\bar{x}$. By (x) and (xi), α_i and $\tilde{\alpha}_i$ are bounded and $\alpha_i T u_i \to T\bar{x}$, $\tilde{\alpha}_i T u_i \to T^2\bar{x}$. Thus $|\alpha_i|$ must be bounded below by a positive constant and a subsequence of $\tilde{\alpha}_i/\alpha_i$ must converge to a constant γ, so that we must have $\gamma T\bar{x} = T^2\bar{x}$, contradicting (i). Q.E.D.

2 Definition. Let T be an operator and E a projection. We say that E is a *subdiagonalizing projection for* T if T leaves the range of E invariant, i.e., if $ETE = TE$.

3 Lemma. *Any operator T in Hilbert space admits a maximal totally ordered set $\mathscr{F}$ of orthogonal subdiagonalizing projections; i.e., a totally ordered set of subdiagonalizing projections not contained in any larger set of subdiagonalizing projections. The family $\mathscr{F}$ contains the strong limit of any monotone-increasing and of any monotone-decreasing sequence of projections in $\mathscr{F}$.*

Proof. Our first statement follows at once from the Hausdorff maximality theorem (I.2.6). Let $\{E_n\}$ be a monotone-increasing sequence of projections in $\mathscr{F}$, let E be in $\mathscr{F}$, and let E_∞ be the strong limit of E_n. If $E_n \leq E$ for all n, then $E_\infty \leq E$; if $E_n \geqq E$ for some E, then $E_\infty \geqq E$. Thus, if E_∞ is adjoined to $\mathscr{F}$, $\mathscr{F}$ remains totally ordered. On the other hand, since $E_n T E_n = T E_n$, we find in the limit $E_\infty T E_\infty = T E_\infty$. Thus T leaves the range of E invariant. It follows from the maximality of $\mathscr{F}$ that E_∞ is in $\mathscr{F}$. The proof for a monotone-decreasing sequence of projections in $\mathscr{F}$ is identical. Q.E.D.

If E and F are orthogonal projections and $E \geqq F$, then $EF = FE = F$. Thus all the projections of a maximal totally ordered family $\mathscr{F}$ of orthogonal projections commute. Let $\{x_n\}$ be a dense set of vectors in Hilbert space and put

$$\varphi(E) = \sum_{n=1}^{\infty} \frac{|Ex_n|^2}{(1+|x_n|^2)2^n}.$$

Then $\varphi(E)$ plainly increases with the projection E. If E, E_1 are in $\mathscr{F}$,

and $\varphi(E) = \varphi(E_1)$, then, since $\mathscr{F}$ is totally ordered, we may assume for the sake of definiteness that $E \leq E_1$. Hence $|Ex_n|^2 = |E_1x_n|^2$ for each n, so that $Ex_n = E_1x_n$ and $E = E_1$. That is, $\varphi(E) = \varphi(E_1)$ implies $E = E_1$. Similarly, $\varphi(E) \leq \varphi(E_1)$ implies $E \leq E_1$. If E_n, E are in $\mathscr{F}$ and $\varphi(E_n)$ increases to the limit $\varphi(E)$, then it follows from what we have already proved that E_n is an increasing sequence of projections and $E_n \leq E$. If E_∞ is the strong limit of E_n, then $E_\infty \leq E$ and $\varphi(E_\infty) = \varphi(E)$. Thus, it follows as above that $E_\infty = E$. This proves that if $\varphi(E_n)$ is increasing with limit $\varphi(E)$, then E_n has the strong limit E. We may show in exactly the same way that if $\varphi(E_n)$ is decreasing with the limit $\varphi(E)$, then E_n has the strong limit E. Since every convergent sequence contains either a monotone-increasing or a monotone-decreasing sequence, it therefore follows that $\varphi(E_n) \to \varphi(E)$ implies $E_n \to E$ strongly. Hence, if we choose a countable set $\{E_i\} \subseteq \mathscr{F}$ such that $\{\varphi(E_i)\}$ is dense in the range of the function φ, then each E in $\mathscr{F}$ is the limit either of an increasing or of a decreasing sequence of projections in $\mathscr{F}$. We shall show below that there exists a Hermitian operator T such that all the projections E_i belong to the spectral resolution of T and such that each projection in the spectral resolution of T is the strong limit of linear combinations of the projections E_i. Suppose for the moment that this has been done. It follows readily that all the projections E in $\mathscr{F}$ belong to the spectral resolution of T. We may consequently use the results of Chapter X on the spectral representation of T to obtain a spectral representation for the maximal family $\mathscr{F}$ of projections.

This may be carried out as follows. By Lemma X.5.8, there exists a sequence x_n, $n \geq 0$, of vectors in our Hilbert space $\mathfrak{H}$, such that $\mathfrak{H}$ is the orthogonal direct sum of the spaces $\mathfrak{H}(x_i)$ where

$$\mathfrak{H}(x_i) = \overline{\mathrm{sp}}\{f(T)x_i \mid f \in C(\sigma(T))\}, \tag{1}$$

and there exists a decreasing sequence $\tilde{e}_n$ of sets such that $(E(e)x_n, x_n) = (E(e\tilde{e}_n)x_0, x_0)$, $n \geq 0$, for all Borel sets e.

We may and shall assume in addition that $|x_0| = 1$.

Put $\lambda(E) = |Ex_0|^2$. It follows as above that $\lambda(E)$ is an increasing function of E, that $\lambda(E_1) = \lambda(E_2)$ implies $E_1 = E_2$ for any pair E_1, E_2 of projections in $\mathscr{F}$, that $\lambda(E_1) \leq \lambda(E_2)$ implies $E_1 \leq E_2$, and that $\lambda(E_n) \to \lambda(E)$ implies $E_n \to E$ strongly. It is plain, on the other hand,

that $E_n \to E$ implies $\lambda(E_n) \to \lambda(E)$. Thus, from Lemma 3, the range of the function λ is a certain closed subset C of $[0, 1]$. Since the mapping $E \to \lambda(E)$ is one-to-one, we may invert it, thereby parametrizing the maximal set $\mathscr{F}$ of subdiagonalizing projections as E_λ, $\lambda \in C$. By our above remarks, E_λ depends continuously on λ, increases as λ increases, and we have $|E_\lambda x_0|^2 = \lambda$.

Let (a, b) be an interval complementary to the closed set C. Then, since $\mathscr{F}$ is maximal, there is no closed subspace $\mathfrak{X}$ of Hilbert space $\mathfrak{H}$ which is invariant under T and which satisfies $E_a\mathfrak{H} \subset \mathfrak{X} \subset E_b\mathfrak{H}$. Consider the mapping $T_0 = (E_b - E_a)T|(E_b - E_a)\mathfrak{H}$. If $\mathfrak{X}_0$ is a closed proper subspace of the Hilbert space $(E_b - E_a)\mathfrak{H}$ which is invariant under T_0, then it is plain that $\mathfrak{X} = \mathfrak{X}_0 \oplus E_a\mathfrak{H}$ is closed, invariant under T, and satisfies $E_a\mathfrak{H} \subset \mathfrak{X} \subset E_b\mathfrak{H}$. Thus, T_0 can admit no proper closed invariant subspaces. Since T_0 is compact, it follows immediately from Theorem 1 that $(E_b - E_a)\mathfrak{H}$ is one-dimensional.

Let g denote a function defined on the interval $[0, 1]$, taking on each of its finitely many values on an interval of $[0, 1]$, and having all its points of discontinuity in the closed set C. Such a function has the form

$$g(s) = \alpha_i, \qquad a_i \leqq s < a_{i+1}, \tag{2}$$

where

$$0 = a_1 < a_2 < \ldots < a_n = 1. \tag{3}$$

Call the class of all such functions $\mathfrak{Z}$. Let U_m be the mapping which sends the above function $g \in \mathfrak{Z}$ into the vector

$$\sum_{i=1}^{n} \alpha_i(E(a_{i+1}) - E(a_i))x_m. \tag{4}$$

Then, in the first place, we have

$$|U_0 g|^2 = \sum_{i=1}^{n} |\alpha_i|^2(a_{i+1} - a_i) = \int_0^1 |g(s)|^2 ds;$$

thus U_0 may be extended to a unique isometry between the closure of $\mathfrak{Z}$ in $L_2[0, 1]$ and the closure in $\mathfrak{H}(x_0)$ of the vectors (4) (for which $m = 0$). It is easily seen that the closure of $\mathfrak{Z}$ in $L_2[0, 1]$ is the set of all functions which are constant on every interval of the complement

of the closed set C; we shall denote this subspace of $L_2[0, 1]$ by the symbol $\tilde{L}_2(C)$. Since each projection in the spectral resolution of T and hence each continuous function of T is a strong limit of linear combinations of the projections E_i, it follows from (1) that the closure in $\mathfrak{H}(x_m)$ of the vectors (4) is $\mathfrak{H}(x_m)$. Thus, by taking $m = 0$, it follows that U_0 may be extended to an isomorphism between $\tilde{L}_2(C)$ and $\mathfrak{H}(x_0)$, which we continue to denote by the letter U_0.

Let S be a bounded operator in $\tilde{L}_2(C)$ which commutes with each projection $U_0^{-1}E_\lambda U_0$. Let 1 denote the function in $\tilde{L}_2(C)$ which is identically equal to 1. If $U_0^{-1}SU_0(1) = h(x)$, then it is evident that $(U_0^{-1}SU_0g)(x) = g(x)h(x)$ for each $g \in \mathfrak{F}$, so that, since $\mathfrak{F}$ is dense in $\tilde{L}_2(C)$, $(U_0^{-1}SU_0g)(x) = g(x)h(x)$ for all $g \in \tilde{L}_2(C)$. In order that S be bounded, it is plainly necessary that h be bounded; in order that S be a projection, it is plainly necessary that $h(x) = 0$ or 1 almost everywhere. In particular, for each projection $E(\tilde{e})$ in the spectral resolution of T, there must exist a Borel set e of $[0, 1]$ such that $(U_0^{-1}E(\tilde{e})U_0g)(x) = \chi_e(x)g(x)$, χ_e denoting the characteristic function of e. If $E(\tilde{f})$ is a second projection of the spectral resolution of T, and $\tilde{f} \subseteqq \tilde{e}$, then since $E(\tilde{e})E(\tilde{f}) = E(\tilde{f})$, we must have $(U_0^{-1}E(\tilde{f})U_0g)(x) = \chi_f(x)g(x)$, where $f \subseteqq e$. On the other hand, it is plain from the definition of the mapping U_0 that $(U_0^{-1}E_\lambda U_0g)(x) = \chi_{[0,\lambda)}(x)g(x)$. It follows from what has been stated just subsequent to formula (1) that there must exist a decreasing family $e_1 \supseteqq e_2 \supseteqq e_3 \ldots$ of Borel subsets of $[0, 1]$ such that

$$(5) \qquad (E_\lambda x_m, x_m) = (E_\lambda E(\tilde{e}_m)x_0, x_0) = \mu([0, \lambda) \cap e_m), \qquad m \geqq 1.$$

where μ denotes Lebesgue measure.

Let (a, b) be an interval complementary to the closed set C. Then $(E_b - E_a)x_m$ is orthogonal to $(E_b - E_a)x_0$ if $m \geqq 1$. Since we have seen above that the range of the projection $E_b - E_a$ is one-dimensional, it follows that $(E_b - E_a)x_m = 0$. Thus $e_m \cap (a, b) = \phi$. The sets $e_1, e_2, \ldots$ consequently are contained in the set C.

It follows from (5) that for $m \geqq 1$ and g in $\mathfrak{F}$ we have

$$\begin{aligned} |U_m g|^2 &= \sum_{i=1}^{n} |\alpha_i|^2 \mu([a_i, a_{i+1}] \cap e_m) \\ &= \int_{e_m} |g(s)|^2 ds. \end{aligned}$$

Thus U_m may be extended to an isometry between the closure of $\mathfrak{Z}$ in the space $L_2(e_m, \mu)$ and the closure of the set of vectors (4). Since each projection in the spectral resolution of T and hence each continuous function of T is a strong limit of linear combinations of the projections E_i, it follows from (1) that the closure of the set of vectors (4) is $\mathfrak{H}(x_m)$. On the other hand, since $e_m \subseteq C$, the closure of $\mathfrak{Z}$ in the space $L_2(e_m, \mu)$ is easily seen to be the entire space $L_2(e_m, \mu)$. Thus U_m may be extended to an isomorphism between $L_2(e_m, \mu)$ and $\mathfrak{H}(x_0)$, which we continue to denote by the letter U_m. It is plain from the definition of U_m that $(U_m^{-1}E_\lambda U_m g)(x) = \chi_{[0,\lambda)}g(x)$, $m \geqq 1$.

Since the whole Hilbert space $\mathfrak{H}$ is the direct sum of the orthogonal subspaces $\mathfrak{H}(x_i)$, the mapping

$$U:[g_0(x), g_1(x), \ldots] \to \sum_{m=0}^{\infty} U_m g_m$$

is an isomorphism of $\tilde{L}_2(C) \oplus L_2(e_1, \mu) \oplus L_2(e_2, \mu) \oplus \ldots$ onto $\mathfrak{H}$. On the other hand, it is clear that $U^{-1}E_\lambda U$ is the mapping

$$[g_0(x), g_1(x), \ldots] \to [\chi_{[0,\lambda)}(x)g_0(x), \chi_{[0,\lambda)}(x)g_1(x), \ldots].$$

The preceding analysis leads us to the following theorem.

4 THEOREM. *Let $\mathcal{F}$ be a maximal totally ordered family of subdiagonalizing orthogonal projections for the compact operator T in Hilbert space. Then there exists a closed set C in* $[0, 1]$ *and a sequence $e_1 \supseteq e_2 \supseteq \ldots$ of Borel sets such that $\mathcal{F}$ is isometrically equivalent to the family $\{E_\lambda | \lambda \in C\}$ of projections in the space*

$$\mathfrak{H}_0 = \tilde{L}_2(C, \mu) \oplus L_2(e_2, \mu) \oplus L_2(e_n, \mu) \oplus \ldots$$

defined by the formula

$$E_\lambda[g_1(x), g_2(x), \ldots] = [\chi_\lambda(x)g_1(x), \chi_\lambda(x)g_2(x), \ldots],$$

where $\chi_\lambda(x)$ denotes the characteristic function of the interval $[0, \lambda)$, *and μ the Lebesgue measure on* $[0, 1]$.

PROOF. To complete the proof of this theorem, we have only to show that if E_i is a countable, totally ordered family of projections, there exists a Hermitian operator T such that:

(a) All the operators E_i belong to the spectral resolution of T.

(b) Every projection in the spectral resolution of T is the strong limit of linear combinations of the projections E_i.

This we do as follows. Let $\mathfrak{A}$ be the commutative B^*-algebra of operators generated by the projections E_i and let Λ be its spectrum. If ω is any element of Λ, i.e., any continuous homomorphism of $\mathfrak{A}$ into the complex number space, let $S(\omega)$ be the sequence $[\omega(E_1), 0, \omega(E_2), 0, \omega(E_3), 0, \ldots]$. Since $E_i^2 = E_i$, $S(\omega)$ is a sequence of zeros and ones, which we may take to be the dyadic-decimal representation of a certain real number $r(\omega)$ lying in the interval $[0, 1]$.

Let $E(e)$ be the spectral resolution of the algebra $\mathfrak{A}$, so that $E(e)$ is a countably additive regular projection-valued Borel measure defined on Λ (cf. Theorem X.2.1). By Theorem X.2.1 we have

$$\int_\Lambda \omega(E_n)E(d\omega) = E_n.$$

Let the Hermitian operator T be defined by

$$T = \int_\Lambda r(\omega)E(d\omega).$$

Then, if E_1 is the spectral measure on the unit interval defined by the formula $E_1(e) = E(r^{-1}e)$, we have

$$T = \int_0^1 rE_1(dr)$$

by the change of measure principle. Hence, by Corollary X.2.7, $E_1(e)$ is the spectral resolution of the Hermitian operator T. If $d_n(r)$ is the nth entry in the dyadic expansion of the real number r, we have

$$\begin{aligned} E_1(\{r|d_{2n+1}(r) = 1\}) &= E(\{\omega|d_{2n+1}(r(\omega)) = 1\}) \\ &= E(\{\omega|\omega(E_n) = 1\}) \\ &= \int_\Lambda \omega(E_n)E(d\omega) = E_n. \end{aligned}$$

Thus, each of the projections E_n is in the spectral resolution of the Hermitian operator T.

On the other hand, let a Borel set $e \subseteq \Lambda$ be given. Choose a dense subset x_n of Hilbert space. Then, since the measure $E(e)$ is regular, there exists for each integer m a closed set $f_m \subseteq e$ and an open set $O_m \supseteq e$ such that

$$|E(e)x_n|^2 - \frac{1}{m} \leq |E(f_m)x_n|^2 \leqq |E(O_m)x_n|^2 \leqq |E(e)x_n|^2 + \frac{1}{m},$$

$$1 \leqq n \leq m.$$

Using the Urysohn theorem (I.5.2), we may find a continuous function $\varphi_m(\omega)$ on Λ such that $0 \leqq \varphi_m(\omega) \leqq 1$ and such that $\varphi_m(\omega) = 0$ if $\omega \notin O_m$, while $\varphi_m(\omega) = 1$ if ω is in f_m. It then follows readily from the formula displayed just above that if

$$\Phi_m = \int \varphi_m(\omega)E(d\omega),$$

we have

$$|(E(e)-\Phi_m)x_n|^2 = \int |\chi_e(\omega)-\varphi_m(\omega)|^2|E(d\omega)x_n|^2$$

$$\leqq \frac{1}{m}, \qquad 1 \leqq n \leq m.$$

Thus $E(e)$ is the strong limit of the operators Φ_m. On the other hand, it follows from Theorem X.2.1 that Φ_m belongs to the algebra $\mathfrak{A}$, so that Φ_m is a limit of linear combinations of products of the operators E_i. Since the projections E_i form a totally ordered family, the product of any finite subcollection of the E_i is the smallest of the E_i (i.e., the projection E_i with the least range). Thus $E(e)$ is a strong limit of linear combinations of the projections E_i.

This proves (a) and (b), and completes the proof of Theorem 4.

We meet the smallest number of inessential technical complications if we state a result on subdiagonalization for the special case of Hilbert-Schmidt operators, in which case a kernel representation of the operator always exists. The following preliminary lemma describes this kernel representation and gives various of its elementary properties.

5 LEMMA. *Let T be a Hilbert-Schmidt operator in the space $\mathfrak{H}_0$ of Theorem 4. Then there exists a unique set $K_{ij}(s, t)$ of kernels, defined and square-integrable on* $[0, 1]\times[0, 1]$, *such that*

(i) $K_{ij}(s, t) = 0$ *if* $s \notin e_i$, $i \geqq 1$ *or if* $t \notin e_j$, $j \geqq 1$.

(ii) $K_{1,j}(s, t) = K_{1,j}(s', t)$ *if s and s' lie in the same interval of the complement of C.*

(iii) $K_{i,1}(s, t) = K_{i,1}(s, t')$ *if t and t' lie in the same interval of the complement of C.*

(iv) $\|K\|^2 = \sum_{i,j=1}^{\infty} \int_0^1 \int_0^1 |K_{ij}(s, t)|^2 ds dt < \infty$.

(v) $T[f_1(s), f_2(s), \ldots] = [g_1(s), g_2(s), \ldots]$,

where

$$g_i(s) = \sum_{j=1}^{\infty} \int_0^1 K_{ij}(s, t) f_j(t) dt,$$

the series converging unconditionally in the topology of L_2. Conversely, if K_{ij} is any family of kernels satisfying (i), . . ., (iv), *then* (v) *defines a Hilbert-Schmidt operator in $\mathfrak{H}_0$ with norm given by* (iv).

Proof. Let $A = [0, 1] \times N$, where N is the set of all integers $n \geq 1$. If we regard N as a measure space, each integer having measure 1, and $[0, 1]$ as carrying its ordinary Borel-Lebesgue measure, we may regard A as a measure space, carrying the product measure ν. Our first step will be to establish that every Hilbert Schmidt operator K in $L_2(A)$ is represented by a unique kernel $K(\cdot, \cdot) \in L_2(A \times A)$ in such a way that

(1) $$\|K\|^2 = \int_A \int_A |K(a, b)|^2 \nu(da) \nu(db) < \infty$$

and

(2) $$(Kf)(a) = \int_A K(a, b) f(b) \nu(db).$$

We shall also establish, for later purposes, that if K is represented by the kernel $K(a, b)$, then the adjoint operator K^* is represented by the kernel $\overline{K(b, a)}$. Conversely, if $K(\cdot, \cdot)$ is a kernel satisfying (1), then (2) defines a Hilbert-Schmidt operator.

To prove these assertions, let $\{\varphi_j\}$ be an orthonormal basis for $L_2(A)$. Then, from the definition of a product measure space every function in $L_2(A \times A)$ may be approximated by linear combinations of characteristic functions of sets of the form $e \times f$, where e and f are measurable subsets of A. Therefore, the set of linear combinations of product functions $\overline{\psi(a)\eta(b)}$ is dense in $L_2(A \times A)$. Thus, the set of functions $\{\varphi_i(a)\overline{\varphi_j(b)}\}$ is an orthonormal basis for $L_2(A \times A)$. Let K be a Hilbert-Schmidt operator in $L_2(A)$. Then, by Corollary 6.3, if we put $C_{ij} = (K\varphi_j, \varphi_i)$, we have $\Sigma |C_{ij}|^2 = \|K\|^2 < \infty$. Thus, there exists in $L_2(A \times A)$ a function $K(a, b)$ whose Fourier coefficients with

respect to the basis $\{\varphi_i(a)\overline{\varphi_j(b)}\}$ are C_{ij}. Plainly (1) is satisfied. We have

$$\left\{\int_A \left|\int_A K(a, b)f(b)\nu(db)\right|^2 \nu(da)\right\}^{1/2} \leq \left\{\int_A \left\{\int_A \left|K(a, b)\right|^2 \nu(da)\right\} \nu(db)\right\}^{1/2} \left\{\int_A |f(b)|\nu(db)\right\}^{1/2}$$

from Theorem III.2.20, Theorem III.11.17, and Schwarz' inequality; thus the integral on the right of (2) defines a bounded operator $\tilde{K}$. It is plain from the definition of the kernel K that

$$(K\varphi_j, \varphi_i) = (\tilde{K}\varphi_j, \varphi_i)$$

for each i and j. Thus, forming linear combinations and by continuity,

$$(Kf, g) = (\tilde{K}f, g), \qquad f, g \in L_2(A),$$

so that $K = \tilde{K}$ and (2) is satisfied.

Since $(K^*\varphi_j, \varphi_i) = \overline{(K\varphi_i, \varphi_j)} = \overline{C_{ji}}$, the kernel representing K^* is the function $\overline{K(b, a)} = K^*(a, b)$ with the Fourier coefficients $\overline{C_{ji}}$.

If $K(a, b)$ is a kernel satisfying the inequality in (1), then we have seen that the right side of (2) defines a bounded operator $\tilde{K}$. Since $(K\varphi_i, \varphi_j)$ are plainly the Fourier coefficients of $K(a, b)$ relative to the orthonormal basis $\{\varphi_i(a)\overline{\varphi_j(b)}\}$, it follows from Corollary 6.3 that $\tilde{K}$ is in HS and that the equality in (1) is satisfied. Consequently, if the kernel K represents the operator 0, we must have $K(a, b) = 0$, $\nu \times \nu$ almost everywhere. Thus all the above preliminary assertions are proved.

It is plain, since A is the product measure space $[0, 1] \times N$, that the elements of $L_2(A)$ may equivalently be regarded as sequences

$$f = [f_1(s), f_2(s), \ldots]$$

of functions in $L_2[0, 1]$, the norm of such a sequence being defined by

$$|f|^2 = \sum_{i=1}^{n} \int_0^1 |f_i(s)|^2 ds.$$

Similarly, functions $K(\cdot, \cdot) \in L_2(A \times A)$ may equivalently be regarded as infinite matrices $K_{ij}(s, t)$ of functions in $L_2([0, 1] \times [0, 1])$. Thus our above conclusions may be reformulated as follows:

If K is a Hilbert-Schmidt operator in $L_2(A)$, there exists a unique set $K_{ij}(s, t)$ of kernels representing K in the sense that

$$K[f_1(s), f_2(s), \ldots] = [g_1(s), g_2(s), \ldots] \tag{3}$$

where

$$g_i(s) = \sum_{j=1}^{\infty} \int_0^1 K_{ij}(s, t) f_j(t) dt, \tag{4}$$

the series converging unconditionally in the topology of L_2. Moreover, (iv) is satisfied. The adjoint operator K^* is represented by the set of kernels

$$K^*_{ij}(s, t) = \overline{K_{ji}(t, s)}$$

Finally, if K is any set of kernels satisfying the inequality in (iv), then (3) and (4) define a Hilbert-Schmidt operator K in $L_2(A)$ satisfying the equality in (iv).

It is plain from Theorem 4 that the Hilbert space $\mathfrak{H}_0$ of Theorem 4 is a subspace of $L_2(A)$. The orthogonal projection E of $L_2(A)$ onto $\mathfrak{H}_0$ is readily verified to be defined by the formula

$$E[f_1(s), f_2(s), \ldots] = [g_1(s), g_2(s), \ldots],$$

where $g_j(s) = \chi_{e_j}(s) f_j(s)$, $j \geqq 2$ and e_j denotes the Borel sets of Theorem 4, and where

$$g_1(s) = f_1(s), \qquad s \in C$$

$$g_1(s) = \frac{1}{\mu(I)} \int_I f_1(t) dt, \qquad s \in I;$$

where I denotes an arbitrary one of the intervals complementary to the closed set C and $\mu(I)$ its Lebesgue measure.

If T is a Hilbert-Schmidt operator in $\mathfrak{H}_0$, then since an orthonormal basis for $L_2(A)$ may be found which consists of the union of an orthonormal basis for $\mathfrak{H}_0$ and an orthonormal basis for $\mathfrak{H}_0^{\perp}$, it is plain from Definition 6.1 that $K = TE$ is a Hilbert-Schmidt operator in $L_2(A)$. We have plainly $EKE = K$, $K|\mathfrak{H}_0 = T$, $K^*|\mathfrak{H}_0 = T^*$. Moreover, from Definition 6.1, $\|K\| = \|T\|$. Thus, the Hilbert-Schmidt operators in $\mathfrak{H}_0$ are simply the restrictions to $\mathfrak{H}_0$ of the class of Hilbert-Schmidt operators in $L_2(A)$ satisfying $EKE = K$, or

equivalently $EK = KE = K$; the mapping of such operators onto their restrictions to $\mathfrak{H}_0$ is isometric.

Since it is obvious that K satisfies $EK = KE = K$ if and only if the representing kernels $K_{ij}(s, t)$ satisfy (i), (ii), and (iii), the present lemma is fully proved. Q.E.D.

We have also proved the following corollary.

6 Corollary. *The adjoint operator of the operator K of the preceding lemma is defined by the set K^*_{ij} of kernels defined by*

$$K^*_{ij}(s, t) = \overline{K_{ji}(t, s)}.$$

The following theorem gives the subdiagonal representation of a Hilbert-Schmidt operator.

7 Theorem. *Let $\mathscr{F}$ and T be as in Theorem 4, and let T be a Hilbert-Schmidt operator. Then the kernels K_{ij} of Lemma 5 satisfy*

$$K_{ij}(s, t) = 0$$

unless either $s \leqq t$ or $i = 1$, $j = 1$, and s and t lie in the same interval of the complement of C. Conversely, if the kernels K_{ij} have this property, then $\mathscr{F}$ is a maximal family of subdiagonalizing orthogonal projections for T.

Moreover, T is a quasi-nilpotent operator if and only if $K_{11}(s, t) = 0$ whenever s and t lie in the same interval of the complement of C.

Proof. Since T leaves the range of E_λ invariant, $(I - E_\lambda)TE_\lambda = 0$. The form for the projection E_λ given by Theorem 4 makes it plain that the kernels representing $(I - E_\lambda)TE_\lambda$ are

$$\tilde{\chi}_\lambda(s)K_{ij}(s, t)\chi_\lambda(t),$$

where χ_λ denotes the characteristic function of the interval $[0, \lambda)$, and $\tilde{\chi}_\lambda$ denotes the characteristic function of the interval $(\lambda, 1]$. Thus, $K_{ij}(s, t) = 0$ if there exists a $\lambda \in C$ such that $s \geqq \lambda \geqq t$. That is, $K_{ij}(s, t) = 0$ unless either $s < t$ or s and t belong to the same interval of the complement of C. Since, by Lemma 5, $K_{ij}(s, t) = 0$ if s and t belong to the same interval of the complement of C unless $i = j = 1$, our first assertion is proved.

Conversely, suppose that the kernels K_{ij} have the property

described in the hypotheses of the present theorem. Then, retracing the steps of the above argument, we can conclude that $(I-E_\lambda)TE_\lambda = 0$ for each λ in C. Hence T leaves the range of each projection E_λ invariant, and the set $\mathscr{F}$ of projections E_λ, $\lambda \in C$, subdiagonalizes T. To prove the second proposition of our theorem, we have only to verify that if E is any other orthogonal projection such that $E \geqq E_\lambda$ or $E_\lambda \geqq E$ for each λ in C, then E is in $\mathscr{F}$. For this purpose, let $\lambda_0 = \sup\,(\lambda | \lambda \in C,\ E \geqq E_\lambda)$. Plainly, $E \geqq E_{\lambda_0}$, while $E \leqq E_\lambda$ if $\lambda > \lambda_0$. If $\lambda_1 = \inf\,(\lambda | \lambda \in C, \lambda > \lambda_0)$, we must therefore have $E \leqq E_{\lambda_1}$. It is clear that $\lambda_1 = \lambda_0$ unless (λ_0, λ_1) is an interval complementary to the closed set C. If $\lambda_1 = \lambda_0$, then $E_{\lambda_0} \leqq E \leqq E_{\lambda_0}$, so that E is in $\mathscr{F}$. If (λ_0, λ_1) is an interval complementary to the closed set C, then it is plain from Theorem 4 that $E_{\lambda_1} - E_{\lambda_0}$ is a projection with a one-dimensional range. Thus, since $E_{\lambda_0} \leqq E \leqq E_{\lambda_1}$, it follows that either $E = E_{\lambda_0}$ or $E = E_{\lambda_1}$, so that E is in $\mathscr{F}$ in this case also, and the second assertion of our theorem is proved.

Next, suppose that T is quasi-nilpotent. Let (a, b) be an interval of the complement of C, and χ its characteristic function. Let $K_{11}(s, t) = c$ for s, t in $[a, b]$. Then, if f is the vector

$$f = [\chi, 0, 0, \ldots],$$

it follows from what has already been proved that we may write

$$Tf = [c\chi + g_1, g_2, \ldots],$$

where $[g_1, g_2, \ldots]$ belongs to the range of the projection E_a. Thus, inductively, since T leaves the range of E_a invariant, we have

$$T^n f = [c^n\chi + g_1^{(n)}, g_2^{(n)}, \ldots],$$

where $[g_1^{(n)}, g_2^{(n)}, \ldots]$ belongs to the range of the projection E_a. Thus

$$(T^n f, f) = c^n(b-a).$$

Since T is quasi-nilpotent, we have $|T^n| = O(\varepsilon^n)$ for each positive ε as $n \to \infty$. Thus we must have $c = 0$, i.e., $K_{11}(s, t) = 0$ if s and t belong to the same interval of the complement of C.

Conversely, we shall suppose that $K_{11}(s, t) = 0$ if s and t belong to the same interval of the complement of C, and prove that T is quasi-nilpotent. Since by Lemmas VII.6.7 and VII.6.8, Theorem

VII.4.5, Lemma 6.2, and Theorem 6.6 the limit in Hilbert-Schmidt norm of a sequence of quasi-nilpotent Hilbert-Schmidt operators is quasi-nilpotent, it follows from Lemma 5 that we may assume without loss of generality that for some finite M, $|K_{ij}(s, t)| \leqq M$ and that for some finite d, $K_{ij}(s, t) = 0$, if $i \geqq d$ or $j \geqq d$. The nth power T^n of T is plainly represented by the set of kernels $K_{ij}^{(n)}$ defined by

$$K_{ij}^{(n)}(s, t) = 0, \qquad \text{if } i \geqq d \qquad \text{or } j \geqq d,$$

$$K_{ij}^{(n)}(s, t) =$$
$$\sum_{i_1, \ldots, i_{n-1}=1}^{d} \int_0^1 \cdots \int_0^1 K_{i, i_1}(s, \sigma_1) K_{i_1, i_2}(\sigma_1, \sigma_2) \ldots K_{i_{n-1}, j}(\sigma_{i_{n-1}}, t) d\sigma_1 \ldots d\sigma_{i_{n-1}},$$
$$1 \leqq i, j \leqq d.$$

Using the bound M for the kernels K_{ij} and the fact that $K_{ij}(s, t) = 0$ unless $s \leqq t$, we may estimate the sum in the preceding equation by

$$(dM)^n \int_{0 \leq \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_{n-1} \leq 1} \cdots \int d\sigma_1 \ldots d\sigma_{n-1} = \frac{(dM)^n}{(n-1)!}.$$

Thus, by Lemmas 6.2 and 5, $|T^n| = o(\varepsilon^n)$ for each $\varepsilon > 0$, and T is quasi-nilpotent. Q.E.D.

8 LEMMA. *Let H be a compact self adjoint operator leaving the range of each of the projections E_λ of Theorem 4 invariant. Then*

$$H[f_1(s), f_2(s), \ldots] = [m(s)f_1(s), 0, 0, \ldots],$$

where $m(s)$ is a bounded function vanishing on C and constant on each of the complementary intervals of C.

PROOF. If $E_\lambda H E_\lambda = H E_\lambda$, then, taking adjoints, we have $E_\lambda H = E_\lambda H E_\lambda = H E_\lambda$. Thus H commutes with any orthogonal projection whose range it leaves invariant. For each Borel set e which either includes or excludes the whole of each interval of the complement of C, define the projection $E(e)$ in the space $\mathfrak{H}_0$ of Theorem 4 by writing

$$E(e)[f_1(s), f_2(s), \ldots] = [\chi_e(s)f_1(s), \chi_e(s)f_2(s), \ldots],$$

χ_e denoting the characteristic function of the set e. Since, from what we have seen, $E(I_j)H = HE(I_j)$ if $I_j = (a, b)$ is an interval of the

complement of C (for $E(I_j) = E_b - E_a$), we have $HE(C) = E(C)H$. Thus, if $\mathfrak{X}$ is the range of $E(C)$, $\mathfrak{X}$ is invariant under H and the restriction $\tilde{H}$ of H to $\mathfrak{X}$ is a Hermitian operator in the Hilbert space $\mathfrak{X}$. We wish to show that $\tilde{H} = 0$.

Suppose that this is not the case. Then, by Corollary X.3.5 and Corollary X.2.3, there is a number λ such that the space $\mathfrak{X}_0 = \{x | \tilde{H}x = \lambda x\}$ is finite-dimensional and non-zero. By Theorem X.2.1, there must consequently exist in $\mathfrak{X}_0$ a vector x such that $E(e)x = x$ or $E(e)x = 0$ for each Borel set e. Since $E(C)x = x$, it follows that if C is divided into the union of 2^n Borel sets e_j each of diameter less than 2^{-n}, we must have $E(e_j)x = x$ for exactly one of them. Thus, for each integer n, there exists a Borel subset $\tilde{e}_n$ of diameter at most 2^{-n} such that $E(\tilde{e}_n)x = x$. Putting $\bar{e}_n = \bigcup_{m>n} \tilde{e}_m$, we may even suppose that the $\tilde{e}_n$ form a monotone-decreasing family of sets. But then it is clear from the definition of $E(e)$ that $E(\tilde{e}_n)y \to 0$ for each y. We consequently obtain a contradiction which proves that $\tilde{H} = 0$, so that $H = HE(C')$.

If I_j is an interval complementary to C, then it is plain from Theorem 4 that $E(I_j)$ has a one-dimensional range. Thus there exists a constant m_j, $|m_j| \leq |T|$, such that $HE(I_j) = m_j E(I_j)$. If we define the function $m(s)$ by $m(s) = m_j$, $s \in I_j$, and $m(s) = 0$, $s \in C$, and put

$$M[f_1(s), f_2(s), \ldots] = [m(s) f_1(s), 0, 0, \ldots]$$

it is plain then that $HE(I_j) = ME(I_j)$, so that $HE([0, 1]-C) = ME([0, 1]-C)$. Since $HE(C) = ME(C) = 0$, it follows by addition that $H = M$, proving the present lemma. Q.E.D.

We recall that if T is an operator in Hilbert space, $(T+T^*)/2$ is called its *real* or *Hermitian part*, and $(T-T^*)/2i$ is called its *imaginary* or *anti-Hermitian part*.

Let us note that the quasi-nilpotent operator

$$J : f(x) \to i \int_0^x f(y) dy$$

in $L_2[0, 1]$ has been shown to be of class $C_{1+\varepsilon}$ for each positive ε, but to lie outside of class C_1. On the other hand, the adjoint of the operator J is

$$J^* : f(x) \to -i \int_x^1 f(y) dy.$$

Thus $(2i)^{-1}(J-J^*)f(x) = 2^{-1}\int_0^1 f(y)dy$; i.e., the anti-Hermitian part of J is a one-dimensional operator. This makes it clear that in order to conclude that $N \in C_p$ if the anti-Hermitian part of N lies in C_p, we must have $p > 1$. In this range, the following theorem states exactly such a result.

9 THEOREM. *If $1 < p < \infty$, and if T is a compact quasi-nilpotent operator whose anti-Hermitian part belongs to the class C_p, then T itself belongs to the class C_p.*

We will prepare for the proof of Theorem 9 with a preliminary series of lemmas and theorems. First we state a convexity theorem for the classes C_p which is in close analogy and even in close relation to the Riesz convexity theorem of Section VI.10.

10 THEOREM. *Let $1 \leqq p, r, p', r' \leqq \infty$; $p > p'$. Let $\mathscr{S}$ be a bounded linear transformation mapping the class C_p of compact operators into the class C_r of compact operators. Suppose that $\mathscr{S}$ is also a bounded linear transformation of the subclass $C_{p'}$ of C_p into the class $C_{r'}$. Then, if $0 < \alpha < 1$, $1/p'' = \alpha/p+(1-\alpha)/p'$, $1/r'' = \alpha/r+(1-\alpha)/r'$, $\mathscr{S}$ is a bounded linear transformation of $C_{p''}$ into $C_{r''}$.*

PROOF. Put

$$\Phi(T, B) = \operatorname{tr}\big((\mathscr{S}T)B\big)$$

for each T in C_p and each B with finite-dimensional range. Lemma 9.14 makes it plain that to prove the desired result, we have only to establish the existence of a constant Γ such that

$$|\Phi(T, B)| \leq \Gamma|T|_{p''}|B|_{q''},$$

where

$$\frac{1}{q''} = \frac{\alpha}{q} + \frac{1-\alpha}{q'}, \qquad \frac{1}{q} + \frac{1}{r} = 1, \qquad \frac{1}{q'} + \frac{1}{r'} = 1.$$

Since from Lemma 9.14 $\Phi(T, B)$ is continuous in T for each B with finite-dimensional range, it follows from Lemma 9.11 that we may assume without loss of generality that T has a finite-dimensional range.

The arguments of the third paragraph of the proof of Lemma 9.6 now show that our theorem may be reduced to the following proposition.

If $\Phi(T, B)$ is a bilinear form on the family of all matrices in d-dimensional Hilbert space, and if

$$|\Phi(T, B)| \leqq \Gamma|T|_p|B|_q$$
$$|\Phi(T, B)| \leqq \Gamma|T|_{p'}|B|_{q'},$$

then

$$|\Phi(T, B)| \leqq \Gamma|T|_{p''}|B|_{q''}, \tag{1}$$

where p, p', p'' and q, q', q'' are as above. Since the set of non-singular matrices is dense in the set of all finite matrices, we may assume in proving (1) that T and B are non-singular. In this case, it was shown in the fourth paragraph of the proof of Lemma 9.14 that $T = U_1D_1U_2$ and $B = V_1D_2V_2$, where U_1, U_2, V_1, V_2 are unitary matrices and where D_1 and D_2 are diagonal matrices. Thus, our theorem is implied by the following statement:

If $\Phi(T, B)$ is a bilinear form on the family of all matrices in d-dimensional Hilbert space, and if

$$|\Phi(U_1D_1U_2, V_1D_2V_2)| \leqq \Gamma|D_1|_p|D_2|_q$$
$$|\Phi(U_1D_1U_2, V_1D_2V_2)| \leqq \Gamma|D_1|_{p'}|D_2|_{q'}$$

for all unitary matrices U_1, U_2, V_1, V_2 and diagonal matrices D_1, D_2, then

$$|\Phi(U_1D_1U_2, V_1D_2V_2)| \leqq \Gamma|D_1|_{p''}|D_2|_{q''}.$$

If we examine the Definition 9.1 of the norms in the class C_p, we see that the above statement is an immediate consequence of the Riesz convexity theorem, Lemma VI.10.7. Q.E.D.

11 Corollary. *If $1 \leqq p < \infty$, if T is in C_p, and if $\{\varphi_i\}$ is an orthonormal set of vectors, then there exists a finite constant Γ_p such that*

$$\left\{\sum_{i=1}^{\infty} |(T\varphi_i, \varphi_i)|^p\right\}^{1/p} \leqq \Gamma_p|T|_p.$$

Proof. The mapping $\mathscr{S}$ which sends T into the transformation

$$(\mathscr{S}T)(x) = \sum_{i=1}^{\infty} (T\varphi_i, \varphi_i)\varphi_i(x, \varphi_i)$$

plainly is bounded, and even of bound 1, as a map of $C_\infty \to C_\infty$.

Lemma 9.7 shows that the series $\sum_{i=1}^{\infty}(T\varphi_i, \varphi_i)$ is unconditionally, and hence absolutely, convergent if T is in C_1. Thus, if T is in C_1, the sequence $\{(T\varphi_i, \varphi_i)\}$ belongs to the sequence space l_1. The closed graph theorem now implies that for some finite constant Γ_1,

$$\sum_{i=1}^{\infty} |(T\varphi_i, \varphi_i)| \leqq \Gamma_1 |T|_1.$$

Thus the above transformation $\mathscr{S}$ is bounded as a mapping of C_1 into C_1. Our result now follows at once from the preceding theorem. Q.E.D.

Let K_{ij} be the set of kernels representing a Hilbert-Schmidt operator K, as in Lemma 5. Put

$\rho(s, t) = 0$ if s and t belong to the same interval of the complement of C;

$\rho(s, t) = \operatorname{sgn}(t-s)$ otherwise;

$\eta(s, t) = \big(\rho(s, t)\big)^2$.

Let ρK be the Hilbert-Schmidt operator represented by the kernels $K_{ij}(s, t)\rho(s, t)$, and let ηK be the Hilbert-Schmidt operator represented by the kernels $K_{ij}(s, t)\eta(s, t)$. It is plain from Lemma 5 that ρ and η are bounded linear transformations of the Hilbert-Schmidt class C_2 into itself. The following lemma shows that a corresponding statement is also valid for $p \neq 2$.

12 Lemma. *Let $1 < p < \infty$. Then the linear transformations ρ and η defined above may be extended to bounded linear mappings of C_p into C_p.*

Proof. We will mimic the proof of the inequality of M. Riesz, Theorem 7.3. Thus, our proof will involve:

(a) consideration of the case $p = 2k$, k integral;

(b) interpolation from (a) to get the general case $\infty > p \geqq 2$;

(c) a "conjugacy" argument to go from (b) to the cases $1 < p \leqq 2$.

The details are as follows. Since $\eta(s, t) = 1$ unless s and t belong to the same interval of the complement of C, it follows from Lemma 5 that $K - \eta K$ is a Hilbert-Schmidt operator represented by kernels $\tilde{K}_{ij}$ satisfying $\tilde{K}_{ij} = 0$ unless $i = j = 1$, $\tilde{K}_{11}(s, t) = 0$ unless s and t

belong to the same interval of the complement of C, $\bar{K}_{11}(s,t) = K_{11}(s,t)$ if s and t belong to the same interval of the complement of C.

Let I_j be an enumeration of the intervals forming the complement of C, and let

$$\begin{aligned} \tilde{\varphi}_j(s) &= \frac{1}{(\mu(I_j))^{1/2}}, \qquad s \in I_j \\ &= 0 \qquad \text{otherwise,} \end{aligned}$$

μ denoting Lebesgue measure, so that $\{\tilde{\varphi}_j\}$ is an orthonormal set of functions. Put

$$\varphi_j = [\tilde{\varphi}_j, 0, 0, \ldots]$$

so that φ_j is an orthonormal set of vectors in the Hilbert space $\mathfrak{H}_0$ of Theorem 4. Then the above description of $K - \eta K$ implies that

$$(K - \eta K)x = \sum_{j=1}^{\infty} (K\varphi_j, \varphi_j)\varphi_j(x, \varphi_j).$$

Thus, by the preceding lemma, $K \to K - \eta K$ is a bounded mapping of C_p into C_p so that $K \to \eta K$ is a bounded mapping of C_p into C_p, $1 \leq p \leq \infty$.

Since, from Lemmas 9.6(c), 9.14, and Definition 9.1, an operator is in C_p if and only if its Hermitian and anti-Hermitian parts are in C_p, all we must do to prove our theorem is demonstrate the existence of a finite constant Γ_p such that $|\rho H|_p \leq \Gamma_p |H|_p$ if $H \in C_p \cap C_2$ and H is Hermitian.

Let H be in C_2 and let H be Hermitian. Then, by Lemma 6, ρH is anti-Hermitian and ηH is Hermitian. Moreover, by Theorem 7, $(\rho + \eta)H$ is quasi-nilpotent. Thus, by Lemma 9.14 and the definition following Corollary 9.8, $\operatorname{tr}\big(((\rho+\eta)H)^{2k}\big) = 0$ for each integer $k \geq 1$. Since the real part of the trace of an operator is the trace of its Hermitian part, it follows that

$$\sum_{j=0}^{k} \binom{2k}{2j} \operatorname{tr}\big((\rho H)^{2j}(\eta H)^{2k-2j}\big) = 0. \tag{1}$$

Since by Lemma 9.14 and Corollary 9.8

$$|\operatorname{tr}((\rho H)^{2j}(\eta H)^{2k-2j})| \leq |\rho H|_{2k}^{2j} |\eta H|_{2k}^{2k-2j},$$

while from Definition 9.1

$$\operatorname{tr}\big((\rho H)^{2k}\big) = |\rho H|_{2k}^{2k},$$

it follows from (1) that

$$(2) \qquad |\rho H|_{2k}^{2k} - \sum_{j=0}^{k-1} \binom{2k}{2j} |\rho H|_{2k}^{2j} |\eta H|_{2k}^{2k-2j} \leqq 0.$$

It follows at once that

$$\frac{|\rho H|_{2k}}{|\eta H|_{2k}} \leqq \alpha,$$

where α is the largest root of the polynomial

$$\alpha^{2k} - \sum_{j=0}^{k-1} \binom{2k}{2j} \alpha^{2j} = 0.$$

Since we have already seen that $H \to \eta H$ is a bounded mapping in C_p, the correctness of the present lemma for $p = 2k$, k integral, follows at once. Theorem 10 immediately implies that the present lemma is correct for any $p \geq 2$, $p < \infty$.

Next, we shall establish the identity

$$(3) \qquad \operatorname{tr}\big((\rho A)B\big) = \operatorname{tr}\big(A(\rho B)\big)$$

for $A, B \in C_2$. This identity would follow at once from the definition of the mapping ρ if only we knew that

$$(4) \qquad \operatorname{tr}(AB) = \sum_{i,j=1}^{\infty} \int_0^1 \int_0^1 A_{ij}(s, t) B_{ji}(t, s)\, ds\, dt$$

for each A and B in C_2, $\{A_{ij}\}$ and $\{B_{ij}\}$ being the sets of kernels representing A and B, respectively, in the sense of Lemma 5. We may write (4) equivalently in the slightly more convenient form

$$(5) \qquad \operatorname{tr}(AB^*) - \sum_{i,j} \int_0^1 \int_0^1 A_{ij}(s, t) \overline{B_{ij}(s, t)}\, ds\, dt = 0.$$

To verify (5), note that its left-hand side defines a Hermitian inner product $[A, B]$ on C_2, such that $[A, A] = 0$ for each A in C_2 by Lemma 5. Thus

$$2\mathscr{R}[A, B] = [A+B, A+B]-[A, A]-[B, B] = 0$$

for all A, B in C_2, and similarly $\mathscr{I}[A, B] = 0$, so that $[A, B] = 0$, proving (5), (4), and (3).

The validity of the present theorem in the range $1 < p \leqq 2$ now follows at once from its validity in the range $2 \leqq p \leqq \infty$ and from Lemma 9.14. Q.E.D.

In what follows, we will use the symbols ρ and η to denote the continuous extension to the classes C_p of operators of the mappings ρ and η of the previous lemma. We will also write $\tau = (\rho+\eta)/2$. We note the following corollary.

13 Corollary. (a) *The mapping τ is a bounded map of C_p into C_p, $1 < p < \infty$.*

(b) *If H is anti-Hermitian, then the anti-Hermitian part of τH is ηH.*

(c) *τH leaves invariant the range of each of the projections E_λ of Theorem 5.*

Proof. Only statement (c) has not been proved in our earlier discussion. To prove (c), we have only to note that both sides of the equation $E_\lambda(\tau A)E_\lambda = (\tau A)E_\lambda$ are continuous in the operator A. Thus, by Lemma 9.11, it is sufficient to prove (c) for operators A with finite-dimensional range. Since, by Theorem 7, (c) is true for each Hilbert-Schmidt operator, the present corollary is obvious.

14 Corollary. *If A in C_p is an operator in the space $\mathfrak{H}_0$ of Theorem 5 and $(A\varphi, \varphi) = 0$ for the characteristic function φ of each of the intervals complementary to the set C, then $A = \eta A$.*

Proof. Let I_i be an enumeration of the intervals complementary to the closed set C, and let φ_j be the vectors of the second paragraph of the proof of Lemma 12. We have seen in the indicated paragraph of that proof that

$$(A-\eta A)x = \sum_{i=1}^{\infty}(A\varphi_i, \varphi_i)\varphi_i(x, \varphi_i)$$

if A is in C_2. Since, from Theorem 9 and Corollary 11, both sides of this equation are continuous in $A \in C_p$, it follows from Lemma 9.14 that the equation holds for all A in C_p. The present corollary is

immediate from this equation. Q.E.D.

Now we are ready to prove Theorem 9.

PROOF OF THEOREM 9. Using Theorem 4, we may suppose without loss of generality that T maps the range of each of the projections E_λ of Theorem 4 into itself. Let (a, b) be an interval of the complement of C, and χ its characteristic function. Then, if f is the vector

$$f = [\chi, 0, 0, \ldots],$$

we may write

$$Tf = [c\chi + g_1, g_2, \ldots],$$

where $[g_1, g_2, \ldots]$ belongs to the range of the projection E_a. Thus, inductively,

$$T^n f \quad [c^n\chi + g_1^{(n)}, g_2^{(n)}, \ldots],$$

where $[g_1^{(n)}, g_2^{(n)}, \ldots]$ belongs to the range of the projection E_a. Thus

$$(T^n f, f) = c^n(b-a).$$

Since T is quasi-nilpotent, we have $|T^n| = o(\varepsilon^n)$ for each positive ε as $n \to \infty$. Therefore we must have $c = 0$. Thus, $(T\varphi_i, \varphi_i) = 0$, φ_i denoting the orthonormal vectors of the second paragraph of the proof of Lemma 12. Let H be the anti-Hermitian part of T. Then $(H\varphi_i, \varphi_i) = 0$ for all i, so that, by Corollary 14, $H = \eta H$. By Corollary 18, ηH is the anti-Hermitian part of τH; thus T and τH have the same anti-Hermitian part, so that $S = T - \tau H$ is Hermitian. By Corollary 18 again, S leaves invariant the range of each of the projections E_λ. It is plain from the definition and continuity of τ that $((\tau H)\varphi_i, \varphi_i) = 0$ for all i. Thus $(S\varphi_i, \varphi_i) = 0$ for all i. But the operator S has the restricted form specified in Lemma 8. Thus we must have $S = 0$. Hence $T = \tau H$, and Theorem 9 follows from Lemma 12. Q.E.D.

It is now quite easy to extend the completeness theorems of Sections 6 and 9 in a useful way. The first three paragraphs of the proof of Theorem 6.29 may easily be seen to show that the validity of the completeness theorem (6.29) for an arbitrary compact operator T follows from the analyticity of $\lambda^N R(\lambda; T^*|\mathfrak{X})$ at the origin, $\mathfrak{X}$ being a subspace of Hilbert space, invariant under T^*, such that the restriction of T^* to $\mathfrak{X}$ is quasi-nilpotent. If we assume that T has an anti-Hermitian part lying in the class C_p, then plainly T^* also has

an anti-Hermitian part H_1 lying in the class C_p. Let E be the orthogonal projection on $\mathfrak{X}$. Then it is easy to see that the anti-Hermitian part of $T^*|\mathfrak{X}$ is the restriction to $\mathfrak{X}$ of EH_1E; thus $T^*|\mathfrak{X}$ has an anti-Hermitian part lying in C_p. By Theorem 9, $T^*|\mathfrak{X}$ itself must lie in C_p. Once this key fact is established, we may continue to reason along the lines of the proof of Theorem 6.29. In this way, and with the generalization noted in the discussion of Theorem 9.29, we obtain the following theorem.

15 THEOREM. *Let $1 < p < \infty$, and let the compact operator T in Hilbert space $\mathfrak{H}$ have an anti-Hermitian part lying in the class C_p. Let $\gamma_1, \ldots, \gamma_s$ be non-overlapping, differentiable arcs in the complex plane starting at the origin. Suppose that each of the s regions into which the plane is divided by these arcs is contained in an angular sector of opening less than π/p. Let $N > 0$ be an integer, and suppose that the resolvent of T satisfies the inequality*

$$|R(\lambda; T)| = O(|\lambda|^{-N})$$

as $\lambda \to 0$ along any of the arcs γ_i. Then the subspace $\mathrm{sp}(T)$ *contains the subspace $T^N\mathfrak{H}$.*

Similarly, by arguing as in the proofs of Corollary 6.30 and Corollary 6.31, we obtain the two following corollaries, which generalize Corollaries 9.30 and 9.31.

16 COROLLARY. *Let the arcs $\gamma_1, \ldots, \gamma_s$ be chosen as in the preceding theorem and suppose that as λ tends to zero along any of these arcs the resolvent of the compact operator T satisfies the inequality $|R(\lambda; T)| = O(|\lambda|^{-1})$. Then the subspace* $\mathrm{sp}(T)$ *coincides with the entire Hilbert space $\mathfrak{H}$.*

→ 17 COROLLARY. *Let $1 < p < \infty$. Let T be a densely defined unbounded operator in Hilbert space $\mathfrak{H}$, with the property that for some λ_0 in the resolvent set of T, the operator $R(\lambda_0; T)$ is compact and has an anti-Hermitian part belonging to the class C_p. Let $\gamma_1, \ldots, \gamma_s$ be non-overlapping differentiable arcs having a limiting direction at infinity, and suppose that no adjacent pair of these arcs forms an angle as great as π/p at infinity. Suppose that the resolvent $R(\lambda; T)$ satisfies an inequality $|R(\lambda; T)| = O(|\lambda|^N)$ along each of these arcs. Then the subspace* $\mathrm{sp}(T)$ *coincides with the entire Hilbert space $\mathfrak{H}$.*

11. Notes and Remarks

Compact groups. Theorem 1.1 is a special case of Theorem 4 stated below. An elementary proof of the existence of Haar measure on a compact group with a countable base was given by von Neumann [12] and is reproduced in Pontrjagin [1; Sec. 25]. The fundamental theorem of Peter and Weyl [1] was first proved for compact Lie groups; see also Loomis [1; Sec. 38], Pontrjagin [1; Sec. 29] and Weil [1; Sec. 21].

The Peter-Weyl Theorem 1.4 is basic to the theory of representations of compact groups. The principal definitions and theorems of this theory are as follows.

Definition: Let G be a topological group, and $\mathfrak{X}$ a B-space. Then a *representation* R of G in $\mathfrak{X}$ is a strongly continuous homomorphism $g \to R(g)$ of G into the group of bounded invertible linear transformations on $\mathfrak{X}$. If $\mathfrak{X}$ is a finite dimensional complex Euclidean space, the representation R is said to be finite-dimensional. If $\mathfrak{X}$ is a Hilbert space and $R(g)$ is a unitary operator for each $g \in G$, then the representation R is said to be *unitary.*

Definition: Let G be a topological group, and R_1 and R_2 two representations of G, both acting in a space $\mathfrak{X}$. Then R_1 and R_2 are said to be *equivalent* if there exists a bounded linear T in $\mathfrak{X}$ with a bounded linear inverse, such that $R_1(g) = T^{-1}R_2(g)T$ for each $g \in G$.

Using the Haar measure on a compact group, it is easy to prove the following theorem.

Theorem: *Any finite dimensional representation of a compact group G is equivalent to a finite dimensional unitary representation of G.*

Definition: Let G be a topological group and R a representation of G in a B-space $\mathfrak{X}$. Then R is said to be *irreducible* if $\mathfrak{X}$ admits no proper closed subspace invariant under all the operators $R(g)$, $g \in G$.

Definition: Let G be a topological group, and R a representation of G in a B-space $\mathfrak{X}$. Let $\mathfrak{X}$ be written as a direct sum $\mathfrak{X} = \mathfrak{X}_1 \oplus \ldots \oplus \mathfrak{X}_n$ of closed subspaces invariant under all the operators $R(g)$, and let us write $R_i(g) = R(g)|\mathfrak{X}_i$, $i = 1, \ldots, n$, so that R_i is a representation of G for each $i = 1, \ldots, n$. Then we write

$R = R_1 \oplus \ldots \oplus R_n$, and say that R is the *direct sum of* $R_1, \ldots, R_n$.

The following theorem is easily proved by induction in case R is unitary, and thus follows in the general case by the theorem stated above.

THEOREM. *Any finite dimensional representation of a compact group G is a direct sum of irreducible representations.*

This theorem shows that in studying finite dimensional representations of a compact group G we may, without loss of generality, confine ourselves to the analysis of irreducible finite dimensional unitary representations. If such a representation acts in a finite dimensional space E^n, then introducing a basis for E^n, we may regard the representation as being described by a set of unitary matrices $\{U_{ij}(g)\}$. The individual entries $U_{ij}(g)$ are called the *matrix elements* of the representation, and depend continuously on g. The sum $\sum_{i=1}^n U_{ii}(g) = \operatorname{tr}(R(g))$ is called the *trace* of the representation.

THEOREM. *Let R and $\tilde{R}$ be two irreducible finite dimensional unitary representations of a compact group G whose Haar measure is μ. Let $\{U_{ij}(g)\}$ and $\{\tilde{U}_{kl}(g)\}$ be the matrix elements of these two representations. Then, if R and $\tilde{R}$ are not equivalent, we have*

$$\int_G U_{ij}(g)\overline{\tilde{U}_{kl}(g)}\mu(dg) = 0, \quad \text{all } i, j, k, l.$$

Moreover, we have

$$\int_G U_{kl}(g)\overline{U_{ij}(g)}\mu(dg) = 0$$

unless $k = i$ and $l = j$.

By $L_2(G)$ we denote the Hilbert space of all functions on G square-integrable with respect to Haar measure. The preceding theorem and the Peter-Weyl Theorem 1.4 together have the following consequence.

THEOREM. *Let G be a compact topological group; let $\{R^{(\alpha)}\}$ be a maximal set of unitary representations of G, no two of which are equivalent. Let $\{U_{ij}^{(\alpha)}(g)\}$ be the corresponding family of matrix elements. Then $\{U_{ij}^{(\alpha)}(\cdot)\}$ is a complete set of orthogonal functions in $L_2(G)$.*

DEFINITION: A set $\{R^{(\alpha)}\}$ of finite-dimensional irreducible unitary representations of a compact G is said to be a *complete set of representations of G* if

(a) No two of the representations $R^{(\alpha)}$ are equivalent.
(b) Any irreducible representation of G is equivalent to one of the representations $R^{(\alpha)}$.

COROLLARY: If G is a compact topological group satisfying the second axiom of countability, and G is not a finite set, then any complete set of representations of G is countable. A complete set of representations of a finite group is finite.

DEFINITION: A *class function* on a compact group G is an element f of $L_2(G)$ such that $f(h) = f(ghg^{-1})$ for almost all h whenever $g \in G$. The *class* of an element h of G is the set $\{ghg^{-1}|g \in G\}$.

The class functions form a closed subspace of $L_2(G)$. The trace of any finite dimensional representation of G is a class function.

THEOREM. *Two irreducible representations of a compact group G are equivalent if and only if they have the same trace, inequivalent if and only if their traces are orthogonal. The traces of a complete set of representations of G form a complete orthogonal basis for the space of class functions in $L_2(G)$.*

COROLLARY: *If G is a finite group, the number of representations in a complete set of representations is equal to the number of distinct classes of G.*

The main aim of the representation theory of compact groups is to display a complete set of representations for a given group explicitly.

Explicit complete sets of representations for many finite groups have been given. For an account of the representations of the group $\pi(n)$ of all permutations of n objects, see Littlewood, *The theory of group characters and matrix representations of groups*, Oxford, Clarendon Press, 1950. A corresponding account of the representation theory of the alternating subgroup $\alpha(n)$ of $\pi(n)$ is to be found in G. Frobenius, *Über die Charaktere der alternierenden Gruppe*, Sitzungsber. Akad. Berlin 1901, p. 303—315.

The infinite compact groups whose representations have been most exhaustively studied are the *compact connected Lie groups*. These are the compact connected groups, satisfying the second axiom of countability, which have a neighborhood of the identity homeomorphic to a domain in a finite dimensional Euclidean space. In this

case, the homeomorphism may be chosen so that, in the "coordinates" in a neighborhood of the identity which the homeomorphism introduces, the basic group operations $h \to h^{-1}$ and $[g, h] \to gh$ are described by functions which are not merely continuous but infinitely often differentiable. The structure of any compact connected Lie group is well understood. Let G be such a group. Then there exists a connected topological group H such that G is the image of H under a homomorphism h such that $h^{-1}(e)$ is a subset of H with no limit points. Moreover, H is the direct sum of a finite number of groups H_j, each of which is either

(1) The additive group of the real axis; or

(2) The group $SU(n)$ of all $n \times n$ complex unitary matrices of determinant 1; or

(8) The group $SpU(n)$ of all $2n \times 2n$ complex unitary matrices V such that $[Vx. Vy] \quad [x, y]$, where $[x, y]$ is the non-singular bilinear form $[x, y] = x_1y_2 \quad y_1x_2 + x_3y_4 + \cdots + x_{2n-1}y_{2n} \quad y_{2n-1}x_{2n}$; or

(4) A group $\widehat{UR}(n)$ admitting a two-to-one homomorphism onto the group $UR(n)$ of all real $n \times n$ unitary matrices of determinant 1; or

(5) One of five other compact groups, known as the *exceptional simple compact groups*, described explicitly in the monograph *Les groups réels simples finis et continus*, Ann. Ec. Normale Sup, 3-ième Série, XXXI, (1914), p. 263—355 of Élie Cartan.

The structure theorem for compact connected Lie groups which we have just stated makes it plain that the irreducible representations of an arbitrary compact connected Lie group may be found once one knows all the irreducible unitary representations of the basic types (1)-(5) of Lie groups which we have listed. Since the additive group of the real axis is Abelian, its irreducible unitary representations are all 1-dimensional, and are of the form $x \to \exp(i\alpha x)$, where $-\infty < \alpha < +\infty$. A complete set $\{R^{(\alpha)}\}$ of each of the groups $SU(n)$ and $SpU(2n)$ is described in H. Weyl's famous book *The Classical Groups, Their Invariants and Representations*, Princeton, 1946. The spaces in which the operators $R^{(\alpha)}$ act are irreducible spaces of tensors; for each tensor χ and each $g \in SU(n)$ or $SpU(2n)$, the transformed tensor $R^{(\alpha)}(g)\chi$ is defined in the manner customary in tensor-analysis. Weyl also describes a complete set of representations of the group $RU(n)$ of $n \times n$ rotation matrices; these representations are

likewise given by irreducible sets of tensors. The group $\widehat{RU}(n)$ has additional representations, which, if one tries to regard them as representations of the rotation group itself, turn out to be double-valued. These representations are the so-called *spin representations* of the group $\widehat{RU}(n)$ (or, by abuse of language, of the rotation group $RU(n)$.) For an account of these representations, cf. R. Brauer and H. Weyl, *Spinors in n dimensions*, Am. Jour. of Math. 57, 1935, p. 425—449, and also P. K. Raševskii, *The theory of spinors*, A. M. S. Translations Ser. 2, v. 6, 1957, p. 1 –110. A complete classification of the irreducible unitary finite-dimensional representations of the five exceptional simple compact groups is given in Weyl's memoir *Theorie der Darstellungen kontinuierlicher halb-einfachen Gruppen durch lineare Transformationen* III, Math. Zeitschift v. 24 (1926) p. 377 395.

The representation theory for groups which are neither compact nor commutative is a good deal more complex, and is still not fully worked out. For certain important non-compact non-commutative groups however, the irreducible unitary representations in Hilbert space have been completely classified, and non-commutative extensions of various important theorems of harmonic analysis, notably the Plancherel theorem, have been given. For an introduction to the representation theory of groups of this type, cf. the book of I. M. Gelfand and M. A. Neumark, *Unitäre Darstellungen der Klassischen Gruppen*, Berlin, Akademie Verlag, 1957. The important particular case of the Lorentz group is treated in Neumark's article *Linear Representations of the Lorentz group*, A. M. S. Translations Ser. 2, v. 6, (1957), p. 379—458.

Almost periodic functions. The main theorem on almost periodic functions is due to H. Bohr (see Bohr [2; Sec. 84] for a different proof). For other proofs of this theorem and additional results see also Loomis [1; Sec. 41], Weil [1; Sec. 33—35] and the references given in IV.16 pertaining to the section on the space AP.

Convolution algebras. The discussion in Section 8 in the text makes free use of the properties of Lebesgue measure on the real axis. The corresponding properties for Haar measure, which are needed in the varous arguments, will be proved here. The reader will thus have a treatment of the subject which is self contained except for the exist-

ence of Haar measure on a locally compact, σ-compact Abelian group. As remarked in the text, the development presented in this section is valid for a general non-discrete locally compact, σ-compact Abelian group. However, there are a few comments that we should make concerning the general non-Abelian case. First of all we shall prove, in Theorem 2, that a locally compact group is automatically a normal topological space, a fact that was occasionally used in the text. Next we shall state the fundamental theorem concerning the existence of Haar measure and prove some of the more important elementary properties of this measure which will give ample evidence to indicate that it behaves very much like Lebesgue measure on the real line.

It should be noted that these groups on which we use a Haar measure are not only locally compact but are σ-compact, i.e., they are the denumerable union of compact sets. We shall discuss briefly the special case when R is compact, and also the case when R is discrete, which was excluded in Section 3. Finally, we give a proof of the celebrated Pontrjagin "Duality Theorem".

In the following we write the group operation as addition, since Sections 3 and 4 deal with Abelian groups.

1 LEMMA. *If R is a locally compact space, and if F is a closed subset of R which does not contain a point p, then there is a real valued continuous function f on R with $f(p) = 0$, $f(F) = 1$, and $0 \leq f(x) \leq 1$, for x in R.*

NOTE. This lemma says that a locally compact space is a completely regular (IV.6.21) topological space.

PROOF. Compactify R by adding a point ∞ and taking neighborhoods of ∞ in $R \cup \{\infty\}$ to be the complements of compact sets in R. Then $R \cup \{\infty\}$ is a compact Hausdorff space and hence (I.5.9) normal, and the set $F_1 = F \cup \{\infty\}$ is closed in $R \cup \{\infty\}$. Let f_1 be a real valued continuous function on $R \cup \{\infty\}$ such that $f_1(p) = 0$, $f_1(F_1) = 1$ and $0 \leq f_1(x) \leq 1$, $x \in R \cup \{\infty\}$. The restriction of f_1 to the space R is the desired function. Q.E.D.

2 THEOREM. *If R is a locally compact group, then R is a normal topological space.*

PROOF. Let K_n be an increasing sequence of compact sets with

$R = \bigcup_{n=1}^{\infty} K_n$. We observe that if A and B are disjoint closed subsets of R and if n is an integer, then there is an open set $U \subseteq R$ such that $A \cap K_n \subseteq U$ and $\overline{U} \cap B = \phi$. This is true since for each $p \in A \cap K_n$ there is an open set $U(p)$ such that $p \in U(p)$ and $\overline{U(p)} \cap B = \phi$; by the compactness of $A \cap K_n$ the desired U is the union of a finite number of such $U(p)$. To prove the normality of R we shall use this remark inductively.

Let F_1 and F_2 be disjoint closed sets in R. We select an open set G_1 in R such that

$$F_1 \cap K_1 \subseteq G_1, \quad \overline{G}_1 \cap F_2 = \phi,$$

and then choose an open set H_1 such that

$$F_2 \cap K_1 \subseteq H_1, \quad \overline{H}_1 \cap (F_1 \cup \overline{G}_1) = \phi.$$

By induction, choose open sets G_n and H_n such that

$$\begin{aligned} &F_1 \cap K_n \subseteq G_n, \qquad \overline{G}_n \cap (F_2 \cup \overline{H}_1 \cup \ldots \cup \overline{H_{n-1}}) = \phi, \\ &F_2 \cap K_n \subseteq H_n, \qquad \overline{H}_n \cap (F_1 \cup \overline{G}_1 \cup \ldots \cup \overline{G}_n) = \phi. \end{aligned}$$

The construction assures that $G_n \cap H_m = \phi$ for all integers n, m. Put $G = \bigcup_{n=1}^{\infty} G_n$ and $H = \bigcup_{n=1}^{\infty} H_n$, so that G and H are disjoint open sets. Since $\cup K_n = B$ it follows that $F_1 \subseteq G$ and $F_2 \subseteq H$ so the normality of B is proved. Q.E.D.

We now turn to the question of the measure theory on a locally compact group.

8 THEOREM. (*Haar*) *If the locally compact topological group B is the denumerable union of compact sets then there exists a non-negative countably additive measure λ defined on the Borel subsets Σ of B such that $\lambda(U) > 0$ for any open set U, $\lambda(K) < \infty$ for any compact set K and $\lambda(x+E) = \lambda(E)$ for any $x \in R$ and $E \in \Sigma$. The measure λ has the regularity property*

$$\sup_{F \subseteq E} \lambda(F) = \lambda(E) = \inf_{G \supseteq E} \lambda(G), \qquad E \in \Sigma,$$

where F is a closed set and G an open set. Furthermore λ is unique in the sense that any other measure satisfying these conditions is a positive multiple of λ.

It may be remarked that this theorem was proved for compact

groups in Theorem 1.1, and that the only use that is made of it is in the case of an Abelian group. We shall not prove this theorem, but refer the reader to Halmos [5; pp. 254—263].

The existence of an invariant measure on a group satisfying the second axiom of countability was first shown by Haar [1], and the question of uniqueness was first discussed by von Neumann [17]. Other proofs of existence or uniqueness have been given by Cartan [1], Kakutani [17], Kakutani and Kodaira [1], Loomis [1, 3], von Neumann [12], Raikov [1, 2], and Weil [1, 2]. Other results concerning measures invariant under transformations are found in Oxtoby and Ulam [1].

We now present some useful properties of Haar measure which, though elementary, are not obvious consequences of the invariance property.

4 Lemma. *Let B be a locally compact, σ-compact, Abelian topological group, Σ its Borel field, and λ its Haar measure. Then $\lambda(E+x) = \lambda(E)$ and $\lambda(-E) = \lambda(E)$ for every E in Σ and x in B.*

Proof. The invariance of Haar measure under right translations follows immediately since B is Abelian. The fact that $\lambda(-E) = \lambda(E)$ is a simple consequence of the uniqueness of Haar measure. Q.E.D.

5 Corollary. *Let B, Σ, and λ be as in Lemma 4. Then $\lambda(B)$ is finite if and only if B is compact. Points in B have positive measure if and only if B is discrete.*

Proof. If B is compact then Theorem 3 implies that $\lambda(B) < \infty$. Conversely, suppose that B is not compact, and let V be a neighborhood of the identity with compact closure. Since no finite collection of translates of V can cover B, we may select a sequence $\{x_n\}$ such that $x_{n+1} \notin \bigcup_{i=1}^{n}(x_i+V)$. Let $U = -U$ be a non-void neighborhood of 0 such that $U+U \subseteq V$. Then the open sets $\{x_n+U\}$ are pairwise disjoint and so $\infty = \sum_{n=1}^{\infty}\lambda(x_n+U) = \lambda\left(\bigcup_{n=1}^{\infty}(x_n+U)\right) \leq \lambda(B)$, proving that $\lambda(B) = \infty$.

If B is discrete, then the set consisting of a single point is open and hence has positive measure. Conversely, if points have positive measure α, then since each point p can be included in an open set U of measure less than $3\alpha/2$, it follows that U can contain no point but p. Thus p is simultaneously open and closed. Q.E.D.

Since the measure space (B, Σ, λ) is a σ-finite measure space the theory of integration as developed in Chapter III may be used as a basis for the theory developed in Sections 3—4. In particular we should notice that the product group $B \times B$ is a locally compact, σ-compact, (Abelian) group if B is. Thus the product group has a Haar measure $\lambda^{(2)}$ defined on its Borel field $\Sigma^{(2)}$. It is natural to expect that the product measure $\lambda \times \lambda$ coincides, up to a constant multiple, with $\lambda^{(2)}$. This fact will be established in Theorem 7.

6 LEMMA. *Let the locally compact Abelian group B be the denumerable union of compact sets. Let λ be a Haar measure in B, and let Σ be the σ-field of Borel sets. Then if f is λ-measurable, the function* g *defined by* $g(x, y) = f(x-y)$ *is $\lambda \times \lambda$ measurable.*

PROOF. Let $\Sigma \times \Sigma$ be the product σ-field of subsets of $B \times B$ and let E be an open subset of R. Then, because of the continuity of the group operations, the set $\{[x, y] | x-y \in E\}$ is open in $B \times B$ and hence is in $\Sigma \times \Sigma$. Now, for an arbitrary subset E of B, let $p(E) = \{[x, y] \in B \times B | x-y \in E\}$. Then $p(\bigcup_{i=1}^{\infty} E_i) = \bigcup_{i=1}^{\infty} p(E_i)$, $p(E') = (p(E))'$, and $p(\phi) = \phi$. Since $p(E) \in \Sigma \times \Sigma$ if E is an open set, it follows immediately that $p(E) \in \Sigma \times \Sigma$ for $E \in \Sigma$. Now let $E \in \Sigma$ and have λ-measure zero. By Fubini's theorem (III.11.9) we have

$$\begin{aligned}\int_{R \times R} \chi_{p(E)}(s, t)(\lambda \times \lambda)(d[s, t]) \\ &= \int_{R \times R} \chi_E(s-t)(\lambda \times \lambda)(d[s, t]) \\ &= \int_R \left\{ \int_R \chi_E(s-t)\lambda(ds) \right\} \lambda(dt) \\ &= \int_R 0 \cdot \lambda(dt) = 0.\end{aligned}$$

Thus, $p(E)$ has $\lambda \times \lambda$-measure zero if E has λ-measure zero. It follows that if E is in the λ-completion of Σ, $p(E)$ is in the λ-completion of $\Sigma \times \Sigma$. Let f be λ-measurable. Let U be an open set of complex numbers, $E = \{x | f(x) \in U\}$ and $D = \{[s, y] | f(s-y) \in U\}$. Then $D = p(E)$, and now the measurability of the function $f(x-y)$ follows immediately from III.6.10. Q.E.D.

The next result is of prime importance when working with Haar measure.

7 THEOREM. *Let Σ be the field of Borel sets in the locally compact,*

σ-compact group B and let λ be a Haar measure in B. Then the product measure $\lambda \times \lambda$ is a Haar measure in $B \times B$.

Proof. Since the product group $B^{(2)} = B \times B$ is locally compact and σ-compact, it has a Haar measure $\lambda^{(2)}$ defined on its Borel field $\Sigma^{(2)}$ and what we shall prove is that for some constant c,

$$(B^{(2)}, \Sigma^{(2)}, \lambda^{(2)}) = c(B, \Sigma, \lambda) \times (B, \Sigma, \lambda).$$

Since it is clear that $\Sigma^{(2)} = \Sigma \times \Sigma$, what will be proved then, is that

$$\text{(i)} \qquad \lambda^{(2)}(E) = c(\lambda \times \lambda)(E), \qquad E \in \Sigma^{(2)},$$

for some constant c independent of E. This condition (i), as is seen from Corollary III.11.6, is a consequence of the assertion that

$$\text{(ii)} \qquad \lambda^{(2)}(A \times B) = c\lambda(A)\lambda(B), \qquad A, B \in \Sigma.$$

Thus we shall endeavor to establish (ii). For every E in $\Sigma^{(2)}$ let $\mu(E) = \lambda^{(2)}(hE)$ where h is the homeomorphic homomorphism in $B^{(2)}$ defined by the equation

$$h([x, y]) = [y, x], \qquad [x, y] \in B^{(2)}.$$

It is readily verified that μ is a Haar measure in $B^{(2)}$ and so $\mu(E) = c\lambda^{(2)}(E)$ for some constant c independent of E. This shows that $\lambda^{(2)}(A \times B) = c\lambda^{(2)}(B \times A)$ for each pair of sets A, B in Σ. Actually $c = 1$, as may be seen by letting A and B be open sets with compact closures so that $0 < \lambda^{(2)}(A \times B) < \infty$. Thus

$$\text{(iii)} \qquad \lambda^{(2)}(A \times B) = \lambda^{(2)}(B \times A), \qquad A, B \in \Sigma.$$

Now fix B in Σ with $0 < \lambda(B) < \infty$, and consider the measure on Σ which assigns to the set A in Σ the value $\lambda^{(2)}(A \times B)$. This measure is readily seen to be a Haar measure in B and so for some constant $c(B)$, independent of A, we have

$$\text{(iv)} \qquad \lambda^{(2)}(A \times B) = c(B)\lambda(A), \qquad A \in \Sigma.$$

If A also satisfies the condition imposed upon B, i.e., $0 < \lambda(A) < \infty$, then it is seen, by interchanging A and B and using (iii) and (iv), that $c(B)\lambda(A) = c(A)\lambda(B)$. Thus for sets A, B in Σ with positive finite λ measures it follows that the ratio

$$\frac{c(A)}{\lambda(A)} = \frac{c(B)}{\lambda(B)}$$

is a constant c independent of A or B, i.e., $c(B) = c\lambda(B)$, which in view of (iv) means that (ii) holds if $0 < \lambda(B) < \infty$. Using the countable additivity of $\lambda^{(2)}$ and λ as well as the σ-compactness of R it follows immediately that (ii) also holds for all B in Σ. Q.E.D.

When discussing a group R with Haar measure λ this theorem enables us to refer to *the* Haar measure on the product group $R \times R$ rather than *a* Haar measure. *The* Haar measure on $R \times R$ will always mean the uniquely defined measure $\lambda \times \lambda$.

We now consider the special form that the results of Sections 3 and 4 take when R is a compact Abelian group, a case studied explicitly in Section 1.

8 THEOREM. *If R is a compact Abelian group, its character group $\hat{R}$ is discrete.*

PROOF. Consider the neighborhood $N(0, R, 1) = \{m \in \hat{R} \mid |[x, m] - [x, 0]| < 1, x \in R\}$ of the identity in $\hat{R}$. If $m \in N(0, R, 1)$ and if $x \in R$ is such that $[x, m] \neq 1$, then it is easily seen that for some integer n we have $\mathscr{R}[nx, m] = \mathscr{R}([x, m]^n) < 0$. This shows that $N(0, R, 1)$ contains only the identity of $\hat{R}$, hence $\hat{R}$ is discrete. Q.E.D.

In this case every subset of $\hat{R}$ is measurable with respect to the measure μ constructed in Lemma 3.6, and, except for a constant factor, its measure is the number of points in the subset if this is finite and ∞ otherwise. Plancherel's theorem asserts that the set of characters forms a complete orthonormal set in $L_2(R)$, which fact was also proved in Theorem 1.6. We leave it to the reader to show that if R is the compact Abelian group of the real numbers modulo 2π, then $\hat{R}$ is algebraically and topologically isomorphic with the additive group of integers and we may write $[x, m] = e^{imx}$. Then the point p_∞ is the point $\infty(=-\infty)$ compactifying the integers and the statement that $(\tau f)(p_\infty) = 0$ for all $f \in L_1(0, 2\pi)$ is the familiar Riemann Lebesgue lemma. The value of the function τf at the character (integer) m is the m-th Fourier coefficient of f, i.e.,

$$(\tau f)(m) = c_m = \frac{1}{2\pi}\int_0^{2\pi} e^{-imx} f(x)\,dx,$$

and the isometry in Plancherel's theorem is the classical equality of Parseval:

$$\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^2\,dx = \sum_{-\infty}^{\infty} |c_m|^2,$$

valid if $f \in L_2(0, 2\pi)$. Since $\hat{R}$ is discrete and therefore contains no perfect sets, Theorem 4.21 asserts that if f and g are in $L_1(0, 2\pi)$ and if each Fourier coefficient of f vanishes whenever the correspondiug one for g does, then f is in the closed linear subspace spanned by the translates of g.

Next we consider the case where B is a discrete Abelian group. Here every subset of B is measurable and, except for a constant factor, the Haar measure of a set is equal to the number of points in the set if this is finite and ∞ otherwise. In this case the operation of convolution by the function $f(0) = 1$, $f(x) = 0$, $x \neq 0$, is the identity on $L_2(B)$ and so the last statement in Lemma 3.3 is false. The situation in this case is actually *simpler*, since there is no longer any need to adjoin an identity and the algebra $\mathfrak{A}$ can merely be taken to be $\overline{\mathfrak{A}}_0$ instead of $\overline{\mathfrak{A}}_0 + \{\alpha I\}$. Thus no point p_∞ needs to be removed and so $\mathscr{M}_0$ can be taken equal to $\mathscr{M}$. Thus we have the following result. which is dual to Theorem 8.

9 THEOREM. *If B is a discrete Abelian group, its character group $\hat{R}$ is compact.*

The results of Sections 3 and 4 carry over for the case of discrete groups and many of them, for example Theorem 3.16, simplify to some extent. A particular case of interest is furnished by the additive group of integers. We leave it to the reader to show that the character group of this group is algebraically and topologically isomorphic with the additive group of real numbers modulo 2π, or, equivalently, with the multiplicative group of complex numbers of unit modulus. Using the second of these realizations for $\hat{R}$ we have $[n, \lambda] = \lambda^n$ where $n \in B$ (so that n is a positive or negative integer) and $\lambda \in \hat{R}$. We now state a result which corresponds to Theorem 4.24 for the case that B is the group of integers; suitable modification of the proof of Theorem 4.24 will yield a proof of the result to follow. We leave the details to the reader.

10 THEOREM. *Suppose that $\phi = \{\alpha_n\}$, $-\infty < n < +\infty$, is a bounded sequence of complex numbers. Let f be the function of the complex variable z defined by*

$$f(z) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} \alpha_n z^{-n}, & |z| > 1, \\ \displaystyle -\sum_{n=0}^{\infty} \alpha_{-n} z^{n}, & |z| < 1. \end{cases}$$

Then a complex number t of modulus 1 is outside $\sigma(\phi)$ if and only if there exists a function g which is analytic in a neighborhood of t and is such that $g(z) = f(z)$ for all z in this neighborhood for which $|z| \neq 1$.

Making use of this theorem and an analog of Theorem 4.22 for the group of integers, we obtain the following interesting result concerning analytic functions.

11 THEOREM. *Let f be a function defined and analytic at every point of the complex sphere except for a finite number of points, $\zeta_1, \ldots, \zeta_r$ lying on the unit circle. Suppose that the coefficients in the Taylor expansion of f in the region $|z| < 1$ and the Laurent expansion of f in the region $|z| > 1$ are bounded. Then there are complex numbers c_k such that*

$$f(z) = \sum_{k=1}^{r} c_k(z-\zeta_k)^{-1}.$$

There are a number of applications of this theorem to operator theory, some of which are given as exercises in Section 5.

We have already noted that the additive group of integers and the multiplicative group of complex numbers of unit modulus (or equivalently, the additive group of real numbers modulo 2π) have the property that each is algebraically and topologically isomorphic with the character group of the other. This is a special case of the well-known "Pontrjagin Duality Theorem" which asserts that if B is a locally compact Abelian group and if $\hat{\hat{R}}$ denotes the character group of $\hat{R}$, then B and $\hat{\hat{R}}$ are algebraically and topologically isomorphic under a natural isomorphism. We shall now give a proof of this theorem, but first it will be convenient to establish a preliminary lemma showing how to embed B homomorphically into $\hat{\hat{R}}$.

12 LEMMA. *The mapping κ of B into $\hat{\hat{R}}$ defined by $[m, \kappa x] = [x, m]$, $m \in \hat{R}$, is a continuous homomorphism.*

PROOF. Clearly $\kappa(0) = 0$, and since

$$\begin{aligned}[m, \kappa(x_1+x_2)] &= [x_1+x_2, m] = [x_1, m][x_2, m] \\ &= [m, \kappa x_1][m, \kappa x_2] = [m, \kappa x_1+\kappa x_2],\end{aligned}$$

the mapping κ is a homomorphism. By Lemma II.1.6 it is sufficient to prove that κ is continuous at 0. To do this let $\hat{N}(0, \hat{K}, \varepsilon)$ be the neighborhood of $0 \in \hat{\hat{R}}$ defined by

$$\{X \in \hat{\hat{R}} \mid |[m, X] - [m, 0]| < \varepsilon, m \in \hat{K}\},$$

where $\hat{K}$ is a compact set in $\hat{R}$ and $\varepsilon > 0$. Let U be a neighborhood of $0 \in B$ with compact closure, and for each m_1 in K consider the neighborhood

$$N\left(m_1, \overline{U}, \frac{\varepsilon}{3}\right) = \left\{m \in \hat{B} \,\middle|\, |[x, m] - [x, m_1]| < \frac{\varepsilon}{3}, x \in \overline{U}\right\}.$$

Since $\hat{K}$ is compact, there exist a finite number of points $m_1, \ldots, m_k$ such that the neighborhoods $N(m_1, \overline{U}, \varepsilon/3), \ldots, N(m_k, \overline{U}, \varepsilon/3)$ cover $\hat{K}$. Now each of the characters m_i is continuous, so we can find a neighborhood V of $0 \in B$, with $V \subseteq U$, such that if $x \in V$ and $i = 1, \ldots, k$ then

$$|[x, m_i] - [0, m_i]| < \frac{\varepsilon}{3}.$$

Hence if $x \in V$ and $m \in \hat{K}$, we have for some choice of $i = 1, \ldots, k$,

$$\begin{aligned} |[m, \kappa x] - [m, 0]| &= |[x, m] - [0, m]| \\ &< |[x, m] - [x, m_i]| + |[x, m_i] - [0, m_i]| + |[0, m_i] - [0, m]| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

This proves that if x is in V, then κx is in $\hat{N}(0, \hat{K}, \varepsilon)$ so that κ is continuous, Q.E.D.

13 Theorem. (*Pontrjagin*) *The mapping κ is an algebraic and topological isomorphism of B onto $\hat{\hat{R}}$.*

Proof. We first show that κ is an algebraic isomorphism. If $\kappa x_0 = 0$ then we must have $[x_0, m] = 1$ for all $m \in B$; that is, to show that κ is an algebraic isomorphism we must show that for each $x_0 \neq 0$ there exists at least one $m_0 \in \hat{R}$ such that $[x_0, m_0] \neq 1$. Assume that this is false for some x_0, and let V be a neighborhood of 0 with compact closure such that $(x_0 + V) \cap V = \phi$. Let f be a continuous function vanishing outside of $x_0 + V$ such that $f(x_0) \neq 0$. Then $f \in L_1 \cap L_2(B)$, and so τf is continuous and in $L_2(\hat{R})$. By the assumption that $[x_0, m] = 1$ for all $m \in \hat{R}$ and Plancherel's theorem we have

$$\int_{\hat{R}} \overline{[x_0, m]} \tau f(m) \overline{\tau f(m)} \mu(dm) = \int_{\hat{R}} |\tau f(m)|^2 \mu(dm) \neq 0.$$

On the other hand, by Corollary 3.17 and Plancherel's theorem we have

$$\begin{aligned}\int_{\hat{R}} \overline{[x_0, m]}\tau f(m)\overline{\tau f(m)}\mu(dm) &= \int_{\hat{R}} \tau f_{x_0}(m)\overline{\tau f(m)}\mu(dm) \\ &= \int_R f(x-x_0)\overline{f(x)}dx.\end{aligned}$$

But this last integral vanishes since f vanishes outside x_0+V and $(x_0+V)\cap V = \phi$. Hence κ is one-to-one, and thus is an algebraic isomorphism of R into $\hat{R}$.

Next we shall show that $\kappa(B)$ is dense in the space $\hat{R}$. If not, then by applying Lemma 4.2 to $\hat{R}$, we find that there exists a function $H \in L_1 \cap L_2(\hat{R})$ with $|H|_2 \neq 0$ but such that $\hat{\tau}H$ vanishes on $\kappa(R)$. If $h = \tau^{-1}H \in L_2(R)$ it follows from Theorem 3.16 that h vanishes almost everywhere on B and hence $|h|_2 = 0$, which contradicts the Plancherel theorem.

To complete the proof, we add the point at infinity to both R and $\hat{R}$ and define $\kappa(\infty) = \infty$, that is, we consider κ as a mapping on $R \cup \{\infty\}$ to the space $\hat{R} \cup \{\infty\}$. The spaces $R \cup \{\infty\}$ and $\hat{R} \cup \{\infty\}$ are both compact Hausdorff and κ is a one-to-one mapping of $R \cup \{\infty\}$ into a dense subset of $\hat{B} \cup \{\infty\}$. If we can show that κ is continuous on $R \cup \{\infty\}$, then its image in $\hat{R} \cup \{\infty\}$ will be compact and thus all of $\hat{R} \cup \{\infty\}$. It will then follow from I.5.8 that κ^{-1} is continuous. Now, it was proved in the preceding lemma that κ is continuous at every point of R. Therefore it only remains to be shown that κ is continuous at ∞.

Let $\{x_\alpha\}$ be a generalized sequence in R which approaches ∞. By Lemma 4.8, given any neighborhood U of $\{\infty\}$, there exists an $F_0 \in L_1(\hat{R})$ such that $(\hat{\tau}F_0)(\hat{x}) = 1$ for $\hat{x} \notin U$. Thus, if we can show that $(\hat{\tau}F_0)(\kappa x_\alpha) \to 0$, it will follow that κx_α is eventually in U, which will prove the continuity of κ at infinity. Thus, it suffices to prove that for any $F \in L_1(\hat{R})$ we have $\hat{\tau}F(\kappa x_\alpha) \to 0$, where

$$\hat{\tau}F(\kappa x_\alpha) = \int_{\hat{R}} \overline{[x_\alpha, m]}F(m)\mu(dm).$$

Now for each α the mapping $F \to \hat{\tau}F(\kappa x_\alpha)$ is a linear functional on $L_1(\hat{R})$ with norm at most 1, so by Theorem II.3.6 it is enough to show that $\hat{\tau}F(\kappa x_\alpha) \to 0$ for F in a dense subset of $L_1(\hat{R})$. Since the

measure in R is regular, the set $\mathscr{K}$ of functions in $L_1 \cap L_2(R)$ which vanish outside of compact sets in R are dense in $L_2(R)$; by Plancherel's theorem $\{\tau f | f \in \mathscr{K}\}$ is dense in $L_2(\hat{R})$ and hence $\{\tau f \cdot \tau g | f, g \in \mathscr{K}\}$ is dense in $L_1(\hat{R})$. Now let f, g be in $\mathscr{K}$ and vanish outside of a compact set C with $C = -C$. Then if $F = \tau f \cdot \overline{\tau g}$ we have by 3.17 and Plancherel's theorem

$$\begin{aligned}\hat{\tau} F(\kappa x_\alpha) &= \int_{\hat{R}} \overline{[x_\alpha, m]} \tau f(m) \overline{\tau g(m)} \mu(dm) \\ &= \int_R f(x - x_\alpha) \overline{g(x)} dx,\end{aligned}$$

but the integral vanishes if $x_\alpha \notin C + C$ and so the proof is complete. Q.E.D.

This theorem was first proved (in the separable case) by Pontrjagin and in general by van Kampen [1]. For other proofs of this theorem see Cartan and Godement [1; p. 95], Loomis [1; p. 151], Pontrjagin [1; Sec. 31—35], Raikov [1], and Weil [1; Sec. 28].

The results of Section 8 are, as we have seen, generalizations both of the theory of Fourier series and of the Fourier integral. For the classical results in these closely related subjects the reader should consult Zygmund [1] for Fourier series and Bochner [6], Titchmarsh [3] and Wiener [4] for the Fourier integral. Treatments of the abstract theories are also given in the treatises of Loomis [1] and Weil [1], and in the papers of Cartan and Godement [1], Raikov [1] and Segal [2]. Much additional information is available, for example concerning positive definite functions, almost periodic functions, and results for non-Abelian groups. The survey article of Mackey [5] and Chapter IX of Loomis [1] will be of considerable aid to the reader. In addition to the papers cited above we cite the following: Ambrose [1], Beurling [1], [2], Gelfand and Raikov [1], [2], Godement [2], [3], [4], Krein [6], Naimark's book, *Normed Rings*, and Segal [3], [4].

Tauberian and closure theorems. The basic results of this section have their foundation in the work of Wiener [4], [5], which was for the case of the real line, although many of his proofs are valid in greater generality. Extensions to the case of locally compact Abelian groups were given independently by Segal [2] and Godement [1]. The discussion in 4.14—4.20 is a slight modification of the method used by Helson [1] to prove Theorem 4.20. Proofs of this result in the

case of the real line have been given by Ditkin [1], Segal [2], and Mandelbrojt and Agmon [1], [2], and for a wide class of groups by Kaplansky [5]. Theorem 4.16 is due to Kaplansky [5]; another proof has been given by Riss [1], the one here being due to Helson [1]. That spectral synthesis is not possible for all functions in L_∞ was shown by L. Schwartz [2] for Euclidean space of three dimensions. It has recently been shown by M. Paul Malliavin that spectral synthesis is not possible for all functions on the real axis. Cf. P. Malliavin, Sur l'impossibilité de la synthèse spectrale sur la droite, *Compt. rend.*, 248 (1959) pp. 2155—2157. For related results, cf. Paul Koosis, Sur un théorème remarquable de M. Malliavin, *Compt. rend.*, 249 (1959) pp. 852—854; P. Malliavin, Sur l'impossibilité de la synthèse spectrale dans un algèbre de functions presque périodiques, *Compt. rend.*, 248 (1959), pp. 1756—1759, and J. P. Kahane, Sur un théorème de Paul Malliavin, ibid., pp. 2943—2944. H. Pollard (*Duke Math. J.*, 20 (1953), pp. 499—511 has shown that spectral synthesis is valid for all L_1 functions satisfying a Hölder condition of order $\geq \frac{1}{2}$. For related results, cf. H. Helson and J. P. Kahane, *Compt. rend.*, 247 (1958), p. 626; J. P. Kahane, *Compt. rend.*, 246 (1958), p. 1949; Y. Katznelson, *Compt. rend.*, 247 (1958), p. 404. For spectral synthesis in topologies weaker than the L_1 topology of L_∞, see Beurling [8].

Attention in Section 4 has been directed to the manifolds in $L_1(R)$, (or dually in $L_\infty(R)$) which are closed under translation of functions. That this is essentially the same as considering ideals in $L_1(R)$ is indicated by Lemma 4.6. It is frequently convenient and suggestive to consider the ideal-theoretic point of view; this is done, for example, in the works of Loomis [1] and Mackey [4], and the papers of Šilov [4] and Mirkil [1], which the reader should consult for an exposition and for references. We have not drawn explicit attention to these questions since we found it convenient to adjoin an identity to the algebra $L_1(R)$ in order to apply the results of Chapter IX, and such adjunction alters the ideal-theoretic structure. However, a few remarks are in order and follow. In a commutative algebra $\mathfrak{B}$ over the complex field which does not possess an identity the ideals of main interest are the *regular* ideals, i.e., ideals I such that $\mathfrak{B}/I$ has an identity. It is seen that I is a regular maximal ideal if and only if $\mathfrak{B}/I$

is isomorphic with the complex field, and it turns out that the regular maximal ideals of $L_1(R)$ are in one-to-one correspondence with the points of $\mathscr{M}_0$, i.e., with all the maximal ideals of the algebra obtained by adjoining an identity to $L_1(R)$ except the point at infinity of $\mathscr{M}$. Now in an algebra with identity every ideal is contained in a maximal ideal, but if an identity is not present this is not true and it becomes an important problem to find when a closed ideal is contained in a regular maximal ideal. Theorem 4.8 settles this question for $L_1(R)$.

In analogy with the decomposition of an integer into a product of prime powers, algebraists study the decomposition of ideals into the intersection of primary ideals. For our purposes we define a closed ideal to be *primary* if it is contained in precisely one regular maximal ideal. Theorem 4.16 may be interpreted as saying that every primary ideal in $L_1(R)$ is a regular maximal ideal. Hence the problem in $L_1(R)$ is to find when a closed ideal is the intersection of the regular maximal ideals which contain it. This is sometimes called the spectral synthesis problem for ideals. While the example of L. Schwartz cited above shows this is not always possible in $L_1(R)$, Theorem 4.20 gives a conditionally affirmative result. Wermer [8] has studied an abstractly characterized class of B-algebras for which every primary ideal is a regular maximal ideal and in which a closed ideal is the intersection of the regular maximal ideals containing it. L. Schwartz [8] and Whitney [1] have also given examples of B algebras in which every closed ideal is the intersection of primary ideals. Beurling [1] introduced, in 1938, classes of subalgebras of $L_1(-\infty, \infty)$ giving conditions under which every closed ideal is contained in a regular maximal ideal. Similar results for a more general class of function algebras were obtained by Wermer [7] who determined the primary ideals in some of these algebras. Other results of a related nature have been obtained by Beurling [2], [3].

Fredholm theory. Location of eigenvalues. The Fredholm theory of operators of Hilbert-Schmidt type presented in Section 6 is due to Carleman [2], to Hille and Tamarkin [1], and to Smithies [1]. Hille and Tamarkin give in addition an assortment of theorems dealing with the asymptotic distribution of the eigenvalues of an integral operator, under various analytical conditions on its kernel. They also prove the theorem of Lalesco, that $\sum_{i=1}^{\infty}|\lambda_i| < \infty$ if $\{\lambda_i\}$ is the se-

quence of eigenvalues of the product of two operators of Hilbert-Schmidt type. Chang [1] and [2] gives related results dealing with the product of an arbitrary finite number of operators of Hilbert-Schmidt type. Nikovič [1] gives some results for a class of integral operators defined by requiring the finiteness of integral expressions in the kernel then generalizing the Hilbert-Schmidt requirement $\iint |K(x, y)|^2 dx dy < \infty$. Exercises 25 through 36 give most known inequalities for the eigenvalues of a finite matrix. The following should be mentioned in this connection. Farnell [1] shows that each eigenvalue λ of A satisfies the inequality

$$|\lambda|^2 \leq \sum_{i=1}^{n} |A\delta_i| \, |A^*\delta_i|$$

where $\delta_1 \ldots \delta_n$ is an orthonormal basis for n-dimensional Hilbert space. Barankin [1] and [2] shows that

$$|\lambda|^2 \leq \max_k \Big(\sum_{i=1}^{n} |a_{ik}| \Big) \Big(\sum_{i=1}^{n} |a_{ki}| \Big),$$

(a_{kl}) being the matrix-elements of A, and gives a number of generalizations of this inequality.

The spaces C_p of Section 9 were introduced by von Neumann and Schatten, see their paper in the *Ann. Math.* 49, 557 (1948). The completeness theorems of Sections 9 and 10 generalize earlier work of Carleman and of Keldyš, and are related to theorems given in various recent Russian papers on the theory of compact operators. See M. G. Krein, "On the Theory of Linear Nonselfadjoint Operators," *Doklady Akad. Nauk S.S.S.R.* 130, 254–256 (1960); I. C. Gohberg and M. G. Krein, "On Completely Continuous Quasinilpotent Operators," *Doklady Akad. Nauk S.S.S.R.* 128. 227 230 (1959); M. G. Krein, "On Spectrally Complete Systems of Generalized Eigenvectors of a Dissipative Operator," *Uspekhi Math. Nauk*, 14, 145–152, 1959. The theory presented in Section 10 is closely related to the theory given in the paper cited above, of Gohberg and Krein. Particular attention has been paid, in the papers cited, to *dissipative* operators, i.e., to operators whose anti-Hermitian part is negative definite. Such operators behave in a particularly simple way in a number of respects. See the cited papers for an account of these special

properties of dissipative operators, as well as for references to other Russian papers on nonselfadjoint compact operators.

The theory of subdiagonalization presented in Section 10 has been extended in a very interesting way by M. S. Livšic. He gives a theory of subdiagonalization for arbitrary bounded operators whose anti-Hermitian part belongs to the trace class C_1. See his paper "On the Spectral Resolution of Nonselfadjoint Operators," *Mat. Sbornik* 31 [73], 149–199 (1954), as well as the expository paper of M. S. Brodskiĭ and M. S. Livšic, "Spectral Analysis of Nonselfadjoint Operators and Unstable Systems," *Uspekhi Mat. Nauk* 13, 1–85 (1958). The method of Livšic is based upon consideration of the *characteristic operator function* of an operator $T = H+iH'$, defined as $W_\lambda(T; F, G) = I+2iF(\lambda I-T)^{-1}G$, where $GF = H'$ is a factorization of the anti-Hermitian part H' of T into the product of two Hilbert-Schmidt operators. As an application of this theory, Brodskiĭ and Livšic establish the very interesting result that the operation $f(x) \to \int_0^x f(y)dy$ in $L_2[0, 1]$ has no invariant subspace other than the obvious invariant subspaces $L_2[a, 1]$, $0 < a < 1$. Thus, this operator is *unicellular*, in the sense that its lattice of invariant subspaces is totally ordered. For all of this, see the papers cited above, of Livšic and of Brodskiĭ and Livšic; these papers also give an extensive bibliography of related Russian and non-Russian work on nonselfadjoint operators.

Singular integrals and inequalities. The general inequalities of Section 7 are due to Calderón and Zygmund [1] and [5]; the particular case $n = 1$ (Hilbert transform) to M. Riesz. Calderón and Zygmund [1] show that if the function Ω satisfies a suitable, rather weak, continuity hypothesis, then the singular integral

$$\varphi(x) = \int_{E^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)dy$$

(i) exists for almost all x if f is in $L_1(E^n)$ or $L_p(E^n)$, $\infty > p > 1$, (cf. Exercise 8.28);

(ii) satisfies $\int_A |\varphi(x)|^{1-\varepsilon} dx < \infty$ if $f \in L_1(E^n)$, $\varepsilon > 0$, and A is bounded;

(iii) satisfies $\left\{\int_A |\varphi(x)| dx\right\} < \infty$ if A is bounded and

$$\int_{E^n} |f(x)|\{1 + \log^+ |f(x)|\} dx < \infty;$$

(iv) satisfies $\int_{E^n} |\varphi(x)| dx < \infty$ if

$$\int_{E^n} |f(x)| \log^+ \{1 + |f(x)| + |x|^n |f(x)|\} dx < \infty.$$

In cases (ii), (iii), and (iv), the singular integrals

$$\lim_{\varepsilon \to 0} \int_{x \geqq \varepsilon} \frac{\Omega(x)}{|x|^n} f(y-x) dx$$

converge in the topology of $L_{1-\varepsilon}(A)$, $L_1(A)$, and $L_1(E^n)$ respectively. These results make it possible to relax the condition

$$\left\{\int_S |\Omega(s)|^{1+\varepsilon} \mu(ds)\right\}^{1/1+\varepsilon} < \infty,$$

of Theorem 7.11 to the following inequality:

$$\int_S |\Omega(s)| \log^+ |\Omega(s)| \mu(ds) < \infty.$$

Calderón and Zygmund also give applications of their results to a number of "potential" kernels of the sort arising in the theory of partial differential equations. Subsequently their inequality has found many important applications in this theory.

In [5], Calderón and Zygmund give similar results for a related, somewhat more general class of integral operators in L_p, $1 < p < \infty$. In [6], they discuss the theory of algebras of singular convolution operators of the type studied in Section 7, in particular, the maximal ideal theory and the question of the existence of inverses.

It is well known that, by applying standard inequalities to functions with values in an appropriate Banach space, the applicability of these inequalities can often be usefully extended. For this reason, it is noteworthy that, whereas M. Riesz' original proof of his well-known inequality for the Hilbert integral uses complex-variable methods and hence cannot be extended to vector-valued functions, a real-variable proof of the sort given by Calderón and Zygmund [1] can be so extended. We indicate here how to take systematic advantage of this fact. We will find that we obtain inequalities of the Calderón-Zygmund-Riesz type sufficiently general so that inequalities of Paley, Littlewood, and Zygmund may be obtained as

corollaries. Our presentation follows the lines of the recent elegant paper of Hörmander ("Translation Invariant Operators", *Acta Math.* 104, 93–139 (1960)). Hörmander gives an extensive bibliography of earlier work.

One of our tools will be the following important interpolation theorem of Marcinkiewicz, generalized to vector-valued functions. Neither the statement nor the proof of this theorem is any different for vector-valued than for scalar-valued functions.

14 THEOREM. *Let (S, Σ, μ) and (S_1, Σ_1, μ_1) be two positive measure spaces; X and X_1 be two B-spaces; and let $1 \leqq p < q < r$, $1 \leqq p_1 < q_1 < r_1$, with $p \leqq p_1$, $r \leqq r_1$, $1/q = \alpha/p + (1-\alpha)/r$ and $1/q_1 = \alpha/p_1 + (1-\alpha)/r_1$ for some α, with $0 < \alpha < 1$. Let T be a linear transformation of the space L_0 of all bounded measurable functions defined on S vanishing outside a set of finite measure and having values in X, into the space of all measurable functions defined on S_1 and having values in X_1. Suppose that there exists a finite constant K such that*

$$(1)\qquad [\mu_1(\{s_1| |(Tf)(s_1)| > a\})]^{1/p_1} < \frac{K}{a}\left\{\int_S |f(s)|^p \mu(ds)\right\}^{1/p}, \qquad f \in L_0, \quad a > 0,$$

and also that

$$(2)\qquad [\mu_1(\{s_1| |(Tf)(s_1)| > a\})]^{1/r_1} < \frac{K}{a}\left\{\int_S |f(s)|^r \mu(ds)\right\}^{1/r}, \qquad f \in L_0, \quad a > 0.$$

Then it follows that there exists a finite constant K' such that

$$(3)\qquad \left\{\int_{S_1} |Tf(s_1)|^{q_1} \mu_1(ds_1)\right\}^{1/q_1} \leqq K' \left\{\int_S |f(s)|^q \mu(ds)\right\}^{1/q}.$$

PROOF. For each measurable function g defined on the measure space (S, Σ, μ) and each $a > 0$, let

$$(4)\qquad (g)_a = \mu(\{s| |g(s)| > a\}).$$

If g is defined on (S_1, Σ_1, μ_1), let $(g)_a$ be defined in an exactly corresponding way. Then $|g|_t^t$, the tth power of the L_t-norm of g, is given by the formula

$$(5)\qquad |g|_t^t = \int |g(s)|^t \mu(ds) = -\int_0^\infty a^t\, d(g)_a = t\int_0^\infty a^{t-1}(g)_a\, da.$$

For each such function g, put

$$(6)\qquad \begin{aligned} g^{(a)}(s) &= g(s) && \text{if } |g(s)| \leq a, \\ g^{(a)}(s) &= a\,\frac{g(s)}{|g(s)|} && \text{if } |g(s)| \geqq a, \\ \tilde g^{(a)}(s) &= g(s) - g^{(a)}(s). \end{aligned}$$

It is then plain that

$$(7)\qquad \begin{aligned} (g^{(a)})_b &= (g)_b, && b \leqq a, \\ (g^{(a)})_b &= 0, && b > a, \\ (\tilde g^{(a)})_b &= (g)_{b+a}. \end{aligned}$$

Moreover, since $f = f^{(a)} + \tilde f^{(a)}$, we have $Tf = (Tf^{(a)}) + (T\tilde f^{(a)})$, and therefore if $b > 0$ we have, by hypothesis,

$$(8)\qquad \begin{aligned} (Tf)_b &\leqq (Tf^{(a)})_{b/2} + (T\tilde f^{(a)})_{b/2} \\ &\leqq K'(b^{-r_1}|f^{(a)}|_r^{r_1/r} + b^{-p_1}|\tilde f^{(a)}|_p^{p_1/p}); \end{aligned}$$

here and in what follows, K' denotes an arbitrary finite constant. Now we put $a = a(b)$, when $a(b)$ is a monotone-increasing function with inverse $b = b(a)$ whose precise form is to be specified below, and use (5), (7), and (8) to obtain

$$(9)\qquad \begin{aligned} |Tf|_{q_1}^{q_1} &\leq K'\int_0^\infty \{b^{q_1-r_1-1}|f^{(a(b))}|_r^{r_1/r} + b^{q_1-p_1-1}|\tilde f^{(a(b))}|_p^{p_1/p}\}db \\ &\leqq K'\left\{\int_0^\infty b^{q_1-r_1-1}\left[\int_0^{a(b)}(f)_c c^{r-1}dc\right]^{r_1/r}db\right. \\ &\quad \left.+\int_0^\infty b^{q_1-p_1-1}\left[\int_0^\infty (f)_{c+a(b)}c^{p-1}dc\right]^{p_1/p}db\right\} \\ &= K'\left\{\int_0^\infty\left[\int_0^{a(b)} b^{r/r_1(q_1-r_1-1)}(f)_c c^{r-1}dc\right]^{r_1/r}db\right. \\ &\quad \left.+\int_0^\infty\left[\int_{a(b)}^\infty b^{p/p_1(q_1-p_1-1)}(f)_c(c-a(b))^{p-1}dc\right]^{p_1/p}db\right\}. \end{aligned}$$

The continuous form of Minkowski's inequality may be written

$$(10)\qquad \int d\beta\left\{\int \varphi(\alpha,\beta)d\alpha\right\}^k \leq \left[\int\left\{\int|\varphi(\alpha,\beta)|^k d\beta\right\}^{1/k}d\alpha\right]^k,$$

and is valid whenever $k \geqq 1$. Using this and (9) we get

$$
\begin{aligned}
|Tf|_{q_1}^{q_1} \leqq K' \Big\{ & \Big[\int_0^\infty (f)_c c^{r-1} \Big[\int_{b(c)}^\infty b^{(q_1-r_1-1)} db\Big]^{r/r_1} dc\Big]^{r_1/r} \\
& + \Big[\int_0^\infty (f)_c \Big[\int_0^{b(c)} (c-a(b))^{p_1/p(p-1)} b^{q_1-p_1-1} db\Big]^{p/p_1} dc\Big]^{p_1/p}\Big\} \\
= K' \Big\{ & \int_0^\infty (f)_c c^{r-1} b(c)^{(q_1-r_1)r/r_1} dc \\
& + \Big[\int_0^\infty (f)_c \Big[\int_0^c (c-a)^{p_1/p(p-1)} b(a)^{q_1-p_1-1} b'(a) da\Big]^{p/p_1} dc\Big]^{p_1/p}\Big\}.
\end{aligned} \tag{11}
$$

Now we put $b(a) = a^\xi$, so that (11) reduces to

$$
\begin{aligned}
|Tf|_{q_1}^{q_1} \leqq K' \Big\{ & \int_0^\infty (f)_c c^{r-1+\xi(q_1-r_1)r/r_1} dc \\
& + \Big[\int_0^\infty (f)_c \Big[\int_0^c (c-a)^{p_1/p(p-1)} a^{\xi(q_1-p_1-1)+\xi-1} da\Big]^{p/p_1} dc\Big]^{p_1/p}\Big\} \\
\leqq K' \Big\{ & \int_0^\infty (f)_c \big[c^{r-1+\xi(q_1-r_1)r/r_1} + c^{p-1+\xi(q_1-p_1)p/p_1}\big] dc\Big\}.
\end{aligned} \tag{12}
$$

Since

$$
\frac{1}{q} = \frac{\alpha}{p} + \frac{1-\alpha}{r}, \qquad \frac{1}{q_1} = \frac{\alpha}{p_1} + \frac{1-\alpha}{r_1}, \tag{13}
$$

it follows that if we determine ξ from the equation

$$
\xi\left(\frac{q_1}{p_1} - 1\right) = \frac{q}{p} - 1 \tag{14}
$$

we have also

$$
\xi\left(\frac{q_1}{p_1} - 1\right) = \frac{q}{r} - 1, \tag{15}
$$

and

$$
r+\xi(q_1-r_1)r/r_1 = p+\xi(q_1-p_1)p/p_1 = q. \tag{16}
$$

Thus Marcinkiewicz' interpolation theorem is proved. Q.E.D.

It is to be noted that if T is a bounded map of $L_p(X)$ into $L_{p_1}(X_1)$, it satisfies condition (1).

Next, following Hörmander, we exhibit a class of convolution kernels which satisfy condition (1) for $p = 1$. These kernels are described in the following theorem.

15 Theorem. *Let X and X_1 be two B-spaces. Let $K(x)$ be a function of x in E^n, having values in the B-space of bounded linear mappings of X into X_1. Suppose that $K(x)$ is integrable over every finite region. Let $q \geqq 1$ and $A > 0$ be given and suppose that there exists a constant $C < \infty$ such that for each t of the form 2^j we have*

$$\left\{\int_{|x| \geqq A} |K(t(x-y))-K(tx)|^q dx\right\}^{1/q} \leqq Ct^{-n/q}, \qquad |y| \leqq \frac{1}{A}. \tag{17}$$

Put

$$(\mathscr{K}f)(x) = \int_{E^n} K(x-y)f(y)dy, \qquad f \in L_0(X). \tag{18}$$

Suppose also that for some p and r with $\infty > p \geqq 1$, $\infty > r \geqq 1$, and $1/p - 1/r = 1-1/q$, we have

$$\left\{\int_{E^n} |(\mathscr{K}f)(x)|^r dx\right\}^{1/r} \leqq C\left\{\int_{E^n} |f(x)|^p dx\right\}^{1/p}. \tag{19}$$

Then

$$\mu\{x \in E^n \mid |(\mathscr{K}f)(x)| > a\}^{1/q} \leqq \frac{C'}{a}\int_{E^n} |f(x)|dx, \qquad f \in L_0, \tag{20}$$

for some finite constant C'.

The proof of Theorem 15 is based upon the following covering lemma.

16 Lemma. *Let a number $s > 0$ be given, and let u be an integrable function defined on E^n with values in the B-space X. Then we may write*

$$u = v + \sum_{k=1}^{\infty} w_k, \tag{21}$$

where

$$|v| + \sum_{k=1}^{\infty} |w_k| \leqq 3|u|, \tag{22a}$$

$$|v(x)| \leqq 2^n s, \qquad x \in E^n, \tag{22b}$$

(22c) *each function w_k vanishes outside a certain cube I_k having a side of length 2^{-n_k}, where n_k is a certain positive or negative integer, all the cubes I_k are disjoint and $\sum_{k=1}^{\infty} m(I_k) \leqq s^{-1}|u|$, and $\int_{I_k} w_k(x)dx = 0$.*

In (22c) we have used $m(e)$ to denote the Lebesgue measure of the set e, and have used $|u|$ to denote the L_1-norm of the integrable function u. Lemma 16 is proved by Hörmander (loc. cit.) for scalar-valued functions. The proof for vector-valued functions is hardly different. We therefore omit the proof, and pass immediately to the proof of Theorem 15.

Proof of Theorem 15. First note that A may be increased without spoiling any of the hypotheses. Thus, we may and shall suppose that A has the form $A = 2^M$. Suppose the w is a function whose integral is zero and which vanishes outside a cube I having a side of length A^{-1}, centered at the origin. Then we have

$$(23) \qquad (\mathscr{K}w)(y) = \int_I \big(K(x-y)-K(x)\big)w(y)dy,$$

so that, applying (17),

$$(24) \qquad \left\{\int_{y\notin A^2 I} |(\mathscr{K}w)(y)|^q dy\right\}^{1/q} \leqq C|w|_1,$$

$|w|_1$ denoting the L_1-norm. Since the condition (17) is invariant under translations and under dilations in any ratio $1:2^j$, it follows that if w' is a function whose integral is zero and which vanishes outside a cube I' having a side whose length is of the form 2^k, then the cube I' may be included in a cube I'' having a side of length 2^{k+2M} such that

$$(25) \qquad \left\{\int_{y\notin I''} |(\mathscr{K}w')(y)|^q dy\right\}^{1/q} \leqq C|w_1|.$$

Let the functions u, v, and w_k be as in the preceding lemma, and let $w'' = \sum_{k=1}^{\infty} w_k$; it follows at once from (25) and (22c) that there exists a set e of measure at most $2^{2nM}s^{-1}|u|_1$ such that

$$(26) \qquad \left\{\int_{y\notin e} |(\mathscr{K}w'')(y)|^q dy\right\}^{1/q} \leqq C|w''|_1 \leqq 3C|u|_1.$$

Since $(W)_t \leqq t^{-q}|W|_q^q$ for any function W, (cf. (4)), it follows at once that for $t > 0$

$$(27) \qquad (\mathscr{K}w'')_t \leqq C'(t^{-q}|u|_1^q + s^{-1}|u_1|);$$

here and in what follows we shall write C' for an arbitrary but finite constant and $|W|_q$ for the L_q-norm of W. Since $u = v + w''$, we deduce at once that

$$(\mathscr{K}u)_t \leq (\mathscr{K}v)_{t/2} + (\mathscr{K}w'')_{t/2}. \tag{28}$$

Since $|v|_\infty \leq C's$ and $|v|_1 \leq |u|_1$, Hölder's inequality gives $|v|_p \leq C's^{1-1/p}|u|_1^{1/p}$ for each $p > 1$. Since we have assumed that $|\mathscr{K}f|_r \leq C'|f|_p$ for each function f in L_0, it follows that

$$|\mathscr{K}v|_r \leq C's^{1-1/p}|u|_1^{1/p}. \tag{29}$$

Thus

$$\begin{aligned}(\mathscr{K}v)_t &\leq C's^{r(1-1/p)}|u|_1^{r/p}t^{-r} \\ &= C''s^{r/q-1}|u|_1^{r/p}t^{-r}.\end{aligned} \tag{30}$$

Thus, using (27), (30), and (28), we find

$$(\mathscr{K}u)_t \leq C'(t^{-q}|u|_1^q + s^{-1}|u|_1 + s^{r/q-1}|u|_1^{r/p}t^{-r}). \tag{31}$$

The positive number s is still at our disposal. Choosing it so as to minimize the expression on the right of (31), we get

$$(\mathscr{K}u)_t \leq C't^{-q}|u|_1^q, \tag{32}$$

and Theorem 15 is proved. Q.E.D.

17 Corollary. *Under the hypotheses of Theorem* 15 *it follows that*

$$|\mathscr{K}f|_{r'} \leq C'|f|_{p'} \tag{33}$$

for each $r > r' \geq 1$, $p > p' \geq 1$ *with* $1/p'-1/r' = 1-1/q$.

Proof. This follows at once from the Marcinkiewicz interpolation theorem and from Theorem 15. Q.E.D.

18 Corollary. *If the kernel* K *of Theorem* 15 *satisfies*

$$\left(\int_{|x|\geq At}\left|\frac{\partial}{\partial x_i}K(x)\right|^q dx\right)^{1/q} \leq Ct^{-1}, \qquad i = 1, \ldots, n \tag{34}$$

for every t *which is a positive or negative power of* 2, *then it necessarily satisfies* (17).

Proof. Equation (34) may be written

$$\left\{\int_{|x|\geq A}\left|\frac{\partial}{\partial x_i}K(tx)\right|^q dx\right\}^{1/q} \leq Ct^{-n/q}, \tag{35}$$

which, by integration over an appropriate path, evidently implies (17). Q.E.D.

19 COROLLARY. *Let the kernel K of Theorem 15 satisfy the equation*

$$O_1K(2x)O_2 = 2^{-n/q}K(x), \qquad x \in E^n, \tag{36}$$

where O_1 and O_2 are operators in X_1 and X, respectively, all of whose powers, both positive and negative, are bounded. Then hypothesis (17) of Theorem 15 is implied by the statement

$$\int_{2\geq|x|\geq 1}\left|\frac{\partial}{\partial x_i}K(x)\right|^q dx < \infty, \qquad i = 1, \ldots, n. \tag{37}$$

PROOF. If t is of the form $t = 2^j$, j being a positive or a negative integer, then (36) and (37) together imply

$$\begin{aligned}\int_{2t\geq|x|\geq t}&\left|\frac{\partial}{\partial x_i}K(x)\right|^q dx \\ &\leq C\int_{2t\geq|x|\geq t} t^{-n}\left|\frac{\partial}{\partial x_i}\left[K\left(\frac{x}{t}\right)\right]\right|^q dx \\ &= Ct^{-1}\int_{2t\geq|x|\geq t}\left|\left[\frac{\partial}{\partial x_i}tK\right]\left(\frac{x}{t}\right)\right| t^{-n}dx \\ &= Ct^{-1}\int_{2\geq|x|\geq 1}\left|\frac{\partial}{\partial x_i}K(x)\right| dx.\end{aligned} \tag{38}$$

Thus

$$\begin{aligned}\int_{\infty\geq|x|\geq 1}\left|\frac{\partial}{\partial x_i}K(x)\right|^q dx &\leq C\sum_{n=1}^{\infty}\frac{1}{2^n}\int_{2\geq|x|\geq 1}\left|\frac{\partial}{\partial x_i}K(x)\right|^q dx \\ &= 2C\int_{2\geq|x|\geq 1}\left|\frac{\partial}{\partial x_i}K(x)\right|^q dx,\end{aligned} \tag{39}$$

so that the present corollary follows easily. Q.E.D.

If $n = 1$, $q = 1$, condition (34) reduces to the condition $\operatorname{var}_{|x|\geq t}(K(x)) = O(t^{-1})$.

The inequality of Paley and Littlewood is not hard to deduce from Corollary 19. We first prove the following generalization of the Calderón–Zygmund theorem to vector-valued functions.

20 THEOREM. *Let (S, Σ, μ) be a positive measure space. Let $\Omega(x)$ be a numerically-valued kernel defined in E^n, homogeneous of order 0, smooth except at $x = 0$, and whose surface integral over the surface of the unit sphere is zero. Let $L_p(L_q)$ denote the L_p-space of functions defined on E^n, with values in $L_q(S)$. Then, for each $1 < p < \infty$ and $1 < q < \infty$, the convolution transform*

$$f(x) \to \mathscr{P} \int \frac{\Omega(x-y)}{|x-y|^n} f(y) dy \tag{40}$$

defines a bounded transformation of $L_p(L_q)$ into itself.

PROOF. We first consider the special case $p = q$. The space $L_p(L_p)$ may in an evident manner be identified with the L_p-space on the product set $E^n \times S$ and the inequality to be proved reduces to

$$\int_S \int_{E^n} \left| \mathscr{P} \int_{E^n} \frac{\Omega(x-y)}{|x-y|^n} f(y, s) dy \right|^p dx ds \le C' \int_S \int_{E^n} |f(x, s)|^p dx ds. \tag{41}$$

This inequality is nothing but an integrated form of the ordinary Calderón-Zygmund inequality.

It follows from the validity of the present theorem for $p = q$ and from Corollaries 17 and 19 that the present theorem must be valid whenever $1 < p < q < \infty$. It is then clear that the validity of our theorem for $1 < q < p < \infty$ will follow from the evident fact that the adjoint of an operator of the form (38) is of the same form, and from the following lemma.

21 LEMMA. *Let X be a B-space and X^* its adjoint space. Let $1 < p, q < \infty$, $p^{-1} + q^{-1} = 1$. Let (S, Σ, μ) be a positive measure space. Then*

$$\sup_{g \in L_q(X^*), |g|_q \le 1} \left| \int g(s) f(s) \mu(ds) \right| = |f|_p, \qquad f \in L_p(X). \tag{42}$$

PROOF. That the right side of (42) bounds the left follows at once from Hölder's inequality, generalized in a trivial way to vector-valued functions. Knowing this, it follows at once that it suffices to

prove the reverse inequality for a dense subset of $L_p(X)$, so that we may assume without loss of generality that f is a simple function, i.e., that $f(s) = x_i$ on each of the disjoint sets e_i, $i = 1, \ldots, n$, and that f vanishes outside the union of these sets. There exist elements x_i^* in X^* such that $x_i^*(x) = |x_i|$ for $i = 1, \ldots, n$, and $|x_i^*| = 1$. Let h be a scalar-valued function and put $g(s) = h(s)x_i^*$ if s is in e_i and $g(s) = 0$ if s is in none of the sets e_i. Then

$$\int g(s)f(s)ds = \int h(s)|f(s)|ds; \tag{43}$$

hence (42) follows from the similar well-known equation for scalar-valued functions. This concludes the proof of Lemma 21, and with it the proof of Theorem 20. Q.E.D.

22 Corollary. *Let (S, Σ, μ) be a positive measure space, and let $1 < p < \infty$, $1 < q < \infty$. Let C be a finite constant, and let $f(x, s)$ be a measurable function defined on the product space $E^1 \times S$ such that*

$$\int_{-\infty}^{+\infty} \left\{\int_S |f(x, s)|^p \mu(ds)\right\}^{q/p} dx \leqq C. \tag{44}$$

Let $\hat{f}(\xi, s)$ be the Fourier transform of the function $f(x, s)$ with respect to the variable x, and let $g(x, s)$ be the function whose Fourier transform is defined by $\hat{g}(\xi, s) = \hat{f}(\xi, s)$, $\xi > 0$; $\hat{g}(\xi, s) = 0$, $\xi < 0$. Then there exists a constant C' depending only on p and q such that

$$\int_{-\infty}^{+\infty} \left\{\int_S |g(x, s)|^p \mu(ds)\right\}^{q/p} dx \leq C'C. \tag{45}$$

Proof. This is merely the special case $X = L_p(S)$, $n = 1$, $\Omega(x) = \operatorname{sgn} x$ of Theorem 20. Q.E.D.

The next corollary generalizes the preceding corollary in a familiar way.

23 Corollary. *Let (S, Σ, μ) be a positive measure space, and let $1 < p < \infty$, $1 < q < \infty$. For each real ξ, let $T(\xi)$ be a bounded operator-valued function from $L_p(S)$ to itself. Suppose that the operator-valued function $T(\xi)$ is uniformly bounded and of bounded variation. Let $f(x)$ be a function of the real variable x with values in the space $L_p(S)$, and let $\hat{f}(\xi)$ be its Fourier transform with respect to the variable x. Let $\mathscr{H}_1$ be the mapping defined by the formula*

$$(\widehat{\mathscr{K}_1 f})(\xi) = T(\xi)\hat{f}(\xi). \tag{46}$$

Then $\mathscr{K}_1$ is a bounded mapping of the space $L_q(L_p(S))$ into itself.

PROOF. For each real ξ_0, let $\mathscr{H}_{\xi_0}$ be the mapping in $L_q(L_p(S))$ defined by the formula

$$\begin{aligned}(\widehat{\mathscr{H}_{\xi_0} f})(\xi) &= \hat{f}(\xi), \qquad \xi > \xi_0, \\ &= 0 \qquad\quad \text{otherwise.}\end{aligned} \tag{47}$$

By Corollary 22, it follows that there is a finite constant C' such that

$$|\mathscr{H}_{\xi_0}| \leq C'; \tag{48}$$

the norm being, of course, the norm of $\mathscr{H}_{\xi_0}$ as an operator mapping $L_q(L_p(S))$ into itself.

On the other hand, it is plain from (46) and (47) that

$$\mathscr{K}_1 = \int_{-\infty}^{+\infty} \mathscr{H}_{\xi_0} dT(\xi_0) + \mathscr{T}(-\infty), \tag{49}$$

$\mathscr{T}(-\infty)$ denoting the mapping $f(x) \to T(-\infty)f(x)$. The present corollary follows at once from (48) and (49). Q.E.D.

If the operator $T(\xi)$ of the preceding corollary is a multiplication operator, the result stated can be significantly improved to give a result, generalizing a theorem of Zygmund, which we shall need in what follows. For reasons of notational simplicity we confine our statement to the special case in which S is countable, so that the space L_p of the preceding corollary reduces to the sequence space l_p.

24 LEMMA. *Let $1 < p < \infty, 1 < q < \infty$. Let $k_n(\xi)$ be a sequence of bounded functions. Suppose that these functions are uniformly bounded and that their variations are uniformly bounded. Let $\mathscr{K}$ be the transformation in $L_q(l_p)$ which maps the vector-valued function whose nth component has the Fourier transform $f_n(\xi)$ into the vector-valued function whose nth component has the Fourier transform $k_n(\xi)f_n(\xi)$. Then $\mathscr{K}$ is a bounded mapping of $L_q(l_p)$ into itself.*

We shall, instead of giving a direct proof of Lemma 24, regard Lemma 24 as the limiting case of Lemma 24′, stated immediately below. The deduction of Lemma 24 from Lemma 24′ is trivial and we therefore omit the details of this deduction.

24′ LEMMA. *Let p, q, k_n be as in the preceding lemma, and, for each N, let $\mathscr{K}_N$ be the transformation in $L_p(l_q)$ which maps the vector whose nth component has the Fourier transform $f_n(\xi)$ into the vector whose nth component has the Fourier transform $k_n(\xi)f_n(\xi)$ for $n \leq N$, and $f_n(\xi)$ for $n > N$. Then there exists a finite constant C' independent of N such that the norm of $\mathscr{K}_N$, regarded as a mapping of $L_q(l_p)$ into itself, is at most C'.*

PROOF OF LEMMA 24′. Subtracting a suitable constant c_n from each of the functions k_n, we may suppose without loss of generality that $k_n(-\infty) = 0$ for each n; here we have used the uniform boundedness of the functions k_n and of their variations to conclude that the constants c_n are uniformly bounded. Similarly, multiplying each of the functions k_n by a suitable positive constant c_n', we may suppose without loss of generality that each of the functions k_n has total variation 1; here we have used the uniform boundedness of the variations $\text{var}(k_n)$ to conclude that the constants c_n' are bounded below.

Let $\mathscr{H}(\xi_1, \ldots, \xi_N)$ be the mapping in $L_q(l_p)$ which maps the vector-valued function f whose nth component has the Fourier transform $f_n(\xi)$ into the vector-valued function g whose nth component has the Fourier transform $g_n(\xi)$ defined by

$$
\begin{aligned}
(50) \qquad & g_n(\xi) = f_n(\xi), && n > N, \\
& g_n(\xi) = f_n(\xi), && \xi < \xi_r, \quad n \leqq N, \\
& = 0, && \xi > \xi_r, \quad n \leqq N.
\end{aligned}
$$

Then it is clear that

$$
(51) \quad \mathscr{H}(\xi_1, \ldots, \xi_N) = \mathscr{M}(\xi_1, \ldots, \xi_N)\mathscr{H}(0, \ldots, 0)\mathscr{M}(-\xi_1, \ldots, -\xi_N),
$$

where $\mathscr{M}(\xi_1, \ldots, \xi_N)$ is the isometry of $L_p(l_q)$ defined by

$$
(52) \qquad \{f_n(x)\} \to \{e^{i\xi_n x} f_n(x)\};
$$

in (52) we have written $\xi_n = 0$, $n \geqq N$ for notational simplicity. Thus Corollary 23 implies that $\mathscr{H}(\xi_1, \ldots, \xi_N)$ has a bound C' independent of N and of $\xi_1, \ldots, \xi_N$.

Formula (50) and the definition of $\mathscr{K}_N$ make it evident that

$$
(53) \qquad \mathscr{K}_N = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \mathscr{H}(\xi_1, \ldots, \xi_N) dk_1(\xi_1) \ldots dk_n(\xi_n);
$$

thus since $\text{var}(k_n) = 1$, $\mathscr{K}_N$ has the same bound C'. This proves Lemma 24′, and, as we have remarked, Lemma 24 follows at once. Q.E.D.

After these preliminaries, we may at once proceed to the proof of the inequality of Paley and Littlewood.

25 THEOREM. *Let $1 < p \leq 2$, let $f(x)$ denote an arbitrary function in $L_p(-\infty, +\infty)$, and let $f_n(x)$ denote the function whose Fourier transform is identical with that of f in the range $2^n < |\xi| < 2^{n+1}$ and vanishes outside this range. Then there exist finite constants C and C' such that*

$$(54) \qquad C\int_{-\infty}^{+\infty}\left(\sum_{n=-\infty}^{+\infty}|f_n(x)|^2\right)^{p/2}dx \leq \int_{-\infty}^{+\infty}|f(x)|^p dx$$
$$\leq C'\int_{-\infty}^{+\infty}\left(\sum_{n=-\infty}^{+\infty}|f_n(x)|^2\right)^{p/2}dx.$$

PROOF. Let $\varphi(\xi)$ be an even function in C^∞ identically equal to 1 for $1 \leq |\xi| \leq 2$, identically equal to zero for $|\xi| \leq 1/2$ or $|\xi| \geqq 3$, chosen so that its integral and first few moments are zero. Let $\hat{K}(\xi)$ be the vector whose nth component is $\varphi(2^n\xi)$; $\hat{K}$ is then a vector-valued function with values in the two-sided Hilbert sequence space l_2. Let

$$(55) \qquad K(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\xi x}\hat{K}(\xi)d\xi;$$

since $U^n\hat{K}(\xi) = \hat{K}(2^n\xi)$, U denoting the unit shift operator, it follows that $K(2^nx) = U^n2^{-n}K(x)$. If ψ is the Fourier transform of φ, then $\psi(x)$ and its first few derivatives vanish at 0, while $|\psi(x)| = O(|x|^{-N})$ at $|x| = \infty$ for any finite n. The nth component of the vector $K(x)$ is $2^n\psi(2^nx)$; thus

$$(56) \qquad |K'(x)|^2 = \sum_{n=-\infty}^{+\infty} |2^n\psi'(2^nx)|^2.$$

Since

$$(57) \qquad |\psi'(x)| \leq \left(\frac{|x|}{1+|x|^2}\right)^2,$$

the series (56) may be compared to

$$\sum_{1}^{+\infty} \frac{2^n}{1+2^{2n}|x|^2} + \sum_{-\infty}^{1} 2^n \cdot 2^{2n}|x|^2. \tag{58}$$

and its sum is therefore bounded on each bounded interval of x. Let the transformation $\mathscr{K}$ be defined by the equation

$$(\widehat{\mathscr{K}f})(\xi) = \hat{K}(\xi)\hat{f}(\xi), \tag{59}$$

or equivalently

$$(\mathscr{K}f)(x) = \int_{-\infty}^{+\infty} K(x-y)f(y)dy. \tag{60}$$

Then $\mathscr{K}$ maps scalar-valued functions into functions with values in l_2. It is plain from Plancherel's theorem that $\mathscr{K}$ is a bounded mapping of the space L_2 of scalar-valued functions into the space $L_2(l_2)$ of square-integrable vector-valued functions. Corollary 19 and Corollary 17 now imply that, for $1 < p < 2$, $\mathscr{K}$ maps the space L_p of scalar-valued functions boundedly into the space $L_p(l_2)$ of vector-valued functions.

Let $\mathscr{M}$ be the mapping in $L_p(l_2)$ which maps the vector-valued function whose nth component has the Fourier transform $\hat{g}_n(\xi)$ into the vector-valued function whose nth component has the Fourier transform $h_n(\xi)$ defined by

$$\begin{aligned} h_n(\xi) &= \hat{g}_n(\xi), \qquad && 2^n < |\xi| < 2^{n+1}, \\ &= 0, && \text{otherwise}. \end{aligned} \tag{61}$$

By Corollary 24, $\mathscr{M}$ is a bounded linear transformation. On the other hand, it is plain from the definition of $\hat{K}(\xi)$, from (59), and from (61) that $\mathscr{M}\mathscr{K}$ maps the function f into the vector-valued function whose nth component is the function f_n of (54). Thus the left-hand inequality in (54) is proved.

To prove the right-hand inequality in (54), we argue similarly, as follows. If G is a function with values in the Hilbert sequence space l_2, whose nth component has the Fourier transform $\hat{g}_n(\xi)$, then put

$$(\mathscr{L}G)(x) = \int_{-\infty}^{+\infty} K(x-y) \cdot G(y)dy; \tag{62}$$

equivalently, $\mathscr{L}G$ is the scalar-valued function whose Fourier transform is defined by

$$\mathscr{L}G(\xi) = \sum_{-\infty}^{+\infty} \hat{K}_n(\xi)\hat{g}_n(\xi). \tag{63}$$

By Plancherel's theorem, $\mathscr{L}$ is a bounded mapping of $L_2(l_2)$ into the space of scalar-valued functions l_2. Thus, by Corollary 19 and Corollary 17, $\mathscr{L}$ is a bounded mapping of $L_p(l_2)$ into L_p. It is clear from (63) and (61) that $\mathscr{L}\mathscr{M}$ maps G into the scalar-valued function f whose Fourier transform is defined by

$$\hat{f}(\xi) = g_n(\xi), \qquad 2^n < |\xi| < 2^{n+1}. \tag{64}$$

This proves the right-hand inequality in (54). Q.E.D.

26 Corollary. *Theorem 25 remains valid in the full range* $1 < p < \infty$.

Proof. We saw in the course of proving Theorem 25 that the mapping $\mathscr{M}\mathscr{K}$ which sends a scalar-valued function with the Fourier transform $f(\xi)$ into the vector-valued function whose nth component has the Fourier transform $f_n(\xi)$ defined by

$$\begin{aligned} f_n(\xi) &= f(\xi), \qquad && 2^n \leqq |\xi| < 2^{n+1}, \\ &= 0, && \text{otherwise}, \end{aligned} \tag{65}$$

is a bounded map of L_p in $L_p(l_2)$ if $1 < p \leqq 2$. The adjoint of this map is evidently the map $\mathscr{L}\mathscr{M}$ of the proof of Theorem 25, regarded as a map from $L_q(l_2)$ into L_q, where $1/p+1/q = 1$. Thus $\mathscr{L}\mathscr{M}$ is bounded even in the range $2 \leq q < \infty$. We may show by a similar "adjointness" argument that the map $\mathscr{M}\mathscr{K}$ is bounded in the extended range. Q.E.D.

27 Corollary. *Theorem 25 remains valid in the full range* $1 < p < \infty$, *and for functions* f *with values in an arbitrary Hilbert space.*

Proof. Note that Theorem 20 goes over with trivial modifications of its proof to functions with values in an arbitrary Hilbert space (or even an arbitrary L_p-space) and, in particular, that Lemma 21 generalizes with hardly any change in its proof to any pair X, Y^* of B-spaces such that

(66) $$\sup_{y^* \in Y^*} |y^*(x)| = |x|, \qquad x \in B;$$

and that in consequence Corollary 22 is valid for functions $f(x, s)$ with values in Hilbert space. Therefore, Corollary 28 generalizes, with hardly any change in its proof, to the space of functions f with values in any space $L_p(\mathfrak{H})$, $\mathfrak{H}$ denoting an arbitrary Hilbert space. Next, it may be noted that Lemma 24 generalizes at once, and with the same proof, to the space $L_p(l_p(\mathfrak{H}))$ of functions whose values lie in the sequence space l_p consisting of all sequences of vectors in the Hilbert space $\mathfrak{H}$, the pth powers of whose norms are summable. This remark permits the ready generalization of Theorem 25 to functions f with values in a Hilbert space $\mathfrak{H}$. The generalized Theorem 25 may be extended from the partial range $1 < p \leq 2$ to the full range $1 < p < \infty$ by use of the generalized form of Lemma 21 stated just above. Q.E.D.

Now we are ready to prove Marcinkiewicz' theorem, and even a slight generalization of it.

28 Theorem. *Let $1 < p < \infty$. Let $L_p(\mathfrak{H})$ denote the L_p space of functions with values in a Hilbert space $\mathfrak{H}$. For each real ξ, let $T(\xi)$ be a bounded operator in $\mathfrak{H}$; suppose that the function $T(\xi)$ is bounded, and that the variations*

(67) $$\operatorname*{var}_{2^n < \xi < 2^{n+1}} (T(\xi))$$

and

(68) $$\operatorname*{var}_{-2^n > \xi > -2^{n+1}} (T(\xi))$$

are uniformly bounded as n ranges over all positive and negative integers. Let f be in $L_p(\mathfrak{H})$, and let $\hat{f}(\xi)$ be its Fourier transform with respect to the variable x. Let $\mathscr{K}_1$ be the mapping defined by the formula

(69) $$(\widehat{\mathscr{K}_1 f})(\xi) = T(\xi)\hat{f}(\xi), \qquad f \in L_p(\mathfrak{H}).$$

Then $\mathscr{K}_1$ is a bounded mapping of the space $L_p(\mathfrak{H})$ into itself.

Proof. Let $\mathfrak{H}^+$ denote the Hilbert space of all (two-sided) square-summable sequences of vectors in $\mathfrak{H}$. Let $T^+(\xi)$ be the mapping which takes the vector $\{x_n\}$ in $\mathfrak{H}^+$ into the vector whose nth com-

ponent is $T(\xi)x_n$ if $2^n \leqq |\xi| < 2^{n+1}$ and zero otherwise. As we have remarked in the course of proving Corollary 27, Corollary 24 generalizes to functions with values in the space $l_2(\mathfrak{H}) = \mathfrak{H}^+$. Thus, we may conclude that the mapping $\mathscr{K}_1^+$ in $L_p(\mathfrak{H}^+)$ defined by the formula

$$(70) \qquad (\widehat{\mathscr{K}_1^+ F})(\xi) = T^+(\xi)\hat{F}(\xi), \qquad F \in L_p(\mathfrak{H}^+),$$

is bounded. Let $\mathscr{A}^+$ and $\mathscr{B}$ be the mappings of $L_p(\mathfrak{H})$ into $L_p(l_2(\mathfrak{H}))$ and $L_p(l_2(\mathfrak{H}))$ into $L_p(\mathfrak{H})$, respectively, defined by the formulae

$$(71) \qquad \begin{aligned} (\mathscr{A}^+ f)_{[n]}(\xi) &= f_n(\xi), \qquad && 2^n \leqq |\xi| < 2^{n+1}, \\ &= 0, && \text{otherwise;} \end{aligned}$$

and

$$(72) \qquad (\mathscr{B}F)(\xi) = F_n(\xi), \qquad 2^n \leqq |\xi| < 2^{n+1}.$$

Then, by Corollary 27, $\mathscr{A}^+$ and $\mathscr{B}$ are bounded. On the other hand, it is plain from (69), (70), (71), and (72) that $\mathscr{B}\mathscr{K}_1^+\mathscr{A}^+ = \mathscr{K}_1$. Thus $\mathscr{K}_1$ is bounded. Q.E.D.

29 Corollary. *In Theorem* 28 *the hypotheses* (67) *and* (68) *may be replaced by the hypothesis*

$$(73) \qquad |T'(\xi)| \leqq \frac{C}{|\xi|}.$$

This corollary gives a vector-valued generalization of a useful theorem of Mihlin.

Theorem 25 above was proved in a slightly more general form by Paley and Littlewood [1]. They prove the discrete analog of a theorem which may be stated in the continuous case as follows:

30 Theorem. *Let $\beta \geqq \alpha > 0$ be positive constants. Let $\{\lambda_m\}$ be an increasing sequence of non-negative real numbers such that $\lambda_0 = 0$ and*

$$(1+\alpha)\lambda_n \leqq \lambda_{n+1} \leqq (1+\beta)\lambda_n.$$

Let β be the σ-field of subsets of the real line generated by intervals of the form $[\pm\lambda_m, \pm\lambda_n]$. For each $f \in L_2(-\infty, +\infty)$, and each $e \in \beta$, let $E(e)f = F^{-1}(\chi_e F(f))$, χ_e denoting the characteristic function of e, and $F: L_2 \to L_2$ the Fourier transform. Then, if $1 < p < \infty$ there exists a constant $K = K(p, \alpha, \beta)$ such that

$$|E(e)f|_p \leq K(p, \alpha, \beta)|f|_p, \qquad e \in \beta, \qquad f \in L_2 \cap L_p.$$

Marcinkiewicz [1] improved this result in various ways, arriving at a slightly generalized discrete analog of Theorem 28, above, which may be stated in the continuous case as follows.

31 THEOREM. *Let α, β, p, $\{\lambda_n\}$ be as in the preceding theorem. Let $M < \infty$. Let φ be a function defined on the real axis, and continuously differentiable for $x \neq \pm\lambda_i$, such that*

(i) $$\int_{\lambda_i}^{\lambda_{i+1}} |\varphi'(x)| dx + \int_{\lambda_{i+1}}^{\lambda_i} |\varphi'(x)| dx \leq M, \qquad i \geqq 0$$

(ii) $$|\varphi(x)| \leq M \qquad -\infty < x < +\infty.$$

Let $H_\varphi f = F^{-1}(\varphi F(f))$ for f in L_2. Then there exists a constant $K(M, p, \alpha, \beta)$ such that

$$|H_\varphi f|_p \leqq K(M, p, \alpha, \beta)|f|_p, \qquad f \in L_2 \cap L_p.$$

Marcinkiewicz also gives an m-dimensional version of his theorem. In the two-dimensional case, the conditions (i) and (ii) of the preceding theorem are replaced by the conditions

(i′) $$\left\{\int_{\lambda_i}^{\lambda_{i+1}}\int_{\lambda_j}^{\lambda_{j+1}} + \int_{\lambda_i}^{\lambda_{i+1}}\int_{-\lambda_{j+1}}^{-\lambda_j} + \int_{-\lambda_{i+1}}^{\lambda_i}\int_{\lambda_j}^{\lambda_{j+1}} + \int_{-\lambda_{j+1}}^{\lambda_i}\int_{\lambda_{j+1}}^{\lambda_j}\right\} \left|\frac{\partial^2}{\partial x_1 \partial x_2}\varphi(x_1, x_2)\right| dx_1 dx_2 \leq M. \qquad 0 \leqq i, j.$$

(ii′) $$\left\{\int_{\lambda_i}^{\lambda_{i+1}} + \int_{\lambda_{i+1}}^{-\lambda_i}\right\}\left\{\left|\frac{\partial}{\partial x}\varphi(x, \lambda_j)\right| + \left|\frac{\partial}{\partial x}\varphi(\lambda_j, x)\right| + \left|\frac{\partial}{\partial x}\varphi(x, -\lambda_j)\right| + \left|\frac{\partial}{\partial x}\varphi(-\lambda_j, x)\right|\right\} dx < M, \quad 0 \leqq i, j.$$

(iii′) $$|\varphi(x_1, x_2)| \leq M, \qquad \infty < x_1, x_2 < +\infty.$$

This will make the proper formulation of the m-dimensional version of the Marcinkiewicz theorem plain, though because of the notational complexity of the $m+1$ conditions which replace (i), (ii), and (iii) in this case, we refrain from citing them explicitly.

It follows readily from the Marcinkiewicz theorem that if γ is real, and if we place $\varphi(t) = |t|^{i\gamma}$ for t in E^m, then there exists a finite constant $K_{p,m}$ depending only on p and m such that

$$|F^{-1}\varphi F(f)|_p \leqq K_{p,m}|f|_p, \qquad f \in L_2 \cap L_p.$$

Thorin [1] made use of this fact to give the following m-dimensional generalization of a result due in the one-dimensional case to Hardy and Littlewood [1].

32 Theorem. *Let f be in $L_p(E^m)$ and let $p < r$. Then the integral*

$$(If)(x) = \int_{E^m} \frac{f(y)}{|x-y|^{m(1-1/p+1/r)}}\, dy$$

exists for almost all x in E^m, and defines a bounded mapping of $L_p(E^m)$ into $L_r(E^m)$.

Thorin proves this theorem by a convexity argument comparable to the argument given by him to prove the Riesz convexity theorem. In this way, he is able to simplify the original Hardy-Littlewood proof, which was based on their method of "rearrangements in decreasing order" even in case $m = 1$. This general method is discussed in Hardy-Littlewood-Pólya [1], Chapter 10.

It should be noted that because of the positivity of the kernel in the preceding theorem of Hardy-Littlewood-Thorin, it may be carried over immediately to singular integrals of the form

$$\int_{E^m} \frac{K(x, y)\, f(y)}{|x-y|^{m(1+1/r-1/p)}}\, dy,$$

K being a bounded and measurable function.

In the paper cited above, Marcinkiewicz also shows how the condition $(1+\alpha)\lambda_n \leqq \lambda_{n+1} \leqq (1+\beta)\lambda_n$ in the fundamental theorem of Littlewood and Paley may be relaxed.

In his paper "The Decomposition of Walsh and Fourier Series", *American Mathematical Society Memoirs*, no. 15 (1956), I. I. Hirschman discusses a number of interesting inequalities related to the inequality of Marcinkiewicz, and to related inequalities of Littlewood and Paley and of Babenko. His memoir should be consulted for a detailed account of these inequalities, and for a bibliography. Later work by Hirschman also relates to these inequalities.

A function f defined in a subset A of E^n is said to satisfy a Hölder condition of exponent ε and constant K if

$$|f(x)-f(y)| \leqq K|x-y|^\varepsilon, \qquad x, y \in A.$$

The set of bounded functions satisfying such a condition forms a B-space under the norm

$$|f| = \sup_{x,y \in A} \frac{|f(x)-f(y)|}{|x-y|^\varepsilon} + \sup_{x \in A} |f(x)|.$$

These B-spaces have been studied extensively in connection with the theory of singular integrals. Singular integrals of Hilbert-Calderón-Zygmund type may be shown under suitable hypotheses to map functions satisfying a Hölder condition of exponent $0 < \varepsilon < 1$ into functions of the same sort. Singular integrals of Hardy-Littlewood-Paley type may be shown under suitable hypotheses to map functions satisfying a Hölder condition of a certain exponent $0 < \varepsilon < 1$ into functions satisfying a Hölder condition with larger exponent. Theorems of this type are particularly useful in the theory of partial differential equations. For a variety of such theorems, see Zygmund [1], pp. 225–228; Friedrichs [1], and Friedrichs [11], pp. 101–121, especially Theorem 9.7, p. 116; Hardy and Littlewood [1].

Many of the inequalities which we have proved have "discrete" analogs in the form of inequalities for certain sums, singular integral inequalities for periodic functions, etc. These may be found in the various references cited. See, in particular, Zygmund [1], Chapters 7 and 9, and also the recently appeared second edition of this work of Zygmund.

CHAPTER XII

Unbounded Operators in Hilbert Space

1. Introduction

In the preceding chapter we have seen how the spectral theory developed in Chapters IX and X may be applied to various problems in mathematical analysis. However, we have not as yet applied this theory to the important class of problems known as boundary value problems. This is because the operators arising in boundary value problems are differential operators and thus not everywhere defined on Hilbert space. In the present chapter we lay the foundations for an extension of the spectral theory of Chapter X which is sufficiently general to cover a variety of applications to self adjoint boundary value problems. The boundary value problems themselves will be discussed in the following chapter and the present discussion will merely give the abstract technical foundation.

The new difficulties arise not only because the differential operators are not everywhere defined but because they are not continuous on their domain of definition, i.e., they are unbounded operators. Since some of the concepts used in Chapter X have no *a priori* meaning for unbounded operators we shall redefine many of these notions in a more general setting. The term *operator* will be used for a linear map between linear spaces. The symbols $\mathfrak{D}(T)$ and $\mathfrak{R}(T)$ are used for the *domain* (or domain of definition) and *range* respectively, of an operator T. Two operators T and U are said to be *equal* (in symbols, $T = U$) if and only if $\mathfrak{D}(T) = \mathfrak{D}(U)$ and $Tx = Ux$ for every x in $\mathfrak{D}(T)$; T is said to be an *extension* of U (in symbols, $T \supseteq U$) if and only if $\mathfrak{D}(T) \supseteq \mathfrak{D}(U)$ and $Tx = Ux$ for every x in $\mathfrak{D}(U)$. The symbol $U \subseteq T$ is sometimes used in place of $T \supseteq U$. Unless stated to the contrary it will be assumed that the domain and range of each operator under discussion is a subset of a Hilbert space which will be designated by the symbol $\mathfrak{H}$. As usual, an operator is said to be *bounded* if the supremum sup $|Tx|$ taken over all x in $\mathfrak{D}(T)$

with $|x| = 1$ is finite and otherwise T is said to be *unbounded*. Thus an operator is unbounded if and only if it is discontinuous at one point (and hence at every point) of its domain. The *graph* $\Gamma(T)$ of an operator T is that subset of $\mathfrak{H} \oplus \mathfrak{H}$ consisting of all points of the form $[x, Tx]$ with x in $\mathfrak{D}(T)$, and T is said to be a *closed operator* if its graph is a closed subset of $\mathfrak{H} \oplus \mathfrak{H}$. The direct sum of Hilbert spaces will always be taken in the sense of Section IV.4 so that $\mathfrak{H} \oplus \mathfrak{H}$ is a Hilbert space with the inner product

$$([x_1, x_2], [y_1, y_2]) = (x_1, y_1)+(x_2, y_2).$$

This inner product may be used to define a new inner product in $\mathfrak{D}(T)$ by means of the formula

$$(x, y)_1 = ([x, Tx], [y, Ty]) = (x, y)+(Tx, Ty), \qquad x, y \in \mathfrak{D}(T).$$

The space $\mathfrak{D}(T)$ with the inner product $(x, y)_1$ is not necessarily a Hilbert space since it may not be complete relative to the norm $(x, x)_1^{1/2}$ but if T is a closed operator then the linear space $\mathfrak{D}(T)$ with inner product $(x, y)_1$ is complete and is therefore a Hilbert space. Furthermore a closed operator T when considered as defined from the Hilbert space $\mathfrak{D}(T)$ with the inner product $(x, y)_1$ to $\mathfrak{H}$ with inner product (x, y), is continuous.

In using the elementary algebraic operations of addition and multiplication on operators which are not everywhere defined one must exercise a bit of caution and so we state the following formal definition.

1 Definition. Let T and U be linear operators and let α be a complex number. Then the operators $T+U$, TU, αT, and T^{-1} are defined as follows:

(a) $\mathfrak{D}(T+U) = \mathfrak{D}(T) \cap \mathfrak{D}(U)$, $\qquad (T+U)x = Tx+Ux$;

(b) $\mathfrak{D}(TU) = \{x|x \in \mathfrak{D}(U), Ux \in \mathfrak{D}(T)\}$, $\qquad (TU)x = T(Ux)$;

(c) if $\alpha = 0$ then $\alpha T \equiv 0$, otherwise $\mathfrak{D}(\alpha T) = \mathfrak{D}(T)$ and $(\alpha T)x = \alpha(Tx)$;

(d) if T is one-to-one then $\mathfrak{D}(T^{-1}) = \mathfrak{R}(T)$ and $T^{-1}y = x$ if $y = Tx$.

The usual associative laws $(A+B)+C = A+(B+C)$ and $(AB)C = A(BC)$ hold, so that sums and products of several operators

may be unambiguously written without the use of parentheses. The distributive law $(A+B)C = AC+BC$ has its familiar form but, since $(B+C)x$ may be in the domain of A without Bx or Cx being in this domain we only have the inclusion $A(B+C) \supseteq AB+AC$ instead of equality.

Just as in the case of a bounded operator the *resolvent set* $\rho(T)$ of an operator T is defined to be the set of all complex numbers λ such that $(\lambda I-T)^{-1}$ exists as an everywhere defined bounded operator. For λ in $\rho(T)$ the symbol $R(\lambda; T)$ will be used for the *resolvent operator* $(\lambda I-T)^{-1}$. The *spectrum* $\sigma(T)$ of T is the complement of the resolvent set $\rho(T)$. The *point spectrum* $\sigma_p(T)$, the *continuous spectrum* $\sigma_c(T)$, and the *residual spectrum* $\sigma_r(T)$ are defined just as they were in Definition X.3.1 for bounded operators. One way in which unbounded operators differ from bounded ones is that the spectrum of an unbounded operator may be the whole plane.

2 LEMMA. *The inverse of a closed operator is closed. A bounded operator is closed if and only if its domain is closed.*

PROOF. If A_1 is the isometric automorphism in $\mathfrak{H} \oplus \mathfrak{H}$ which maps $[x, y]$ into $[y, x]$ then $\Gamma(T^{-1}) = A_1\Gamma(T)$ which shows that T is closed if and only if T^{-1} is closed. If B is a bounded closed operator and if $\{x_n\}$ is a Cauchy sequence in $\mathfrak{D}(B)$ then $\{[x_n, Bx_n]\}$ is a Cauchy sequence in the closed set $\Gamma(B)$ and hence it has a limit $[x, Bx]$ in $\Gamma(B)$. Thus the sequence $\{x_n\}$ converges to the point x in $\mathfrak{D}(B)$ which proves that $\mathfrak{D}(B)$ is closed. Conversely, if the domain of the bounded operator B is closed and if $\{[x_n, Bx_n]\}$ is a Cauchy sequence in $\Gamma(B)$ then the limit $x = \lim x_n$ exists in $\mathfrak{D}(B)$ and, since B is continuous, $[x_n, Bx_n] \to [x, Bx]$ which proves that $\Gamma(B)$ is closed. Q.E.D.

3 LEMMA. *Let T be a closed operator. Then the sets $\rho(T)$, $\sigma_p(T)$, $\sigma_c(T)$, and $\sigma_r(T)$ are disjoint and their union is the whole plane. The resolvent set $\rho(T)$ is open and the resolvent $R(\lambda; T)$ is an analytic function of λ and satisfies the resolvent equation*

$$R(\lambda; T)-R(\mu; T) = (\mu-\lambda)R(\lambda; T)R(\mu; T), \qquad \lambda, \mu \in \rho(T).$$

PROOF. It is clear from the definitions that $\rho(T)$, $\sigma_p(T)$, $\sigma_c(T)$, and $\sigma_r(T)$ are disjoint sets and that if a point λ is not in any of these sets the inverse $(\lambda I-T)^{-1}$ must exist as an everywhere defined and

unbounded operator. Thus to prove the first conclusion it will suffice to show that if $(\lambda I-T)^{-1}$ exists with domain $\mathfrak{H}$ then λ is in $\rho(T)$. It follows from Lemma 2 that $(\lambda I-T)^{-1}$ is closed and hence it follows from the closed graph theorem (II.2.4) that $(\lambda I-T)^{-1}$ is bounded since it is everywhere defined. An examination of the proof of Lemma VII.3.2 where the facts that $\rho(T)$ is open and that $R(\lambda; T)$ is analytic are proved for bounded operators will make it clear that these same facts hold for unbounded operators. Finally, the resolvent equation follows by subtracting the first of the following equations from the second

$$(\lambda I-T)R(\lambda; T)R(\mu; T) = R(\mu; T),$$
$$(\mu I-T)R(\lambda; T)R(\mu; T) = R(\lambda; T). \qquad \text{Q.E.D.}$$

The Hilbert space adjoint T^* of a bounded operator T in Hilbert space has been defined by the identity $(Tx, y) = (x, T^*y)$. We shall need to use the notion of the Hilbert space adjoint of an operator which is not necessarily bounded and this concept is formulated in the following definition.

4 Definition. If the domain $\mathfrak{D}(T)$ of the operator T is dense in $\mathfrak{H}$ then the *domain* $\mathfrak{D}(T^*)$ consists, by definition, of all y in $\mathfrak{H}$ for which (Tx, y) is continuous for x in $\mathfrak{D}(T)$. Since $\mathfrak{D}(T)$ is dense in $\mathfrak{H}$ there is (IV.4.5) a uniquely determined point y^* in $\mathfrak{H}$ such that $(Tx, y) = (x, y^*)$ for every x in $\mathfrak{D}(T)$. The *Hilbert space adjoint* or simply the *adjoint* T^* is defined on $\mathfrak{D}(T^*)$ by the equation $T^*y = y^*$. In other words

$$(Tx, y) = (x, T^*y), \qquad x \in \mathfrak{D}(T),\ y \in \mathfrak{D}(T^*).$$

In this definition the domain $\mathfrak{D}(T)$ is required to be dense in $\mathfrak{H}$ in order that the point y^* corresponding to a point y in $\mathfrak{D}(T^*)$ be uniquely defined. Thus, *whenever we mention the adjoint of an operator T, it is tacitly assumed that $\mathfrak{D}(T)$ is dense.* Similarly whenever the inverse T^{-1} is mentioned it is tacitly assumed that T is one-to-one.

We recall that the orthocomplement of a set $\mathfrak{A}$ in $\mathfrak{H}$ is defined as the set $\{x | x \in \mathfrak{H}, (x, \mathfrak{A}) = 0\}$. This orthocomplement is denoted by $\mathfrak{H} \ominus \mathfrak{A}$ or by $\mathfrak{A}^\perp$. The set $\mathfrak{A}^\perp$ is clearly a closed linear manifold and if $\mathfrak{A}$ is itself a closed linear manifold then $\mathfrak{A}$ and $\mathfrak{A}^\perp$ are complementary manifolds, i.e., $\mathfrak{H} = \mathfrak{A} \oplus \mathfrak{A}^\perp$ (IV.4.4).

5 Lemma. *Let the isometric automorphisms A_1 and A_2 in $\mathfrak{H} \oplus \mathfrak{H}$ be defined by the equations*

$$A_1[x, y] = [y, x], \ A_2[x, y] = [y, -x].$$

Then

$$\Gamma(T^{-1}) = A_1\Gamma(T), \qquad \Gamma(T^*) = (A_2\Gamma(T))^\perp,$$
$$A_1A_2 = -A_2A_1, \qquad A_1^2 = I, \qquad A_2^2 = -I.$$

Proof. We have $[y, y^*] \in \Gamma(T^*)$ if and only if

$$0 = (Tx, y)-(x, y^*) = ([Tx, -x], [y, y^*]), \quad x \in \mathfrak{D}(T).$$

Hence $\Gamma(T^*) = (A_2\Gamma(T))^\perp$. The proofs of the remaining statements are left to the reader. Q.E.D.

6 Lemma. *Let T be an operator in Hilbert space. Then*

(a) *if $\mathfrak{D}(T)$ is dense then T^* is a closed linear operator;*

(b) *if T^{-1} exists with dense domain then $(T^*)^{-1}$ exists and $(T^*)^{-1} = (T^{-1})^*$;*

(c) *if B is an everywhere defined bounded operator then*

$$(T+B)^* = T^*+B^*, \qquad (BT)^* = T^*B^*;$$

(d) $$\mathfrak{R}(T)^\perp = \{y|y \in \mathfrak{D}(T^*), T^*y = 0\}.$$

Proof. Since an orthocomplement is closed, statement (a) follows from Lemma 5.

To prove (b) we note first that $T^*y = 0$ implies that $(Tx, y) = 0$ for every x in $\mathfrak{D}(T)$. Since the manifold $\mathfrak{R}(T) = \mathfrak{D}(T^{-1})$ is dense this means that $y = 0$ and thus that $(T^*)^{-1}$ exists. Now, by Lemma 5.

$$\begin{aligned}\Gamma((T^*)^{-1}) &= A_1\Gamma(T^*) = A_1(A_2\Gamma(T))^\perp \\ &= (A_1A_2\Gamma(T))^\perp = (-A_2A_1\Gamma(T))^\perp \\ &= (A_2A_1\Gamma(T))^\perp = (A_2\Gamma(T^{-1}))^\perp = \Gamma((T^{-1})^*),\end{aligned}$$

and thus $(T^*)^{-1} = (T^{-1})^*$.

To prove (c) we have, since B is everywhere defined, $\mathfrak{D}(T+B) = \mathfrak{D}(T)$. Since B is continuous and $((T+B)x, y) = (Tx, y)+(Bx, y)$ we see that $\mathfrak{D}(T^*) = \mathfrak{D}((T+B)^*)$. Thus, for x in $\mathfrak{D}(T) = \mathfrak{D}(T+B)$ and y in $\mathfrak{D}(T^*) = \mathfrak{D}((T+B)^*)$ we have

$$(x, (T+B)^*y) = \big((T+B)x, y\big) = (Tx, y)+(Bx, y)$$
$$= (x, T^*y)+(x, B^*y) = \big(x, (T^*+B^*)y\big),$$

which proves that $(T+B)^* = T^* + B^*$.

To prove that $(BT)^* = T^*B^*$ we suppose that x is in $\mathfrak{D}(T)$ and that y is in $\mathfrak{D}\big((BT)^*\big)$. Then $(x, (BT)^*y) = (BTx, y) = (Tx, B^*y)$ which shows that $B^*y \in \mathfrak{D}(T^*)$ and therefore that $y \in \mathfrak{D}(T^*B^*)$. This equation also shows that for y in $\mathfrak{D}\big((BT)^*\big)$ we have $T^*B^*y = (BT)^*y$ and this proves that $T^*B^* \supseteq (BT)^*$. Similarly, if x is in $\mathfrak{D}(T)$ and y is in $\mathfrak{D}(T^*B^*)$ then $(x, T^*B^*y) = (Tx, B^*y) = (BTx, y)$ which shows that y is in $\mathfrak{D}\big((BT)^*\big)$ and that $(BT)^*y = T^*B^*y$. This proves that $(BT)^* \supseteq T^*B^*$ and completes the proof of (c).

Finally we note that the statement (d) follows immediately from the identity

$$(Tx, y) = (x, T^*y), \quad x \in \mathfrak{D}(T), \quad y \in \mathfrak{D}(T^*),$$

and this completes the proof of the lemma. Q.E.D.

Most of the considerations in this chapter and the next will be directed towards an operator which is either symmetric or self adjoint according to the following definition.

7 Definition. The operator T is said to be *symmetric* if $(Tx, y) = (x, Ty)$ for every pair x, y of points in $\mathfrak{D}(T)$. It is said to be *self adjoint* if $T = T^*$.

An operator T may be symmetric without having a dense domain but if $\mathfrak{D}(T)$ is dense so that T^* is defined then the notion of symmetry is equivalent to the inclusion $T^* \supseteq T$. Of course if T is a bounded everywhere defined operator then the statements $T^* \supseteq T$ and $T^* = T$ are equivalent and thus a bounded operator is symmetric if and only if it is self adjoint. If T is an everywhere defined symmetric operator then $T^* \supseteq T$ and thus $T^* = T$. By Lemma 6(a) T is closed and by the closed graph theorem (II.2.4), T is bounded. Thus an *everywhere defined symmetric operator is bounded and self adjoint.*

Even though symmetry and self adjointness are the same for bounded operators, an unbounded symmetric operator need not be self adjoint. As an example consider the operator id/dt on the domain $\mathfrak{D}(id/dt)$ in $L_2(0, 1)$ consisting of those functions f with a continuous derivative and with $f(0) = f(1) = 0$. Since

$$
\begin{aligned}
(if', g) &= \int_0^1 if'(t)\overline{g(t)}dt \\
&= \int_0^1 f(t)\overline{ig'(t)}dt + if(1)\overline{g(1)} - if(0)\overline{g(0)} \\
&= (f, ig') \qquad\qquad f, g \in \mathfrak{D}\left(i\frac{d}{dt}\right),
\end{aligned}
$$

it is seen that id/dt on the dense domain $\mathfrak{D}(id/dt)$ is symmetric. However, this operator is not self adjoint for it is clear from the above equations that any function g with a continuous first derivative has the property that

$$\left(i\frac{d}{dt}f, g\right) = \left(f, i\frac{d}{dt}g\right), \qquad f \in \mathfrak{D}\left(i\frac{d}{dt}\right),$$

and thus any such g, even though it fails to vanish at one of the endpoints 0 or 1, is in the domain of the adjoint of id/dt.

The problem, suggested by the preceding example, of finding self adjoint extensions of a given symmetric operator will be treated systematically in Section 4.

2. The Spectral Theorem for Unbounded Self Adjoint Operators

In this section the spectral theory developed in Section X.2 for bounded self adjoint operators will be extended to cover the case of unbounded self adjoint operators. In particular it will be shown that every self adjoint operator has a unique self adjoint regular countably additive resolution of the identity in terms of which an operational calculus may be given. This will be shown by first proving that the resolvent $R(\alpha; T) = (\alpha I - T)^{-1}$ of a self adjoint operator T is defined for all non-real α and is itself a normal operator to which the theory in Section X.2 may be applied.

1 Lemma. *Let T be a symmetric operator and α a non-real scalar. Then $(\alpha I - T)^{-1}$ exists and*

$$|x| \leqq \frac{|(\alpha I - T)x|}{|\mathscr{I}(\alpha)|}, \qquad x \in \mathfrak{D}(T).$$

Proof. If $\mathscr{I}(\alpha)$ and $\mathscr{R}(\alpha)$ are the imaginary and real parts of α and if $x \in \mathfrak{D}(T) = \mathfrak{D}(\alpha I - T)$ then

$$\begin{aligned}|(\alpha I-T)x|^2 &= \big((\alpha I-T)x, (\alpha I-T)x\big)\\ &= \big(\mathscr{I}(\alpha)x, \mathscr{I}(\alpha)x\big)+\big((\mathscr{R}(\alpha)I-T)x, (\mathscr{R}(\alpha)I-T)x\big)\\ &\geqq \big(\mathscr{I}(\alpha)x, \mathscr{I}(\alpha)x\big) = |\mathscr{I}(\alpha)|^2|x|^2,\end{aligned}$$

from which the desired conclusions follow immediately. Q.E.D.

2 LEMMA. *The spectrum of a self adjoint operator T is real and the resolvent is a normal operator with $R(\alpha; T)^* = R(\bar{\alpha}; T)$ and*

$$|R(\alpha; T)| \leqq |\mathscr{I}(\alpha)|^{-1}, \qquad \mathscr{I}(\alpha) \neq 0.$$

PROOF. Let α be a non-real scalar. The preceding lemma shows that $(\alpha I-T)^{-1}$ exists as a bounded operator. To prove that α is in $\rho(T)$ it will therefore suffice to prove that its domain is closed and has orthocomplement zero. Since $T = T^*$ it follows from Lemma 1.6(a) that T is closed and thus that $\alpha I-T$ is closed. Lemma 1.2 shows that $(\alpha I-T)^{-1}$ is closed and, since it is a bounded operator, its domain must be closed (Lemma 1.2). By Lemma 1.6(d)

$$\big(\mathfrak{D}((\alpha I-T)^{-1})\big)^{\perp} = \big(\Re(\alpha I-T)\big)^{\perp} = \{y|(\alpha I-T)^*y = 0\}.$$

Now $(\alpha I-T)^* = \bar{\alpha}I-T$ and, since $\mathscr{I}(\bar{\alpha}) \neq 0$ we see (Lemma 1) that $\bar{\alpha}I-T$ is one-to-one. Thus $\{y|(\alpha I-T)^*y = 0\} = \{0\}$. This completes the proof of the statement that $\alpha \in \rho(T)$ and proves therefore that the spectrum is real. Since $(\alpha I-T)^* = \bar{\alpha}I-T$ it follows from Lemma 1.6(b) that $R(\bar{\alpha}; T) = R(\alpha; T)^*$ and thus that $R(\alpha; T)$ is normal. The final inequality is a corollary of the preceding lemma. Q.E.D.

3 THEOREM. *Let T be a self adjoint operator. Then its spectrum is real and there is a uniquely determined regular countably additive self adjoint spectral measure E defined on the Borel sets of the plane, vanishing on the complement of the spectrum, and related to T by the equations*

(a) $$\mathfrak{D}(T) = \{x|x \in \mathfrak{H}, \int_{\sigma(T)} \lambda^2(E(d\lambda)x, x) < \infty\},$$

and

(b) $$Tx = \lim_{n\to\infty}\int_{-n}^{n} \lambda E(d\lambda)x, \qquad x \in \mathfrak{D}(T).$$

PROOF. The reality of the spectrum was established in Lemma 2. Consider the homeomorphism $\mu = h(\lambda)$ of the compact complex sphere which is given by the equation $\mu = (i-\lambda)^{-1}$. We shall show first that h maps $\sigma(T)\cup\{\infty\}$ onto $\sigma(R(i; T))$. Let $\lambda \neq i$ be a point in $\rho(T)$ and let $A = (i-\lambda)^2R(\lambda; T)+(i-\lambda)I$. It follows readily from the resolvent equation (Lemma 1.3) that $(\mu I-R(i; T))A = I$ and thus that μ is in $\rho(R(i; T))$. If $\lambda = i$ then $\mu = \infty$ and therefore, in this case also, μ is not in the spectrum of the bounded operator $R(i; T)$. Conversely, let $0 \neq \mu \in \rho(R(i; T))$, let $A = (\mu I-R(i; T))^{-1}$, and let $B = \mu R(i; T)A$. Then B is one-to-one and its range is $\mathfrak{D}(T)$. Thus the equations

$$\begin{aligned}(\lambda I-T)B &= [(\lambda-i)I+(iI-T)]B\\ &= [-R(i; T)+\mu I]A = I,\end{aligned}$$

show that λ is in $\rho(T)$. On the other hand we cannot have $\mu = 0 \in \rho(R(i; T))$ for this would imply that $R(i; T)^{-1} = iI-T$ is a bounded everywhere defined operator, which case we exclude here since the theorem has already been established for bounded operators in Chapter X. This shows that the homeomorphism h maps $\rho(T)$ onto $\rho(R(i; T))\cup\{\infty\}$ and thus it maps $\sigma(T)\cup\{\infty\}$ onto $\sigma(R(i; T))$.

For every Borel set δ in the complex plane let $E(\delta) = E_1(h(\delta))$ where E_1 is the resolution of the identity for the normal operator $R(i; T)$. We note that $E_1(\{0\}) = 0$ for if $0 \neq x = E_1(\{0\})x$ then $R(i; T)x = \int_{\{0\}}\lambda E_1(d\lambda)x = 0$ which contradicts the fact that $R(i; T)$ has an inverse. This shows that if δ is the finite complex plane we have $E(\delta) = I$ and thus E is a spectral measure. The spectral measure E is self adjoint, countably additive and regular since E_1 has these properties (Corollary X.2.4). Furthermore it is clear that $E(\sigma(T)) = I$ and thus that $E(\delta) = 0$ for $\delta \subseteq \rho(T)$. Since the spectrum of T is real it follows that an integral over the real axis with respect to the measure E is the same as if taken over the spectrum $\sigma(T)$.

Now let

$$\mathfrak{D}_0 = \{x \mid \int_{\sigma(T)}\lambda^2(E(d\lambda)x, x) < \infty\}.$$

It is clear that $E(\delta)\mathfrak{H} \subseteq \mathfrak{D}_0$ for bounded Borel sets δ. Also, for a bounded set δ, we have $|\int_\delta \lambda E(d\lambda)x|^2 = \int_\delta \lambda^2(E(d\lambda)x, x)$ and thus

$$\text{(i)} \qquad \mathfrak{D}_0 = \{x \mid \lim_{n\to\infty} \int_{-n}^{n} \lambda E(d\lambda)x \text{ exists}\}.$$

It will next be shown that $\mathfrak{D}(T) \subseteq \mathfrak{D}_0$. If x is in $\mathfrak{D}(T)$ then, since $\mathfrak{D}(T) = R(i; T)\mathfrak{H}$, x has the form $x = R(i; T)y$ and

$$\int_{-n}^{n} \lambda E(d\lambda)x = \int_{-n}^{n} \lambda E(d\lambda)R(i; T)y.$$

By the change of measure principle

$$\text{(ii)} \qquad R(i; T)y = \int \mu E_1(d\mu)y = \int \frac{E(d\lambda)y}{i-\lambda},$$

where both integrals are taken over the whole plane, and so

$$\int_{-n}^{n} \lambda E(d\lambda)x = \int_{-n}^{n} \frac{\lambda}{i-\lambda} E(d\lambda)y \to \int_{-\infty}^{\infty} \frac{\lambda}{i-\lambda} E(d\lambda)y.$$

This shows that x is in $\mathfrak{D}_0$ and hence that $\mathfrak{D}(T) \subseteq \mathfrak{D}_0$.

From (ii) and Theorem X.1.1 we have

$$R(i; T)\int_{-n}^{n} (i-\lambda)E(d\lambda) = E([-n, n])$$

which shows that $E([-n, n])\mathfrak{H} \subseteq \mathfrak{D}(T)$. Now

$$\begin{aligned} TE([-n, n]) &= [iI-(iI-T)]R(i; T)\int_{-n}^{n} (i-\lambda)E(d\lambda) \\ &= \int_{-n}^{n} \lambda E(d\lambda) \end{aligned}$$

and thus it follows from (i) that for x in $\mathfrak{D}_0$ the sequence $\{TE([-n, n])x\}$ converges. Since T is closed and $E([-n, n])x \to x$ it follows that x is in $\mathfrak{D}(T)$ and $Tx = \lim_n \int_{-n}^{n} \lambda E(d\lambda)x$. This proves that $\mathfrak{D}_0 \subseteq \mathfrak{D}(T)$ and thus that $\mathfrak{D}_0 = \mathfrak{D}(T)$.

To see that E is unique let F be another spectral measure with the properties of E. It follows from (a) that $F([-m, m])x$ is in $\mathfrak{D}(T)$ and thus, from (b), that

$$\begin{aligned} (iI-T)F([-m, m])x &= \lim_{n\to\infty} \int_{-n}^{n} (i-\lambda)F(d\lambda)F([-m, m])x \\ &= \lim_{n\to\infty} \int_{-n}^{n} (i-\lambda)F(d\lambda \cap [-m, m])x \\ &= \int_{-m}^{m} (i-\lambda)F(d\lambda)x. \end{aligned}$$

Since, by Theorem X.1.1, we have

$$F([-m, m]) = \left[\int_{-m}^{m} (i-\lambda)F(d\lambda)\right]\left[\int_{-m}^{m} \frac{F(d\lambda)}{i-\lambda}\right]$$

it follows that

$$F([-m, m]) = (iI-T)F([-m, m])\int_{-m}^{m} \frac{F(d\lambda)}{i-\lambda},$$

$$= (iI-T)\int_{-m}^{m} \frac{F(d\lambda)}{i-\lambda},$$

so that

$$R(i; T)F([-m, m]) = \int_{-m}^{m} \frac{F(d\lambda)}{i-\lambda}.$$

By letting $m \to \infty$ we see that

$$R(i; T) = \int_{-\infty}^{\infty} \frac{F(d\lambda)}{i-\lambda}.$$

Thus, by the change of measure principle,

$$R(i; T) = \int \lambda F(h^{-1}(d\lambda)).$$

Corollary X.2.7 shows that $F(h^{-1}(\delta)) = E_1(\delta)$ for every Borel set δ. Thus $F(\delta) = E_1(h(\delta)) = E(\delta)$ and E is unique. Q.E.D.

4 Definition. The unique spectral measure associated with a self adjoint operator T as in the preceding theorem is called the *resolution of the identity for T.*

It is clear that for bounded self adjoint operators this notion coincides with that of Definition X.2.5.

For every bounded Borel function f defined on the real axis, or on the spectrum of the self adjoint operator T, we may define the bounded normal operator $f(T)$ by the equation

$$f(T) = \int_{\sigma(T)} f(\lambda)E(d\lambda)$$

where E is the resolution of the identity for T. According to Theorem X.1.1 the map $f \to f(T)$ is a $*$-homomorphism of the algebra of

bounded Borel functions into an algebra of normal operators in Hilbert space and thus the above formula defines an operational calculus. Theorem 3 suggests how this operational calculus may be extended to a calculus for unbounded operators $f(T)$ associated with unbounded functions f on $\sigma(T)$. The formal definition is as follows.

5 Definition. Let E be the resolution of the identity for the self adjoint operator T and let f be a complex Borel function defined E-almost everywhere on the real axis. Then the operator $f(T)$ is defined by the equations

$$\mathfrak{D}(f(T)) = \{x \mid \lim_n f_n(T)x \text{ exists}\}$$

where

$$f_n(\lambda) = f(\lambda), \qquad |f(\lambda)| \leqq n; \qquad f_n(\lambda) = 0, \qquad |f(\lambda)| > n,$$

and

$$f(T)x = \lim_n f_n(T)x, \qquad x \in \mathfrak{D}(f(T)).$$

It is clear from Theorem 3 that if $f(\lambda) \equiv \lambda$ then $f(T) = T$ but it is not clear that if f is the polynomial $\alpha_0+\alpha_1\lambda+ \ldots +\alpha_n\lambda^n$ then $f(T)$ is the polynomial $\alpha_0 I+\alpha_1 T + \ldots +\alpha_n T^n$ as defined in Definition 1.1 or as in Definition VII.9.6. This is the case, as will be shown in Corollary 8 below, so that the symbol $f(T)$ for a polynomial f is unambiguously defined.

6 Theorem. *Let E be the resolution of the identity for the self adjoint operator T and let f be a complex Borel function defined E-almost everywhere on the real axis. Then $f(T)$ is a closed operator with dense domain. Moreover*

(a) $\mathfrak{D}(f(T)) = \left\{x \mid \int_{-\infty}^{\infty} |f(\lambda)|^2 (E(d\lambda)x, x) < \infty\right\}$,

(b) $(f(T)x, y) = \int_{-\infty}^{\infty} f(\lambda)(E(d\lambda)x, y), \quad x \in \mathfrak{D}(f(T)), \quad y \in \mathfrak{H}$,

(c) $|f(T)x|^2 = \int_{-\infty}^{\infty} |f(\lambda)|^2 (E(d\lambda)x, x), \quad x \in \mathfrak{D}(f(T))$,

(d) $f(T)^* = \bar{f}(T)$,

(e) $R(\alpha; T) = \int_{-\infty}^{\infty} \frac{E(d\lambda)}{\alpha-\lambda}, \quad \alpha \in \rho(T)$.

Proof. It is convenient to use the notation $f_n(T)$ introduced in Definition 5 and to let $e_n = \{\lambda \mid |f(\lambda)| \leq n\}$. Then, for x in $\mathfrak{D}(f(T))$, we have from Corollary X.2.9(iv),

$$|f(T)x|^2 = \lim_n |f_n(T)x|^2 = \lim_n \int_{e_n} |f(\lambda)|^2 (E(d\lambda)x, x)$$
$$= \int_{-\infty}^{\infty} |f(\lambda)|^2 (E(d\lambda)x, x).$$

This proves (c) and also shows that the domain $\mathfrak{D}(f(T))$ is contained in the manifold $\{x \mid \int |f(\lambda)|^2 (E(d\lambda)x, x) < \infty\}$. On the other hand if x is in this manifold and if $m > n$, we have

$$|f_m(T)x - f_n(T)x|^2 = \int_{e_m - e_n} |f(\lambda)|^2 (E(d\lambda)x, x) \to 0$$

and (a) is established.

It is clear that for each $n = 1, 2, \ldots$, we have $E(e_n)\mathfrak{H} \subseteq \mathfrak{D}(f(T))$ and thus, since $E(e_n)x \to x$ for every x in $\mathfrak{H}$, the domain $\mathfrak{D}(f(T))$ is dense in $\mathfrak{H}$. To see that $f(T)$ is closed let $\{x_n\} \subseteq \mathfrak{D}(f(T))$, $x_n \to x_0$, and $f(T)x_n \to y_0$. Then, for every integer m,

$$f_m(T)x_0 = \lim_{n\to\infty} f_m(T)x_n = \lim_{n\to\infty} E(e_m)f(T)x_n = E(e_m)y_0.$$

Thus

$$y_0 = \lim_m E(e_m)y_0 = \lim_m f_m(T)x_0 = f(T)x_0.$$

It follows that x_0 is in $\mathfrak{D}(f(T))$ and that $f(T)$ is closed.

To prove (b) let x be in $\mathfrak{D}(f(T))$, y in $\mathfrak{H}$, and let $\mu(e)$ be the total variation of the set function $(E(\cdot)x, y)$ on the set e. By the complex form of the Radon-Nikodým theorem (III.10.7) there is a Borel measurable function φ with $\mu(e) = \int_e \varphi(\lambda)(E(d\lambda)x, y)$ for every Borel set e. It follows from Theorem III.2.20 that $|\varphi(\lambda)| = 1$ for μ-almost all λ and thus we may and shall assume that $|\varphi(\lambda)| = 1$ for all λ. Let $f_1(\lambda) = |f(\lambda)|\varphi(\lambda)$ so that by part (a), we have $\mathfrak{D}(f_1(T)) = \mathfrak{D}(f(T))$. For x in $\mathfrak{D}(f_1(T))$ and y in $\mathfrak{H}$ it follows from Corollaries III.10.6 and III.6.17 that

$$(f_1(T)x, y) = \lim_{n\to\infty}\int_{-\infty}^{\infty} |f_n(\lambda)|\varphi(\lambda)(E(d\lambda)x, y)$$
$$= \lim_{n\to\infty}\int_{-\infty}^{\infty} |f_n(\lambda)|\mu(d\lambda)$$
$$= \int_{-\infty}^{\infty} |f(\lambda)|\mu(d\lambda).$$

This proves the existence of the integral in (b) from which it follows that

$$(f(T)x, y) = \lim_{n}\int_{-\infty}^{\infty} f_n(\lambda)(E(d\lambda)x, y)$$
$$= \lim_{n}\int_{e_n} f(\lambda)(E(d\lambda)x, y) = \int_{-\infty}^{\infty} f(\lambda)(E(d\lambda)x, y).$$

which completes the proof of (b).

To prove (d) let x, y be in $\mathfrak{D}(\bar{f}(T)) = \mathfrak{D}(f(T))$. Then

$$(\bar{f}(T)x, y) = \int_{-\infty}^{\infty}\bar{f}(\lambda)(E(d\lambda)x, y) = \overline{\int_{-\infty}^{\infty}f(\lambda)(E(d\lambda)y, x)} = (x, f(T)y).$$

Thus $\bar{f}(T) \subseteqq f(T)^*$ and to prove (d) it will suffice to show that $\mathfrak{D}(f(T)^*) \subseteqq \mathfrak{D}(\bar{f}(T))$. If y is in $\mathfrak{D}(f(T)^*)$ then, for any x in $\mathfrak{H}$ and any integer m,

$$(x, \bar{f}_m(T)y) = (f_m(T)x, y) = (f(T)E(e_m)x, y) = (x, E(e_m)f(T)^*y)$$

and so $\bar{f}_m(T)y = E(e_m)f(T)^*y \to f(T)^*y$ which shows that y is in $\mathfrak{D}(\bar{f}(T))$ and proves (d).

Finally, to prove (e) we observe from Corollary X.2.7 that the spectral measure E_n defined by the equation $E_n(e) = E(e_n e)$ is the resolution of the identity for the restriction of T to the Hilbert space $\mathfrak{H}_n = E(e_n)\mathfrak{H}$. Also, since $R(\alpha; T)(\alpha I - T)E(e_n) = E(e_n)$, it is seen that the restriction of the resolvent to $\mathfrak{H}_n$ is the resolvent of the restriction of T to $\mathfrak{H}_n$. Thus, by Corollary X.2.8,

$$R(\alpha; T)x = \lim_{m} R(\alpha; T)E(e_m)x$$
$$= \lim_{m}\int_{e_m}\frac{E(d\lambda)x}{\alpha-\lambda} = \int_{-\infty}^{\infty}\frac{E(d\lambda)x}{\alpha-\lambda}, \qquad \lambda \in \rho(T),$$

which completes the proof of the theorem. Q.E.D.

7 COROLLARY. *Let T be a self adjoint operator and let f, g be complex Borel functions defined E-almost everywhere on the real axis. Then, for every scalar α and every Borel set e of real numbers, the operators $f(T)$, $g(T)$ of Definition 5 have the properties*

(a) $(\alpha f)(T) = \alpha f(T)$;

(b) $(f+g)(T) \supseteq f(T)+g(T)$;

(c) $\mathfrak{D}(f(T)g(T)) = \mathfrak{D}((fg)(T)) \cap \mathfrak{D}(g(T))$, $(fg)(T) \supset f(T)g(T)$;

(d) $f(T)E(e) \supseteq E(e)f(T)$.

PROOF. Statements (a) and (b) follow directly from Definition 5 and Theorem 6(b). To prove (c) let x be in $\mathfrak{D}(g(T))$ and $g(T)x$ in $\mathfrak{D}(f(T))$. Since

$$\begin{aligned}\int_{-\infty}^{\infty} |f(\lambda)g(\lambda)|^2(E(d\lambda)x, x) &= \lim_{n\to\infty} \lim_{m\to\infty} \int_{-\infty}^{\infty} |f_n(\lambda)g_m(\lambda)|^2(E(d\lambda)x, x) \\ &= \lim_{n\to\infty} \lim_{m\to\infty} |f_n(T)g_m(T)x|^2 = \lim_{n\to\infty} |f_n(T)g(T)x|^2 = |f(T)g(T)x|^2 < \infty,\end{aligned}$$

we see that $\mathfrak{D}(f(T)g(T)) \subseteq \mathfrak{D}((fg)(T)) \cap \mathfrak{D}(g(T))$. On the other hand if x is in $\mathfrak{D}(g(T))$ and $\int_{-\infty}^{\infty} |f(\lambda)g(\lambda)|^2(E(d\lambda)x, x) < \infty$, then the argument above shows that

$$\lim_{n\to\infty} |f_n(T)g(T)x|^2 < \infty.$$

By Theorem 6(c)

$$|f_n(T)g(T)x|^2 = \int_{-\infty}^{\infty} |f_n(\lambda)|^2(E(d\lambda)g(T)x, g(T)x),$$

so that $\int_{-\infty}^{\infty} |f(\lambda)|^2(E(d\lambda)g(T)x, g(T)x) < \infty$. It follows from Theorem 6(a) that $g(T)x$ is in $\mathfrak{D}(f(T))$. Thus

$$\mathfrak{D}(f(T)g(T)) = \mathfrak{D}((fg)(T)) \cap \mathfrak{D}(g(T)).$$

By Theorem 6(b) the integral

$$((fg)(T)x, y) = \int_{-\infty}^{\infty} f(\lambda)g(\lambda)(E(d\lambda)x, y)$$

exists for each y in $\mathfrak{H}$. Consequently, by the dominated convergence theorem (III.6.16),

$$\begin{aligned}(f(T)g(T)x, y) &= \lim_{n\to\infty} \lim_{m\to\infty} \int_{-\infty}^{\infty} f_n(\lambda)g_m(\lambda)(E(d\lambda)x, y) \\ &= ((fg)(T)x, y), \qquad y \in \mathfrak{H}.\end{aligned}$$

Finally, to prove (d) we observe that if x is in $\mathfrak{D}(f(T))$, then $E(e)f(T)x = \lim_n E(e)f_n(T)x = \lim_n f_n(T)E(e)x$ which shows that $E(e)x$ is in $\mathfrak{D}(f(T))$ and that $f(T)E(e)x = E(e)f(T)x$. Q.E.D.

8 Corollary. *If T is a self adjoint operator and if f is the polynomial $f(\lambda) = \alpha_0 + \ldots + \alpha_m\lambda^m$ then the operator $f(T)$ of Definition 5 is the same as the operator $f(T)$ of Definition* VII.9.6. *This operator $f(T)$ is also the same as the operator $\alpha_0 I + \ldots \alpha_m T^m$ as defined in Definition* 1.1.

Proof It is clear that the sum $\alpha_0 I + \ldots + \alpha_n T^n$ defined in accordance with Definition 1.1 is the same as the operator $f(T)$ of Definition VII.9.6. Let $f_1(T)$ be this operator and let $f_2(T)$ be the operator corresponding to f in accordance with Definition 5. Corollary 7 shows that

$$[*] \qquad f_2(T) \supseteq f_1(T).$$

Let e be a bounded Borel set of reals so that, by Theorem 8, $E(e)\mathfrak{H} \subseteq \mathfrak{D}(T)$ and $TE(e) = \int_e \lambda E(d\lambda)$. Thus $TE(e)\mathfrak{H} \subseteq E(e)\mathfrak{H} \subseteq \mathfrak{D}(T)$ which shows that $E(e)\mathfrak{H} \subseteq \mathfrak{D}(T^2)$. Similarly it may be shown that $E(e)\mathfrak{H} \subseteq \mathfrak{D}(T^n)$ for every integer n from which we conclude that $E(e)\mathfrak{H} \subseteq \mathfrak{D}(f_1(T))$. Thus if $e_n = (\lambda \mid |f(\lambda)| \leqq n)$ it follows from [*] that $f_2(T)E(e_n) \supseteq f_1(T)E(e_n)$. But since $E(e_n)\mathfrak{H} \subseteq \mathfrak{D}(f_1(T))$ we have $f_2(T)E(e_n) = f_1(T)E(e_n)$. Now let x be in $\mathfrak{D}(f_2(T))$ so that $E(e_n)x \to x$ and

$$f_1(T)E(e_n)x = f_2(T)E(e_n)x \to f_2(T)x.$$

But, by Theorem VII.9.7, the operator $f_1(T)$ is closed and so x is in $\mathfrak{D}(f_1(T))$. This proves that $\mathfrak{D}(f_2(T)) \subseteq \mathfrak{D}(f_1(T))$ and in view of [*] that $f_1(T) = f_2(T)$. Q.E.D.

9 Theorem. *Let E be the resolution of the identity for the self adjoint operator T and let f be a complex Borel function defined E-almost everywhere on the real axis. Then*

(a) $|f(T)| = \underset{\lambda\in\sigma(T)}{E\text{-ess sup}}\ |f(\lambda)|$;

(b) $\sigma(f(T))$ *is the intersection of all sets $\overline{f(\delta)}$ where δ varies over the Borel subsets of $\sigma(T)$ with $E(\delta) = I$;*

(c) *if f is real then $f(T)$ is self adjoint and its resolution of the*

identity is given in terms of the resolution of the identity for T by the formula

$$E(\delta; f(T)) = E(f^{-1}(\delta)),$$

where δ is an arbitrary Borel set.

PROOF. Let $e_n = \{\lambda \mid |f(\lambda)| \leq n\}$ so that

$$E\text{-ess}\sup_{\lambda\in e_n} |f(\lambda)| \to E\text{-ess}\sup_{\lambda\in\sigma(T)} |f(\lambda)|.$$

Thus, using Corollary X.2.9, it is seen that

$$|f(T)| \geqq |f(T)E(e_n)| = E\text{-ess}\sup_{\lambda\in e_n} |f(\lambda)| \to E\text{-ess}\sup_{\lambda\in\sigma(T)} |f(\lambda)|,$$

and hence that $|f(T)| \geqq E\text{-ess}_{\lambda\in\sigma(T)}\sup |f(\lambda)|$.

Conversely, for x in $\mathfrak{D}(f(T))$, we have

$$\begin{aligned}|f(T)x| &= \lim_n |f(T)E(e_n)x| \\ &\leqq \lim_n \left(E\text{-ess}\sup_{\lambda\in e_n} |f(\lambda)|\right)|x| \leqq \left(E\text{-ess}\sup_{\lambda\in\sigma(T)} |f(\lambda)|\right)|x|,\end{aligned}$$

and thus $|f(T)| \leqq E\text{-ess}_{\lambda\in\sigma(T)}\sup |f(\lambda)|$. This establishes equation (a)

Let $g(\lambda) = (\alpha - f(\lambda))^{-1}$ where α is such that the set $f^{-1}(\alpha) = \{\lambda \mid f(\lambda) = \alpha\}$ has E-measure zero. By (a)

$$|g(T)| = E\text{-ess}\sup_{\lambda\in\sigma(T)} |\alpha - f(\lambda)|^{-1}$$

and thus $g(T)$ is bounded if and only if $(\alpha - f(\lambda))^{-1}$ is E-essentially bounded. Since $g(T)$ is closed and has a dense domain it is bounded if and only if it is everywhere defined (Lemma 1.2). Thus if g is E-essentially bounded, α is in $\rho(f(T))$. Conversely, if α is in $\rho(f(T))$, then

$$(\alpha I - f(T))E(f^{-1}(\alpha)) = \int_{f^{-1}(\alpha)} (\alpha - f(\lambda))E(d\lambda) = 0,$$

which shows that $E(f^{-1}(\alpha)) = 0$ and hence that g is E-almost everywhere defined. Since α is in $\rho(f(T))$ we have $|g(T)| < \infty$ and the above equation shows that g is E-essentially bounded. Thus α is in $\rho(f(T))$ if and only if g is E-essentially bounded. Statement (b) may now be proved just as the corresponding fact for bounded operators was proved in Corollary X.2.9.

To prove (c) we note that the self adjointness of $f(T)$ follows

from Theorem 6 (d). Now let $E_1(\delta) = E(f^{-1}(\delta))$. By the change of measure principle we have, using the notation f_n of Definition 5,

$$\text{(i)} \quad \int_{-n}^{n} \mu E_1(d\mu) = \int_{-\infty}^{\infty} \mu \chi_{[-n,n]}(\mu) E_1(d\mu)$$
$$= \int_{-\infty}^{\infty} f(\lambda) \chi_{[-n,n]}(f(\lambda)) E(d\lambda) = f_n(T).$$

Similarly

$$\int_{-n}^{n} \mu^2 E_1(d\mu) = \int_{-\infty}^{\infty} f(\lambda)^2 \chi_{[-n,n]}(f(\lambda)) E(d\lambda)$$

and so for each vector x

$$\text{(ii)} \quad \int_{-\infty}^{\infty} \mu^2 (E_1(d\mu)x, x) = \int_{-\infty}^{\infty} f(\lambda)^2 (E(d\lambda)x, x).$$

It follows from (ii) and Theorem 6 (a) that

$$\mathfrak{D}(f(T)) = \left\{ x \,\Big|\, \int_{-\infty}^{\infty} \mu^2 (E_1(d\mu)x, x) < \infty \right\}.$$

From (i) and Definition 5 we have, for x in $\mathfrak{D}(f(T))$,

$$f(T)x = \lim_n \int_{-n}^{n} \mu E_1(d\mu)x.$$

Thus it is seen from Theorem 3 that $E_1 = E(\cdot\,; f(T))$. Q.E.D.

Just as in the case of a bounded self adjoint operator the resolution of the identity may be calculated explicitly in terms of the resolvent of T.

10 Theorem. *If E is the resolution of the identity of the self adjoint operator T and if (a, b) is the open interval $a < \lambda < b$ then, in the strong operator topology,*

$$E((a, b)) = \lim_{\delta \to 0+} \lim_{\varepsilon \to 0+} \frac{1}{2\pi i} \int_{a+\delta}^{b-\delta} [R(\mu - \varepsilon i; T) - R(\mu + \varepsilon i; T)] d\mu.$$

Proof. The proof is nearly the same as the proof of the corresponding theorem (X.6.1) for bounded operators. Corollary X.2.9(v) should be used in place of Corollary X.2.8 (iii) and, instead of referring to IX.3.15, the fact that $g(T) = R(\alpha; T)$ if $g(\lambda) = (\alpha - \lambda)^{-1}$ should be inferred from Theorem 6. Q.E.D.

11 THEOREM. *If E is the resolution of the identity of the self adjoint operator T and if F is a continuous scalar function defined on the real line, then, in the strong operator topology,*

$$F(T)E((a, b)) = \lim_{\delta\to 0+}\lim_{\varepsilon\to 0+}\frac{1}{2\pi i}\int_{a+\delta}^{b-\delta} F(\mu)[R(\mu-\varepsilon i; T)-R(\mu+\varepsilon i; T)]d\mu,$$

for every finite open interval (a, b).

PROOF. The method of proof used in the preceding theorem may also be used to prove the present theorem if it is shown that the function

$$G(\delta, \varepsilon, \lambda) = \frac{1}{2\pi i}\int_{a+\delta}^{b-\delta} F(\mu)\left[\frac{1}{\mu-\varepsilon i-\lambda}-\frac{1}{\mu+\varepsilon i-\lambda}\right]d\mu = \frac{1}{\pi}\int_{a+\delta}^{b-\delta}\frac{F(\mu)\varepsilon d\mu}{(\mu-\lambda)^2+\varepsilon^2}$$

is bounded in λ uniformly for small positive δ and ε and

$$[*] \qquad \lim_{\delta\to 0+}\lim_{\varepsilon\to 0+} G(\delta, \varepsilon, \lambda) = F(\lambda)\chi_{(a,b)}(\lambda), \qquad -\infty < \lambda < \infty,$$

where $\chi_{(a,b)}$ is the characteristic function of (a, b). If M is a bound for $|F(\mu)|$ on the interval $a < \mu < b$ then

$$|G(\delta, \varepsilon, \lambda)| \leqq M\frac{1}{\pi}\left[\arctan\frac{b-\delta-\lambda}{\varepsilon}-\arctan\frac{a+\delta-\lambda}{\varepsilon}\right] \leqq M$$

and so G is uniformly bounded for small positive δ and ε. Let $0 < \delta < (b-a)/2$ and suppose first that λ is not in the interval (a, b). Then the integrand approaches zero uniformly as ε approaches zero and so the equation [*] holds on the complement of (a, b). Now let λ be a point in (a, b) and choose δ and ε so that $a+\delta < \lambda-\sqrt{\varepsilon} < \lambda+\sqrt{\varepsilon} < b-\delta$. Then

$$G(\delta, \varepsilon, \lambda) = \frac{1}{\pi}\left[\int_{a+\delta}^{\lambda-\sqrt{\varepsilon}}+\int_{\lambda-\sqrt{\varepsilon}}^{\lambda+\sqrt{\varepsilon}}+\int_{\lambda+\sqrt{\varepsilon}}^{b-\delta}\right]\frac{F(\mu)\varepsilon d\mu}{(\mu-\lambda)^2+\varepsilon^2}.$$

An elementary calculation shows that

$$\left|\frac{1}{\pi}\left[\int_{a+\delta}^{\lambda-\sqrt{\varepsilon}}+\int_{\lambda+\sqrt{\varepsilon}}^{b-\delta}\right]\frac{F(\mu)\varepsilon d\mu}{(\mu-\lambda)^2+\varepsilon^2}\right|$$
$$\leqq \frac{M}{\pi}\left[\arctan\frac{b-\delta-\lambda}{\varepsilon}-\arctan\frac{a+\delta-\lambda}{\varepsilon}-2\arctan\frac{1}{\sqrt{\varepsilon}}\right]$$

and hence that the sum of these two integrals approaches zero with ε. Now let $M(\varepsilon)$ be a bound for $|F(\mu)-F(\lambda)|$ on the interval $\lambda-\sqrt{\varepsilon}<\mu<\lambda+\sqrt{\varepsilon}$. Since F is continuous, $M(\varepsilon)$ approaches zero with ε and we have

$$\frac{1}{\pi}\int_{\lambda-\sqrt{\varepsilon}}^{\lambda+\sqrt{\varepsilon}}\frac{F(\mu)\varepsilon d\mu}{(\mu-\lambda)^2+\varepsilon^2}$$

$$=\frac{F(\lambda)}{\pi}\int_{\lambda-\sqrt{\varepsilon}}^{\lambda+\sqrt{\varepsilon}}\frac{\varepsilon d\mu}{(\mu-\lambda)^2+\varepsilon^2}+\frac{1}{\pi}\int_{\lambda-\sqrt{\varepsilon}}^{\lambda+\sqrt{\varepsilon}}\frac{(F(\mu)-F(\lambda))\varepsilon d\mu}{(\mu-\lambda)^2+\varepsilon^2}$$

and

$$\left|\frac{1}{\pi}\int_{\lambda-\sqrt{\varepsilon}}^{\lambda+\sqrt{\varepsilon}}\frac{F(\mu)\varepsilon d\mu}{(\mu-\lambda)^2+\varepsilon^2}-F(\lambda)\frac{2}{\pi}\arctan\frac{1}{\sqrt{\varepsilon}}\right|\leq M(\varepsilon)\frac{2}{\pi}\arctan\frac{1}{\sqrt{\varepsilon}}$$

which shows that

$$\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{\lambda-\sqrt{\varepsilon}}^{\lambda+\sqrt{\varepsilon}}\frac{F(\mu)\varepsilon d\mu}{(\mu-\lambda)^2+\varepsilon^2}=F(\lambda).$$

This completes the proof of equation [*]. Q.E.D.

We conclude the present section with the following interesting theorem which follows easily from the results of Section XI.6.

12 THEOREM. *Let T be an unbounded self adjoint operator in Hilbert space $\mathfrak{H}$. Suppose that $\sigma(T)$ is a countable set of points with no finite limit point, and that if $\{\lambda_n\}$ is an enumeration of $\sigma(T)$, each point being repeated a number of times equal to the dimension of $E(T, \lambda_n)\mathfrak{H}$, we have*

$$\sum_{n=1}^{\infty}\lambda_n^{-2}<\infty.$$

Then, if B is an arbitrary bounded operator, the set of all vectors x which satisfy an equation $(T+B-\mu I)^\nu x=0$ for some integer ν and some complex μ, is a set fundamental in Hilbert space.

PROOF. By Theorem 6 and Corollary 7, there exists an orthonormal basis $\{\varphi_n\}$ for Hilbert space such that $T\varphi_n=\lambda_n\varphi_n$. (To obtain the set $\{\varphi_n\}$, we have only to take together a family of orthonormal bases for each of the separate spaces $E(T,\lambda_m)\mathfrak{H}$). It follows from Definition XI.6.1 that $(T-\lambda I)^{-1}$ is of Hilbert-Schmidt class for every

$\lambda \in \rho(T)$. By Lemma 2.2, there exists a finite constant K such that $|R(\lambda; T)| < \frac{1}{2}$, $|BR(\lambda; T)| < \frac{1}{2}$ and $|R(\lambda; T)B| < \frac{1}{2}$ for $|\mathscr{I}\lambda| \geqq K$. Then, if $|\mathscr{I}\lambda| \geqq K$, it follows from Lemma VII.6.1 that $I - BR(\lambda; T)$ has a bounded everywhere defined inverse $A(\lambda)$ of norm at most 2. We have

$$(\lambda I - T - B)R(\lambda; T)A(\lambda) = I.$$

Thus

$$(\lambda I - T - B)R(\lambda; T)A(\lambda)(\lambda I - T - B)x = (\lambda I - T - B)x$$

for each $x \in \mathfrak{D}(T) = \mathfrak{D}(T+B)$. If we put

$$y = R(\lambda; T)A(\lambda)(\lambda I - T - B)x - x,$$

it follows that $(\lambda I - T - B)y = 0$, so that, multiplying on the left by $R(\lambda; T)$, $(I - R(\lambda; T)B)y = 0$. Since $|R(\lambda; T)B| < \frac{1}{2}$ for $|\mathscr{I}\lambda| \geqq K$, it follows that $y = 0$ if $|\mathscr{I}\lambda| \geqq K$. Thus, for $|\mathscr{I}\lambda| \geqq K$, we have

$$\begin{aligned} (\lambda I - T - B)R(\lambda; T)A(\lambda)x &= x, \qquad x \in \mathfrak{H} \\ R(\lambda; T)A(\lambda)(\lambda I - T - B)x &= x, \qquad x \in \mathfrak{D}(T+B), \end{aligned}$$

for $|\mathscr{I}\lambda| > K$. This shows that if $|\mathscr{I}\lambda| \geq K$, then $\lambda \notin \sigma(T+B)$, and $R(\lambda; T+B) = R(\lambda; T)A(\lambda)$. Consequently,

$$|R(\lambda; T+B)| \leq \tfrac{1}{2} \cdot 2 \qquad \text{for} \qquad |\mathscr{I}\lambda| \geq K.$$

and the present theorem follows immediately from Corollary XI.6.31. Q.E.D.

3. Spectral Representation of Unbounded Self Adjoint Transformations

The purpose of this section is to extend the notions of spectral representation and ordered representation as presented in Section X.5 to the case of an unbounded self adjoint transformation. A completely satisfactory extension will be given in Theorem 5 below. We shall then make a more detailed investigation of the important case in which the Hilbert space $\mathfrak{H}$ has the form $L_2(S, \Sigma, \nu)$, where ν is a σ-finite positive measure and shall show that, under certain conditions, the linear isometry U which determines the spectral representation may be given a useful form. More explicitly, there are

measurable kernels W_α such that, in the norm of $L_2(\mu_\alpha)$, we have

$$(Uf)_\alpha(\lambda) = \lim_{n\to\infty} \int_{S_n} f(s)\overline{W_\alpha(s, \lambda)}\nu(ds), \qquad f \in L_2(S, \Sigma, \nu).$$

where (S_n) is an increasing sequence of sets of finite ν-measure covering S. This particular form of the spectral representation theorem will have important applications in later chapters where ν will be Lebesgue measure on an open subset of Euclidean space and T will be a differential operator. In this latter case the functions $W_\alpha(\cdot, \lambda)$ have an interpretation as eigenfunctions of the differential operator T and important expansion theorems may be obtained.

In order to describe concisely and clearly the representation of an unbounded self adjoint operator, we collect here some of the notation and terminology which will be used throughout the section.

The symbol E will be used for the resolution of the identity for the self adjoint operator T in the Hilbert space $\mathfrak{H}$. For a vector a in $\mathfrak{H}$ *the symbol* $\mathfrak{H}_a$ *will be used to denote the subspace of* $\mathfrak{H}$ *consisting of all vectors of the form* $F(T)a$ *where* F *varies over all Borel measurable functions for which* a *is in* $\mathfrak{D}(F(T))$. The symbol μ_a will denote the regular measure which is defined on the family $\mathscr{B}$ of Borel sets in the plane by the equation

$$\mu_a(\delta) = (E(\delta)a, a), \qquad \delta \in \mathscr{B}.$$

The symbol $\sum_a \mathfrak{H}_a$ will, as usual, be used for the direct sum of the Hilbert spaces $\mathfrak{H}_a$ (cf. IV.4.18 and IV.4.19).

1 Lemma. *The space* $\mathfrak{H}_a$ *is a Hilbert space which is equivalent, under the mapping* $F(T)a \leftrightarrow F$, *to the space* $L_2(\mu_a)$.

Proof. It has been seen in Theorem 2.6(a), (c) that

$$\mathfrak{D}(F(T)) = \{a \mid \int |F(\lambda)|^2 \mu_a(d\lambda) < \infty\},$$

and

$$|F(T)a|^2 = \int |F(\lambda)|^2 \mu_a(d\lambda), \qquad a \in \mathfrak{D}(F(T)),$$

from which the lemma is apparent. Q.E.D.

2 Lemma. *There is a set* A *in* $\mathfrak{H}$ *for which* $\mathfrak{H} = \sum_{a\in A} \mathfrak{H}_a$.

PROOF. It is clear from Zorn's lemma that there is a maximal set A in $\mathfrak{H}$ for which the spaces $\mathfrak{H}_a$, $a \in A$, are orthogonal. Thus to prove the lemma it suffices to observe that no $x \neq 0$ is orthogonal to each of the spaces $\mathfrak{H}_a$. Indeed, if $x \neq 0$ is orthogonal to the space $\mathfrak{H}_a$ then, for a bounded Borel function F and a point y in $\mathfrak{H}_a$ we see, from Theorem 2.6(d), that $(F(T)x, y) = (x, \bar{F}(T)y) = 0$ so that $F(T)x$ is orthogonal to $\mathfrak{H}_a$. From this it follows that if x is orthogonal to each of the spaces $\mathfrak{H}_a$, then $\mathfrak{H}_x$ is orthogonal to each of the spaces $\mathfrak{H}_a$ and this contradicts the maximality of the family A and proves that $\mathfrak{H} = \sum \mathfrak{H}_a$. Q.E.D.

If x is in $\mathfrak{H}$, then by x_a we denote the component of x in the subspace $\mathfrak{H}_a$. Thus $x = \sum_{a \in A} x_a$.

3 LEMMA. *For every Borel function F we have $\mathfrak{D}(F(T)) = \{x | x_a \in \mathfrak{D}(F(T)), a \in A; \sum_a |F(T)x_a|^2 < \infty\}$, and $(F(T)x)_a = F(T)x_a$ for every x in $\mathfrak{D}(F(T))$ and a in A.*

PROOF. We first assert that

$$(1) \qquad (\mathfrak{D}(F(T)))_a = \mathfrak{H}_a \cap \mathfrak{D}(F(T)).$$

Clearly $\mathfrak{H}_a \cap \mathfrak{D}(F(T)) \subseteq (\mathfrak{D}(F(T)))_a \subset \mathfrak{H}_a$ and so to establish (1) it suffices to show that $(\mathfrak{D}(F(T)))_a \subseteq \mathfrak{D}(F(T))$. We note that since $|E(\delta)x|^2 = \sum_a |E(\delta)x_a|^2$ we have $(E(\delta)x_a, x_a) \leq (E(\delta)x, x)$ and thus

$$\int |F(\lambda)|^2 (E(d\lambda)x_a, x_a) \leq \int |F(\lambda)|^2 (E(d\lambda)x, x).$$

Hence it follows from Theorem 2.6(a) that $(\mathfrak{D}(F(T)))_a \subset \mathfrak{D}(F(T))$ and (1) is established.

Now let F_n be the truncated function of Definition 2.5 so that the operator $F_n(T)$ is bounded and thus

$$|F_n(T)x|^2 = \sum_a |F_n(T)x_a|^2, \qquad x \in \mathfrak{H}.$$

If x is in $\mathfrak{D}(F(T))$ then, by (1), x_a is also in $\mathfrak{D}(F(T))$ and we have

$$\lim_{n \to \infty} F_n(T)x = F(T)x, \qquad \lim_{n \to \infty} F_n(T)x_a = F(T)x_a.$$

Thus, for any finite set $\pi \subseteq A$, we have $\sum_{a \in \pi} |F(T)x_a|^2 \leq |F(T)x|^2$, $x \in \mathfrak{D}(T)$, which shows that $\sum_a |F(T)x_a|^2 < \infty$. It will next be shown that x is in $\mathfrak{D}(F(T))$ provided that x_a is in $\mathfrak{D}(F(T))$ and

$\sum_\alpha |F(T)x_\alpha|^2 < \infty$. Let $\{\alpha_k\} \subseteq A$ be such that $x_\alpha = 0$ if $\alpha \notin \{\alpha_k\}$ (cf. IV.4.10). Since x_α is in $\mathfrak{H}_\alpha$ there is a Borel function g such that a is in $\mathfrak{D}(g(T))$ and $x_\alpha = g(T)a$. Hence $F(T)x_\alpha = F(T)g(T)a$ and thus $F(T)x_\alpha$ is in $\mathfrak{H}_\alpha$ which shows that the terms of the sequence $\{F(T)x_{\alpha_k}\}$ are orthogonal. Since

$$\left|\sum_{k=1}^{n} F(T)x_{\alpha_k}\right|^2 \leq \sum_\alpha |F(T)x_\alpha|^2 < \infty,$$

the series $\sum_{k=1}^{\infty} F(T)x_{\alpha_k}$ converges. By Theorem 2.6 the operator $F(T)$ is closed and thus x is in $\mathfrak{D}(F(T))$ and

$$F(T)x = \sum_{k=1}^{\infty} F(T)x_{\alpha_k} = \sum_\alpha F(T)x_\alpha.$$

Since $F(T)x_\alpha$ is in $\mathfrak{H}_\alpha$ this shows that $(F(T)x)_\alpha = F(T)x_\alpha$. Q.E.D.

4 Definition. Let T be a self adjoint operator in a Hilbert space $\mathfrak{H}$ and let $\{\mu_\alpha\}$ be a family of finite positive measures defined on the Borel sets of the plane and vanishing on the complement of the spectrum of T. Let U be an isomorphism of $\mathfrak{H}$ onto all of $\sum_\alpha L_2(\mu_\alpha)$ which preserves inner products and let V be the self adjoint operator UTU^{-1} in $\sum_\alpha L_2(\mu_\alpha)$. The transformation U is a *spectral representation of $\mathfrak{H}$ onto $\sum_\alpha L_2(\mu_\alpha)$ relative to T* if the following conditions are satisfied:

(a) for every Borel function F defined on the spectrum of T we have

$$\mathfrak{D}(F(V)) = \{\xi \mid \xi \in \sum_\alpha L_2(\mu_\alpha), \quad \sum_\alpha \int_{\sigma(T)} |F(\lambda)\xi_\alpha(\lambda)|^2 \mu_\alpha(d\lambda) < \infty\};$$

(b) $$(F(V)\xi_\alpha)(\lambda) = F(\lambda)\xi_\alpha(\lambda), \qquad \xi \in \mathfrak{D}(f(V))$$

for μ_α-almost all λ.

Remark. It is clear that if T is self adjoint, then so is $V = UTU^{-1}$. Indeed, for $\xi = Ux$, $\eta = Uy$ where x, y are in $\mathfrak{D}(T)$, we have

$$\begin{aligned}(V\xi, \eta) &= (VUx, Uy) = (UTx, Uy) \\ &= (Tx, y) = (x, Ty) = (Ux, UTy) \\ &= (Ux, VUy) = (\xi, V\eta),\end{aligned}$$

which shows that V is symmetric and that $\mathfrak{D}(V^*) = \mathfrak{D}(V)$.

5 THEOREM. *Every Hilbert space admits a spectral representation relative to an arbitrary self adjoint operator defined in it.*

PROOF. The set A and the measures μ_α may be defined as in Lemmas 1 and 2. The map U is defined in the natural way, i.e., the component $(Ux)_\alpha$ of Ux is the function in $L_2(\mu_\alpha)$ corresponding to x_α under the isomorphism U_α of Lemma 1.

Since U and U^{-1} both have bounded everywhere defined inverses, the operator $\lambda I - V = U(\lambda I - T)U^{-1}$ has a bounded everywhere defined inverse if and only if $\lambda I - T$ does, and

$$R(\lambda; V) = UR(\lambda; T)U^{-1}, \qquad \lambda \in \rho(T) = \rho(V).$$

If C, E are the resolutions of the identity for V, T respectively, it follows from Theorem 2.10 that for an open interval δ we have $C(\delta) = UE(\delta)U^{-1}$. Thus, this equation holds for all Borel sets δ and therefore for any Borel function F and any vector $\xi = Ux$ we have

$$\int |F(\lambda)|^2 (C(d\lambda)\xi, \xi) = \int |F(\lambda)|^2 (E(d\lambda)x, x).$$

It follows from Theorem 2.6(a) that $\mathfrak{D}(F(V)) = U\mathfrak{D}(F(T))$ and the stated form for $\mathfrak{D}(F(V))$ follows from Lemma 3.

Also for a vector $\xi = Ux$ in $\mathfrak{D}(F(V))$ we have

$$(F(V)\xi, \xi) = \int F(\lambda)(C(d\lambda)\xi, \xi) = \int F(\lambda)(E(d\lambda)x, x) = (F(T)x, x)$$

which shows that $F(V) = UF(T)U^{-1}$. This equation together with the equation $Ux_\alpha = (Ux)_\alpha$ and the equation $(F(T)x)_\alpha = F(T)x_\alpha$, proved in Lemma 3, shows that

$$\begin{aligned}(F(V)\xi)_\alpha = (UF(T)U^{-1}\xi)_\alpha &= UF(T)x_\alpha \\ &= UF(T)\xi_\alpha(T)a = F(\cdot)\xi_\alpha(\cdot).\end{aligned}$$

Q.E.D.

The next result shows that any spectral representation of $\mathfrak{H}$ may be realized in the above fashion. Its statement uses the symbols μ_a and $\mathfrak{H}_a$ as defined in the paragraph preceding Lemma 1.

6 LEMMA. *Let U be a spectral representation of $\mathfrak{H}$ onto $\Sigma_\alpha L_2(v_\alpha)$ relative to a self adjoint operator T. Then to each α corresponds an a in $\mathfrak{H}$ such that $v_\alpha = \mu_a$, $\mathfrak{H}$ is a direct sum of subspaces $\mathfrak{H}_a$, and U maps $\mathfrak{H}_a$ onto $L_2(v_\alpha)$.*

PROOF. For each α let ξ^α be the element of $\Sigma_\alpha L_2(\nu_\alpha)$ which is defined by the equations

$$(\xi^\alpha)_\beta(\lambda) = 0, \qquad \beta \neq \alpha,$$
$$= 1, \qquad \beta = \alpha.$$

Let $a = U^{-1}\xi^\alpha$. For every Borel set δ we have

$$\nu_\alpha(\delta) = \int_{\sigma(T)} \chi_\delta(\lambda)\nu_\alpha(d\lambda) = (\chi_\delta\xi^\alpha, \xi^\alpha)$$
$$= (U(E(\delta)a), Ua) = (E(\delta)a, a).$$

The remaining statements follow at once from Lemmas 1 and 2. Q.E.D.

We now turn our attention to the problem of giving an analytical representation of the map U. This problem will be discussed under the assumption that T is a self adjoint operator in a Lebesgue space $L_2(S, \Sigma, \nu)$ which is restricted by the following condition, which is assumed to hold throughout the remainder of the present section.

7 HYPOTHESIS. Let T be a self adjoint operator in the Hilbert space $L_2(S, \Sigma, \nu)$, where (S, Σ, ν) is a positive measure space. Let E be the resolution of the identity for T. We assume that there exists an increasing sequence $\{S_n\}$ covering S, each element of which has finite measure, and that for bounded sets e the range of $E(e)$ contains only functions which are ν-essentially bounded on each of the sets S_n.

8 LEMMA. *Let T be a self adjoint operator in the Hilbert space $L_2(S, \Sigma, \nu)$ where (S, Σ, ν) is a positive measure space. Let every element in $\bigcap_{n=1}^{\infty} \mathfrak{D}(T^n)$ be ν-essentially bounded on each set in an increasing sequence of sets of finite measure which covers S. Then Hypothesis 7 is satisfied.*

PROOF. If e is a bounded Borel set it follows from Theorem 2.6(a) that

$$E(e)L_2(S, \Sigma, \nu) \subseteq \bigcap_{n=1}^{\infty} \mathfrak{D}(T^n). \qquad \text{Q.E.D.}$$

In the next chapter we shall study the case where T is a self adjoint extension of an ordinary differential operator. In this case S is a domain in Euclidean space and ν is Lebesgue measure. We will

see that every function in $\mathfrak{D}(T)$ is continuous and thus bounded on every compact subset of S and so Lemma 8 will show that Hypothesis 7 is satisfied for such differential operators.

9 LEMMA. *Under Hypothesis 7, there is, for each g in $L_2(S, \Sigma, \nu)$, a function W defined on the Cartesian product of S and the real number system R which is measurable with respect to the product of ν and the measure $\mu = (E(\cdot)g, g)$, and which has the property that for every bounded Borel set e of reals and every F in $L_2(\mu)$ we have*

$$\text{(i)} \qquad \nu\text{-}\operatorname*{ess\,sup}_{s\in S_n} \int_e |W(s, \lambda)|^2 \mu(d\lambda) < \infty,$$

$$\text{(ii)} \qquad (E(e)F(T)g)(s) = \int_e W(s, \lambda)F(\lambda)\mu(d\lambda).$$

PROOF. If F is in $L_2(\mu)$, then, by Theorem 2.6(a), g is in $\mathfrak{D}(F(T))$. Let $e_n = [-n, +n]$. Let F vanish outside e_n. Then the restriction of $F(T)g = E(e_n)F(T)g$ to S_n is in $L_\infty(S_n, \nu)$ by Hypothesis 7. Since F vanishes outside e_n it follows from Theorem 2.6(c) that the map $A_n : F \to F(T)g$ is bounded as a map of $L_2(e_n, \mu)$ into $L_2(S, \Sigma, \nu)$. Thus it is closed as a map of $L_2(e_n, \mu)$ into $L_\infty(S_n, \nu)$. Hence, by the closed graph theorem (II.2.4), A_n is a continuous map of $L_2(e_n, \mu)$ into $L_\infty(S_n, \nu)$. The B-space adjoint A_n^* of A_n maps $L_\infty^*(S_n, \nu) = L_1^{**}(S_n, \nu)$ into $L_2(e_n, \mu)$. Since $L_1(S_n, \nu)$ is isometrically imbedded in $L_1^{**}(S_n, \nu)$, we may consider the restriction B_n of A_n^* to $L_1(S_n, \nu)$, as a continuous linear mapping of $L_1(S_n, \nu)$ into $L_2(e_n, \mu)$. Since $L_2(e_n, \mu)$ is separable and reflexive, it follows from Theorem VI.8.10 and the remark immediately preceding it that there exists a ν-essentially unique, ν-measurable bounded function V_n defined on S_n with values in $L_2(e_n, \mu)$ such that

$$B_n f = \int_{S_n} f(s)V_n(s)\nu(ds), \qquad f \in L_1(S_n, \nu).$$

By Theorem III.11.17 it follows that there exists a $\nu \times \mu$-essentially unique, $\nu \times \mu$ measurable complex valued function W_n defined on $S_n \times e_n$ such that

$$(B_n f)(\lambda) = \int_{S_n} f(s)W_n(s, \lambda)\nu(ds), \qquad f \in L_1(S_n, \nu), \qquad \lambda \in e_n,$$

and such that

$$\nu\text{-ess}\sup_{s\in S_n}\int_{e_n}|W_n(s,\lambda)|^2\mu(d\lambda) = \nu\text{-ess}\sup_{s\in S_n}|V_n(s)|^2 < \infty.$$

Now suppose that f is in $L_1(S_{n+1}, \nu)$ and that $f(s) = 0$ for $s \notin S_n$; and also that F is in $L_2(e_{n+1}, \mu)$ with $F(\lambda) = 0$ for $\lambda \notin e_n$. Then

$$\begin{aligned}\int_{e_{n+1}}(B_{n+1}f)(\lambda)F(\lambda)\mu(d\lambda) &= \int_{S_{n+1}} f(s)(A_{n+1}F)(s)\nu(ds)\\ &= \int_{S_n} f(s)\big(F(T)g\big)(s)\nu(ds)\\ &= \int_{S_n} f(s)(A_n F)(s)\nu(ds)\\ &= \int_{e_n}(B_n f)(\lambda)F(\lambda)\mu(d\lambda).\end{aligned}$$

Thus, $(B_{n+1}f)(\lambda) = (B_n f)(\lambda)$ μ-almost everywhere on e_n. Consequently

$$(B_n f)(\lambda) = \int_{S_n} f(s)W_{n+1}(s,\lambda)\nu(ds), \qquad f \in L_1(S_n,\nu), \qquad \lambda \in e_n.$$

This fact, together with the uniqueness of the kernel W_n, shows that $W_n(s,\lambda) = W_{n+1}(s,\lambda)$ for $\nu \times \mu$-almost all $[s,\lambda]$ in $S_n \times e_n$. Thus, if we put $W(s,\lambda) = W_n(s,\lambda)$ for $[s,\lambda]$ in $S_n \times e_n$, we obtain a well-defined $\nu \times \mu$-measurable kernel defined on $S \times R = \bigcup_{n=1}^{\infty} S_n \times e_n$ with property (i). If F is in $L_2(\mu)$, and $G = F\chi_e$, where χ_e is the characteristic function of a Borel subset of e_n, we have for each f in $L_1(S_n, \nu)$

$$\begin{aligned}\int_{S_n}\big(E(e)F(T)g\big)(s)f(s)\nu(ds) &= \int_{S_n}\big(G(T)g\big)(s)f(s)\nu(ds)\\ &= \int_{S_n}(A_nG)(s)f(s)\nu(ds) = \int_{e_n}G(\lambda)(B_nf)(\lambda)\mu(d\lambda)\\ &= \int_{e_n}G(\lambda)\left\{\int_{S_n}f(s)W_n(s,\lambda)\nu(ds)\right\}\mu(d\lambda)\\ &= \int_{S_n}f(s)\left\{\int_e W(s,\lambda)F(\lambda)\mu(d\lambda)\right\}\nu(ds),\end{aligned}$$

by Fubini's theorem and (i). Thus

$$\big(E(e)F(T)g\big)(s) = \int_e W(s,\lambda)F(\lambda)\mu(d\lambda)$$

ν-almost everywhere in S_n, and since $\bigcup S_n = S$ this equality must hold ν-almost everywhere in S. Q.E.D.

10 Definition. Let W be a measurable function on the product

$S \times \Lambda$ of two measure spaces (S, Σ, ν) and (Λ, B, μ), and let h be a ν-measurable function on S. We say that the integral $\int_S h(s)W(s, \lambda)\nu(ds)$ *exists in the mean square sense in* L_1 (Λ, B, μ) if there is an increasing sequence $\{S_n\}$ of sets of finite ν-measure which covers S and is such that for each n, $h(\cdot)W(\cdot, \lambda)$ is ν-integrable on S_n for μ-almost all λ in Λ and the function F_n defined by the equation

$$F_n(\lambda) = \int_{S_n} h(s)W(s, \lambda)\nu(ds)$$

is in $L_1(\Lambda, B, \mu)$ and converges in $L_1(\Lambda, B, \mu)$ as $n \to \infty$. If $F_n \to F$ in $L_1(\Lambda, B, \mu)$ we write

$$F(\lambda) = \int_S h(s)W(s, \lambda)\nu(ds).$$

REMARK. In the applications to ordinary and partial differential equations to be made in Chapters XIII and XIV, S will be a domain in Euclidean n-space, Σ will be the field of Borel subsets of S, ν will be Lebesgue measure, and W will, for almost all λ, be square integrable over every compact subset of S. In this case the limit F of the preceding definition is independent of the sequence $\{S_n\}$ provided that the sets S_n are compact.

11 THEOREM. *Let (S, Σ, ν) be a positive measure space and let $\{S_n\}$ be an increasing sequence of sets of finite measure covering S. Let U be a spectral representation of $L_2(S, \Sigma, \nu)$ onto $\sum_{a \in A} L_2(\mu_a)$ relative to the self adjoint operator T in $L_2(S, \Sigma, \nu)$. Let E be the resolution of the identity for T and suppose that for each bounded Borel set e of real numbers the range of the projection $E(e)$ contains only functions which are ν-essentially bounded on each of the sets S_n. Then for each element a in A there is a function W_a defined on the Cartesian product of S with the real line and having the properties:*

(a) *W_a is measurable with respect to the product measure $\nu \times \mu_a$;*

(b) *for each bounded Borel set e on the real line we have*

$$\nu\text{-ess sup}_{s \in S_n} \int_e |W_a(s, \lambda)|^2 \mu_a(d\lambda) < \infty, \qquad n \geq 1;$$

(c) $$(Uf)_a(\lambda) = \int_S f(s)\overline{W_a(s, \lambda)}\nu(ds), \qquad f \in L_2(S, \Sigma, \nu),$$

the integral existing in the mean square sense in $L_2(\mu_a)$.

REMARK. The following proof shows that, for f in $L_2(S, \Sigma, \nu)$, the function $(Uf)_a$ is the limit in $L_2(\mu_a)$ of the sequence $\{\int_{S_n} f(s)\overline{W_a(s, \cdot)}\nu(ds)\}$. Hence this limit is independent of the choice of the sequence $\{S_n\}$.

PROOF. By Lemma 6 the space $\mathfrak{H} = L_2(S, \Sigma, \nu)$ is the direct sum $\sum_{a \in A} \mathfrak{H}_a$ where A is a subset of $\mathfrak{H}$ and

$$U\mathfrak{H}_a = L_2(\mu_a), \qquad \mu_a(e) = \big(E(e)a, a\big), \qquad a \in A,$$

for every Borel set e of real numbers.

For the kernels W_a we take those associated with the elements a in A as in Lemma 9 so that the statements (a) and (b) are satisfied. From (b) it follows that

$$\int_{S_n} \int_e |W_a(s, \lambda)|^2 \mu_a(d\lambda)\nu(ds) < \infty, \qquad n \geq 1,$$

for each bounded Borel set e of the real axis. Thus, if the function f in $\mathfrak{H}$ vanishes outside S_n, the quantity

$$(V_a f)(\lambda) = \int_{S_n} f(s)\overline{W_a(s, \lambda)}\nu(ds)$$

satisfies the inequalities

$$\int_e |(V_a f)(\lambda)|^2 \mu_a(d\lambda) \leq |f|^2 \int_e \left\{\int_{S_n} |W_a(s, \lambda)|^2 \nu(ds)\right\} \mu_a(d\lambda) < \infty,$$

by Fubini's theorem and the Schwarz inequality. It follows that the integral defining $V_a f$ exists μ_a-almost everywhere and $V_a f$ is in $L_2(e, \mu_a)$ for each bounded Borel set e of the real axis. Now let U_a be the isomorphism mapping $\mathfrak{H}_a$ onto $L_2(\mu_a)$ as given in Lemma 1. Thus the ath components f_a and $(Uf)_a$ of f and Uf, in their expansions determined by the direct sum decompositions $\sum \mathfrak{H}_a$ and $\sum L_2(\mu_a)$ respectively, satisfy the equation $(Uf)_a = U_a f_a$. Now if the function F in $L_2(\mu_a)$ vanishes outside the bounded Borel set e we have, by Fubini's theorem and Lemma 9,

$$\begin{aligned}
(V_a f, F) &= \int_e (V_a f)(\lambda)\overline{F(\lambda)}\mu_a(d\lambda) \\
&= \int_{S_n} f(s) \left\{\overline{\int_e W_a(s, \lambda)F(\lambda)\mu_a(d\lambda)}\right\} \nu(ds) \\
&= \int_{S_n} f(s)\overline{\big(E(e)F(T)a\big)(s)}\nu(ds) = (f, F(T)a) \\
&= (f, U_a^{-1}F) = (f_a, U_a^{-1}F) \\
&= (U_a f_a, F) = \big((Uf)_a, F\big).
\end{aligned}$$

This shows that $(V_\alpha f)(\lambda) = (Uf)_\alpha(\lambda)$ for μ_α almost all λ in each bounded Borel set e so that $(V_\alpha f)(\lambda) = (Uf)_\alpha(\lambda)$ for μ_α-almost all λ. If f is an arbitrary function in $L_2(S, \Sigma, \nu)$ and $f_n = f\chi_{S_n}$ we have $f_n \to f$, so that, by continuity,

$$\int_{S_n} f(s)\overline{W_\alpha(S, \cdot)}\nu(ds) = (Uf_n)_\alpha \to (Uf)_\alpha.$$

Thus $\int_S f(s)\overline{W_\alpha(s, \lambda)}\nu(ds)$ exists in the mean square sense and equals $(Uf)_\alpha(\lambda)$, proving (c). Q.E.D.

Using the notation of the preceding proof we let $F = (Uf)_\alpha$ so that, by Lemma 9,

$$\begin{aligned}\int_{-n}^{n} (Uf)_\alpha(\lambda)W_\alpha(\cdot, \lambda)\mu_\alpha(d\lambda) &= E([-n, n])F(T)\alpha \\ &\to F(T)\alpha = U_\alpha^{-1}F = f_\alpha.\end{aligned}$$

Thus the integral $\int_{-\infty}^{\infty} (Uf)_\alpha(\lambda)W_\alpha(s, \lambda)\mu_\alpha(d\lambda)$ exists in the mean square sense and equals $f_\alpha(s)$. Hence the expansion $f = \sum_{\alpha\in A} f_\alpha$ determined by the direct sum decomposition $\mathfrak{H} = \sum \mathfrak{H}_\alpha$ takes the form stated in the following corollary.

12 Corollary. *With the notation and hypothesis of the preceding theorem we have*

$$f(s) = \sum_{\alpha\in A} \int_{-\infty}^{\infty} (Uf)_\alpha(\lambda)W_\alpha(s, \lambda)\mu_\alpha(d\lambda), \qquad f \in L_2(S, \Sigma, \nu),$$

the integrals existing in the mean square sense in $L_2(S, \Sigma, \nu)$ and the series converging in the norm of $L_2(S, \Sigma, \nu)$.

13 Corollary. *The preceding theorem and corollary remain true if the hypothesis concerning the range of $E(e)$ is replaced by the assumption that every function f in $\bigcap_{n=1}^{\infty} \mathfrak{D}(T^n)$ is ν-essentially bounded on each of the sets S_n.*

Proof. This follows from Lemma 3. Q.E.D.

14 Corollary. *In addition to the hypothesis of Theorem 11 suppose that $L_2(S, \Sigma, \nu)$ is separable and that for $n \geqq 1$ the set $\mathfrak{D}^{(n)} = \{f | f \in \mathfrak{D}(T),\ f(s) = 0 = (Tf)(s),\ \nu\text{-almost everywhere on } S_n'\}$ is dense in $L_2(S_n, \Sigma, \nu)$. Let $T^{(n)}$ be the restriction of T to $\mathfrak{D}^{(n)}$ and let*

$$\begin{aligned} W_a^{(n)}(s, \lambda) &= W_a(s, \lambda), \qquad && s \in S_n, \\ &= 0, && s \in S_n'. \end{aligned}$$

Then, for μ_a-almost all λ, the vector $W_a^{(n)}(\cdot, \lambda)$ is in $\mathfrak{D}((T^{(n)})^*)$ and

$$(T^{(n)})^* W_a^{(n)}(\cdot, \lambda) = \bar{\lambda} W_a^{(n)}(\cdot, \lambda).$$

Proof. It follows from Lemma 9(i) and the Fubini theorem that for almost all λ the function $W_a^{(n)}(\cdot, \lambda)$ is in $L_2(S, \Sigma, \nu)$ for each $n \geqq 1$. Since $L_2(S, \Sigma, \nu)$ is separable, there is a countable set $\{[f_j, Tf_j]\}$ dense in the graph of $T^{(n)}$. Thus, by Theorem 11,

$$\begin{aligned} (Tf_j, W_a^{(n)}(\cdot, \lambda)) &= \int_S (Tf_j)(s)\overline{W_a(s, \lambda)}\nu(ds) \\ &= (U(Tf_j))_a(\lambda) = \lambda(Uf_j)_a(\lambda) \\ &= \lambda \int_S f_j(s)\overline{W_a(s, \lambda)}\nu(ds) = \lambda(f_j, W_a^{(n)}(\cdot, \lambda)) \end{aligned}$$

for every λ outside a μ_a-null set N_j. Hence for every λ not in the μ_a-null set $N = \bigcup_{j=1}^{\infty} N_j$ we have $(Tf_j, W_a^{(n)}(\cdot, \lambda)) = \lambda(f_j, W_a^{(n)}(\cdot, \lambda))$. Since the vectors $[f_j, Tf_j]$ are dense in the graph $\Gamma^{(n)}$ of $T^{(n)}$ we have

$$(T^{(n)}f, W_a^{(n)}(\cdot, \lambda)) = \lambda(f, W_a^{(n)}(\cdot, \lambda)), \qquad f \in \mathfrak{D}(T^{(n)}), \qquad \lambda \notin N.$$

The desired conclusion follows immediately since $\mathfrak{D}(T^{(n)})$ is dense in $L_2(S_n, \Sigma, \nu)$. Q.E.D.

We shall now describe the spectral multiplicity theory for unbounded self adjoint operators. Just as in the case of a bounded normal operator (cf. Theorem X.5.10) it will be seen that a separable Hilbert space has an ordered representation relative to a given unbounded self adjoint operator in it and that this representation is uniquely determined up to an equivalence. Also, just as in the case of a bounded normal operator (cf. Theorem X.5.12), an ordered representation determines the unitary equivalence class of the operator. In fact this multiplicity theory for unbounded self adjoint operators may be readily derived from the corresponding theory for bounded operators.

15 Definition. Let μ be a positive measure defined on the family $\mathscr{B}$ of Borel sets of the complex plane and let $\{e_n\}$ be a decreasing sequence of Borel sets whose first element e_1 is the entire plane. Let

$$\mu_n(e) = \mu(e \cap e_n), \qquad e \in \mathscr{B}, \qquad n = 1, 2, \ldots.$$

A spectral representation of a Hilbert space $\mathfrak{H}$ onto $\sum_{n=1}^{\infty} L_2(\mu_n)$ relative to a self adjoint operator T in $\mathfrak{H}$ is said to be an *ordered representation* of $\mathfrak{H}$ relative to T. The measure μ is called the *measure of the ordered representation*. The sets e_n will be called the *multiplicity sets of the ordered representation*. If $\mu(e_k) > 0$ and $\mu(e_{k+1}) = 0$ then the ordered representation is said to have *multiplicity k*. If $\mu(e_k) > 0$ for all k, the representation is said to have *infinite multiplicity*. Two ordered representations U and $\tilde{U}$ of $\mathfrak{H}$ relative to T and $\tilde{T}$ respectively, with measures μ and $\tilde{\mu}$, and multiplicity sets $\{e_n\}$ and $\{\tilde{e}_n\}$ will be called *equivalent* if $\mu \cong \tilde{\mu}$ and $\mu(e_n \Delta \tilde{e}_n) = 0 = \tilde{\mu}(e_n \Delta \tilde{e}_n)$ for $n = 1, 2, \ldots$.

16 THEOREM. *A separable Hilbert space $\mathfrak{H}$ has an ordered representation U relative to a given self adjoint operator T in $\mathfrak{H}$ and every ordered representation of $\mathfrak{H}$ relative to T is equivalent to U. Moreover two self adjoint operators in $\mathfrak{H}$ are unitarily equivalent if and only if the corresponding ordered representations of $\mathfrak{H}$ relative to the operators are equivalent.*

PROOF. Let E, E_1 be the resolutions of the identity for T and its resolvent $R(i; T)$ respectively. We recall (cf. proof of Theorem 2.3) that

$$E(\delta) = E_1(h(\delta)), \qquad \delta \in \mathscr{B},$$

where $\mathscr{B}$ is the family of Borel sets in the plane and where $h(\lambda) = (i-\lambda)^{-1}$. It follows that

$$\begin{aligned} \overline{\text{sp}}\left\{f(R(i; T))x \mid f \in C(\sigma(R(i; T)))\right\} &= \overline{\text{sp}}\left\{E_1(\delta)x \mid \delta \in \mathscr{B}\right\} \\ = \overline{\text{sp}}\left\{E(\delta)x \mid \delta \in \mathscr{B}\right\} &= \overline{\text{sp}}\left\{F(T)x \mid F \in L_2((E(\cdot)x, x))\right\}. \end{aligned}$$

These equations show that a representation of $\mathfrak{H}$ relative to $R(i; T)$ induces a representation relative to T and conversely. Lemma 6 shows that $\mathfrak{H}$ is mapped onto spaces of the form $\sum L_2((E(\cdot)a_n, a_n))$ and $\sum L_2((E_1(\cdot)a_n, a_n))$ by representations of $\mathfrak{H}$ relative to T and $R(i; T)$ respectively. Since the identities

$$\begin{aligned} (E(e)a_n, a_n) &= (E(e \cap h^{-1}(e_n))a_n, a_n), \qquad e \in \mathscr{B}, \\ (E_1(e)a_n, a_n) &= (E_1(e \cap e_n)a_n, a_n), \qquad e \in \mathscr{B}, \end{aligned}$$

are equivalent, a representation of $\mathfrak{H}$ relative to T is ordered if and

only if the corresponding representation relative to $R(i; T)$ is ordered. Furthermore, the corresponding multiplicity sets are $h^{-1}(e_n)$ and e_n. Finally we observe that two unbounded self adjoint operators T_1 and T_2 are unitarily equivalent if and only if $R(i; T_1)$ and $R(i; T_2)$ are unitarily equivalent. Thus the present theorem follows from Theorems X.5.10 and X.5.12. Q.E.D.

The section will be concluded by relating the multiplicity theory for unbounded self adjoint operators to the representation given in Theorem 11. The following lemma is, in the case of scalar functions, a well known theorem of Lusin.

17 Lemma. *Let μ be a finite positive regular measure on the Borel sets of a topological space R. Then, for every B-space valued μ-measurable function f on R and every $\varepsilon > 0$ there is a Borel set σ in R with $\mu(\sigma) < \varepsilon$ and such that the restriction of f to the complement of σ is continuous.*

Proof. If the restrictions $f|\sigma$, $g|\delta$ are continuous then so is the restriction $(\alpha f + \beta g)|\sigma \cap \delta$ and thus the class of measurable functions having the required property is a linear manifold. Furthermore, the characteristic function χ_e of a Borel set e is in this manifold. This follows from the regularity of μ, for there is an open set σ containing e and a closed set δ contained in e with $\mu(\sigma - \delta) < \varepsilon$. Clearly the restriction of χ_e to the complement of $\sigma - \delta$ is continuous. Thus every μ-simple function has the desired property. Since $\mu(R) < \infty$ every μ-measurable function f is the limit in μ-measure of a sequence $\{f_n\}$ of μ-simple functions. In view of Corollary III.6.3 it may be supposed that the sequence $\{f_n\}$ converges μ-uniformly to f. This means that $f_n(\lambda)$ converges to $f(\lambda)$ uniformly on the complement of a set δ_1 with measure $\mu(\delta_1) < \varepsilon/2$. Now for $n \geq 2$ there are sets δ_n with $\sum_{n=2}^{\infty} \mu(\delta_n) < \varepsilon/2$ and such that the restriction of f_n to δ_n' is continuous. Thus, by Corollary I.7.7, the restriction of f to the set $\delta' = \bigcap_{n \geq 1} \delta_n'$ is continuous and the complement $\delta = \bigcup_{n \geq 1} \delta_n$ of this set has measure $\mu(\delta) < \varepsilon$. Q.E.D.

18 Lemma. *Let $f_1, \ldots, f_m$ be linearly independent functions defined on the union S of an increasing sequence $\{S_n\}$ of sets. Then, for all sufficiently large n, the set $f_1|S_n, \ldots, f_m|S_n$ of restrictions is linearly independent.*

PROOF. Suppose, on the contrary, that for a sequence $n_i \to \infty$ there are constants $\alpha_{i1}, \ldots, \alpha_{im}$ with

$$\text{(i)} \qquad \chi_{S_{n_i}}(\alpha_{i1}f_1+\ldots+\alpha_{im}f_m)=0, \qquad \sum_{j=1}^{m}|\alpha_{ij}|=1.$$

By choosing a subsequence which, for simplicity of notation, we suppose has already been done, it may be assumed that the limits

$$\alpha_j = \lim_{i\to\infty} \alpha_{ij},\ 1 \leq j \leq m,$$

exist. Thus from (i) we have

$$\alpha_1 f_1+\ldots+\alpha_m f_m = 0, \qquad \sum_{j=1}^{m}|\alpha_j| = 1,$$

a contradiction which proves the lemma. Q.E.D.

19 THEOREM. *In addition to the hypotheses of Theorem* 11 *suppose that the Hilbert space $L_2(S,\Sigma,\nu)$ is separable. Let U be an ordered representation of $L_2(S,\Sigma,\nu)$ relative to the self adjoint operator T and let μ be its measure and e_n, $1 \leq n \leq k$, its multiplicity sets. Let W_n, $n = 1, 2, \ldots$, be the corresponding kernels as given in Theorem* 11. *Then for every positive integer n which does not exceed the multiplicity of the ordered representation, the set $\{W_1(\cdot,\lambda), \ldots, W_n(\cdot,\lambda)\}$ in the space of ν-measurable functions is linearly independent μ-almost everywhere on e_n.*

PROOF. Let $\mu_n(e) = \mu(e\cap e_n) = (E(e\cap e_n)a_n, a_n)$. Since W_n is measurable with respect to the product of ν and μ_n, it follows from Lemma 9 (i) and the Fubini theorem that for μ_n-almost all λ the function $W_n(\cdot,\lambda)$ is ν-measurable. Thus for μ-almost all λ in e_n the function $W_n(\cdot,\lambda)$ is ν-measurable. It will be convenient to introduce the space $G(S,\Sigma,\nu)$ of all ν-measurable functions f on S which are square integrable on each of the sets S_n. The set $G(S,\Sigma,\nu)$ is a linear metric space under the norm

$$|f| = \sum_{n=1}^{\infty}\frac{1}{2^n}\frac{|(f|S_n)|_2}{1+|(f|S_n)|_2}$$

where

$$|(f|S_n)|_2 = \left(\int_{S_n}|f(s)|^2\nu(ds)\right)^{1/2}.$$

It is clear that $G(S, \Sigma, \nu)$ is a linear manifold in the space $M(S, \Sigma, \nu)$ of all ν-measurable functions on S and that a set of functions in $G(S, \Sigma, \nu)$ is linearly independent in $M(S, \Sigma, \nu)$ if and only if it is linearly independent in $G(S, \Sigma, \nu)$.

The linear independence of $W_1(\cdot, \lambda), \ldots, W_n(\cdot, \lambda)$ will be proved by induction on n. The case $n = 1$ is simply the assertion that $W_1(\cdot, \lambda) \neq 0$ μ-almost everywhere. If this statement is false, there is a bounded Borel set e with $\mu(e) = |E(e)a_1|^2 \neq 0$ and such that $W_1(\cdot, \lambda) = 0$ for all λ in e. Then, by Theorem 11(c), $(Uf)_{a_1}(\lambda) = 0$ for all λ in e and all f in $L_2(S, \Sigma, \nu)$. Since U is a spectral representation relative to T it follows from Definition 4 that $\big(E(e)f\big)_{a_1} = 0$ for all f in $L_2(S, \Sigma, \nu)$. In particular it is seen, by placing $f = a_1$, that $E(e)a_1 = \big(E(e)a_1\big)_{a_1} = 0$, which contradicts the fact that $\mu(e) \neq 0$ and proves the theorem in case $n = 1$.

Now suppose that the theorem is proved for $n < p$ where p is a positive integer at most equal to the multiplicity of the ordered representation. By the inductive hypothesis, the functions $W_1(\cdot, \lambda), \ldots, W_{p-1}(\cdot, \lambda)$ are linearly independent for μ-almost all λ in e_{p-1}, and hence for μ-almost all λ in e_p. We now require the following fact.

(i) The subset σ_0 of e_p consisting of all λ in e_p for which the functions $W_1(\cdot, \lambda), \ldots, W_p(\cdot, \lambda)$ are linearly dependent is μ-measurable (cf. III.2.10).

To prove statement (i) let σ_m be the set of all λ in e_p such that the restrictions $\tilde{W}_1(\cdot, \lambda), \ldots, \tilde{W}_p(\cdot, \lambda)$ of the functions $W_i(\cdot, \lambda)$ to S_m are linearly dependent. If λ is in $e_p - \sigma_0$, then by Lemma 18, λ is in $e_p - \sigma_m$ for some m. On the other hand, $\sigma_0 \subseteq \sigma_m$ for all m. Thus $\sigma_0 = \bigcap_{m=1}^{\infty} \sigma_m$, and it suffices to prove each of the sets σ_m is μ-measurable.

Let Z denote a countable dense subset of the linear manifold in $L_2(S_m, \Sigma, \nu)$ spanned by the functions $\tilde{W}_1(\cdot, \lambda), \ldots, \tilde{W}_p(\cdot, \lambda)$ with λ in e_p. There exist at most countably many linearly independent p-tuples $[z_1, \ldots, z_p]$ in Z, and for each such p-tuple we may choose continuous linear functionals $y_1^*, \ldots, y_p^*$ on $L_2(S_m, \Sigma, \nu)$ such that $\det\,(y_i^* z_j) = 1$. It follows readily that there is a sequence (x_i^*) of continuous linear functionals on $L_2(S_m, \Sigma, \nu)$ such that if $\lambda \in e_p$, and if the functions $\tilde{W}_1(\cdot, \lambda), \ldots, \tilde{W}_p(\cdot, \lambda)$ are linearly independent, then $\det\,\big(x_{n_i}^* \tilde{W}_j(\cdot, \lambda)\big) \neq 0$ for some p-tuple $[x_{n_1}^*, \ldots, x_{n_p}^*] \subseteq (x_i^*)$. On the

other hand, if λ is in e_p and the functions $\tilde{W}_1(\cdot, \lambda), \ldots, \tilde{W}_p(\cdot, \lambda)$, are linearly dependent, there are non-zero numbers $\beta_i(\lambda)$ such that

$$\sum_{i=1}^{p} \beta_i(\lambda)\tilde{W}_i(\cdot, \lambda) = 0,$$

so it is clear that every determinant det $\left(x_{n_i}^* \tilde{W}_j(\cdot, \lambda)\right)$ must vanish. In other words σ_m is precisely the set of those λ in e_p such that det $\left(x_{n_i}^* \tilde{W}_j(\cdot, \lambda)\right) = 0$ for every p-tuple $[x_{n_1}^*, \ldots, x_{n_p}^*] \subseteq (x_i^*)$. Thus σ_m is the intersection of a sequence of measurable sets, and it follows that σ_m is μ-measurable, completing the proof of statement (i).

To complete the proof of the theorem, suppose that the functions $W_1(\cdot, \lambda), \ldots, W_p(\cdot, \lambda)$ are not linearly independent for μ-almost all λ in e_p. It follows from statement (i) that there is a μ-measurable subset σ_0 of e_p, such that $\mu(\sigma_0) \neq 0$, and such that for each λ in σ_0 the set $\{W_1(\cdot, \lambda), \ldots, W_{p-1}(\cdot, \lambda)\}$ is linearly independent but the set $\{W_1(\cdot, \lambda), \ldots, W_p(\cdot, \lambda)\}$ is linearly dependent. By neglecting a set of measure zero we may assume that σ_0 is a Borel set. Thus there exist unique complex-valued functions α_i on σ_0 such that

$$\text{(ii)} \qquad W_p(\cdot, \lambda) = \sum_{i=1}^{p-1} \alpha_i(\lambda) W_i(\cdot, \lambda), \qquad \lambda \in \sigma_0.$$

We will show that the coefficient functions α_i are μ-measurable, and this fact will lead us to a contradiction.

Let $\varepsilon > 0$, and let $\varepsilon_1, \varepsilon_2, \ldots$ be a sequence of positive numbers such that $\sum_{j=1}^{\infty} \varepsilon_j < \varepsilon$. Using Lemma 17, we find for each j a Borel set $c_j \subseteq \sigma_0$ such that $\mu(\sigma_0 - c_j) < \varepsilon_j$, and such that the restrictions of the functions $W_1(\cdot, \lambda), \ldots, W_p(\cdot, \lambda)$ to S_j are continuous as functions on c_j to $L_2(S_j, \Sigma, v)$. Let $c = \bigcap_{j=1}^{\infty} c_j$. Then $\mu(\sigma_0 - c) < \varepsilon$, and it is evident from the definition of the norm in $G(S, \Sigma, v)$ that $W_1(\cdot, \lambda), \ldots, W_p(\cdot, \lambda)$ are continuous as functions on c to $G(S, \Sigma, v)$. Let λ_0 be fixed in c. Then, by Lemma 18, there is an m such that the restrictions $\tilde{W}_1(\cdot, \lambda_0), \ldots, \tilde{W}_{p-1}(\cdot, \lambda_0)$ of $W_1(\cdot, \lambda_0), \ldots, W_{p-1}(\cdot, \lambda_0)$ to S_m are linearly independent functions in $L_2(S_m, \Sigma, v)$. Since every finite dimensional subspace of a B-space is closed by Corollary IV.3.2, it follows from Corollary II.3.18 of the Hahn-Banach theorem that there are continuous linear functionals $y_1^*, \ldots, y_{p-1}^*$ in $L_2(S_m, \Sigma, v)$ such that $y_i^*(W_j(\cdot, \lambda_0)) = \delta_{ij}$, $i = 1, \ldots, p-1$. Define the continuous

linear functional F_i on $G(S, \Sigma, \nu)$ by placing $F_i(f) = y_i^*(f|S_m)$ for f in $G(S, \Sigma, \nu)$, $i = 1, \ldots, p-1$. Then $F_i(W_j(\cdot, \lambda_0)) = \delta_{ij}$, $i, j = 1, \ldots, p-1$. It follows by continuity that there exists a neighborhood $N(\lambda_0)$ of λ_0 such that the determinant $\det\big(F_i(W_j(\cdot, \lambda))\big)$ does not vanish for λ in $N(\lambda_0) \cap c$. Thus, the functions α_j, as solutions of the equations

$$\sum_{j=1}^{p-1} \alpha_j(\lambda) F_i\big(W_j(\cdot, \lambda)\big) = F_i\big(W_p(\cdot, \lambda)\big),$$

are continuous in λ for λ in $N(\lambda_0) \cap c$. Since $\bigcup_{\lambda_0 \in c} N(\lambda_0) \supseteq c$, the functions α_j are continuous on c. Since $\mu(\sigma_0 - c) < \varepsilon$, and ε is arbitrarily small, the functions α_j are μ-measurable on σ_0.

It is now easy to obtain a contradiction. Let σ_1 be a bounded Borel subset of σ_0 such that $\mu(\sigma_1) \neq 0$ and such that functions α_i are bounded on σ_1. Since $\sigma_1 \subseteq e_p$ it follows from the equation (ii), the formula (ii) of Lemma 9, and the fact that $\mu_i(e) = \mu_p(e)$ for $e \subseteq e_p$ and $i = 1, \ldots, p$, that

$$\begin{aligned} E(\sigma_1)a_p &= \int_{\sigma_1} W_p(\cdot, \lambda)\mu_p(d\lambda) \\ &= \int_{\sigma_1} \sum_{i=1}^{p-1} \alpha_i(\lambda) W_i(\cdot, \lambda)\mu_p(d\lambda) \\ &= \sum_{i=1}^{p-1} \int_{\sigma_1} \alpha_i(\lambda) W_i(\cdot, \lambda)\mu_i(d\lambda) \\ &= \sum_{i=1}^{p-1} \alpha_i(T)E(\sigma_1)a_i. \end{aligned}$$

Since the manifolds $\mathfrak{H}_{a_i}$ are orthogonal, we have $E(\sigma_1)a_p = 0$. However $\sigma_1 \subset e_p$, so

$$0 = (E(\sigma_1)a_p, a_p) = \big(E(\sigma_1)a_1, a_1\big) = \mu(\sigma_1) \neq 0,$$

a contradiction. Q.E.D.

4. The Extensions of a Symmetric Transformation

In order to apply the spectral theorem we must in many cases first find a self adjoint extension of a given symmetric operator. To find such an extension is neither trivial, nor even always possible.

In the theory of *bounded* operators, we have only to verify symmetry ($T^* \supseteq T$), for if T is everywhere defined and symmetric, then $T^* = T$. But if T is unbounded the situation is quite different. Consider, as an example, an operator which will be studied in greater detail in the next chapter: the differential operator $iD = i(d/dt)$ in the space $L_2(0, 1)$. How are we to choose its domain? A natural first guess is to choose as domain the collection $\mathfrak{D}_1$ of all functions with one continuous derivative. If f and g are any two such functions, we have

$$\begin{aligned}(iDf, g) &= \int_0^1 if'(t)\overline{g(t)}dt \\ &= \int_0^1 f(t)\overline{ig'(t)}dt + i\big(f(1)\overline{g(1)} - f(0)\overline{g(0)}\big) \\ &= (f, iDg) + i\big(f(1)\overline{g(1)} - f(0)\overline{g(0)}\big).\end{aligned}$$

The two additional "boundary" terms on the right ruin the symmetry. To restore it, it is natural to take for our second choice of domain for iD the collection $\mathfrak{D}_2$ of all functions f with one continuous derivative such that $f(0) = f(1) = 0$. This choice of domain will make the operator iD symmetric, but still not self adjoint. It is easy, for instance, to see that the set $\mathfrak{D}_1$ will be contained in the domain of the adjoint $(iD)^*$, even if we take $\mathfrak{D}_2$ to be the domain of iD. All these difficulties crop up even without taking into account any of the wide variety of possible "local pathologies" (non-differentiability, etc.) of the functions of L_2.

Faced with obstacles of this sort, it is not unreasonable to demand an abstract theory which can provide some unified guidance in the concrete cases to be discussed. Our general abstract problem will be to determine all symmetric, and thus all self adjoint, extensions of a given symmetric operator. In the present section, we discuss this problem by the method of Calkin, which is adapted to our later application to differential equations.

The basic idea of the theory is to start with a symmetric operator T and to find a self adjoint operator by enlarging its domain. We begin by narrowing down the field of search in the next lemma.

1 Lemma. *Let T_1 be an operator with dense domain.*

(a) *If the operator T_2 is an extension of the operator T_1 then T_1^* is an extension of T_2^*.*

(b) *If T_1 is symmetric then every symmetric extension T_2 of T_1, and, in particular, every self adjoint extension of T_1, satisfies $T_1 \subseteq T_2 \subseteq T_2^* \subseteq T_1^*$.*

Proof. If $T_1 \subseteq T_2$ and $y \in \mathfrak{D}(T_2^*)$, then $(x, T_2^* y) = (T_2 x, y) = (T_1 x, y)$ for any $x \in \mathfrak{D}(T_1)$. Hence $y \in \mathfrak{D}(T_1^*)$ and $(x, T_2^* y) = (x, T_1^* y)$. It follows that $T_1^* \supseteq T_2^*$. This proves (a) and (b) is an immediate corollary. Q.E.D.

It follows from Lemma 1(b) that to obtain the most general symmetric extension of a symmetric operator T with dense domain, we have only to restrict T^* to some suitably chosen subdomain of $\mathfrak{D}(T^*)$. Keeping this basic principle firmly in mind, we embark upon a systematic study of the linear subspaces of $\mathfrak{D}(T^*)$.

Throughout the rest of this section, T will denote an unbounded symmetric operator with dense domain.

Into the linear space $\mathfrak{D}(T^*)$ we introduce two convenient bilinear forms.

2 Definition. If x, y are in $\mathfrak{D}(T^*)$, we define

(a) $(x, y)^* = (x, y) + (T^*x, T^*y)$,

(b) $\{x, y\} = i\{(T^*x, y) - (x, T^*y)\}$.

The form $(x, y)^*$ is simply the inner product which $\mathfrak{D}(T^*)$ inherits from $\mathfrak{H} \oplus \mathfrak{H}$ if we identify $\mathfrak{D}(T^*)$ with $\Gamma(T^*)$ by the map $x \leftrightarrow [x, T^*x]$. The significance of the form $\{x, y\}$ is indicated by the following lemma; its special importance will emerge even more clearly in the next chapter.

3 Lemma. *A subspace $\mathfrak{D}$ of $\mathfrak{D}(T^*)$ containing $\mathfrak{D}(T)$ is the domain of a symmetric extension T_1 of T if and only if $\{x, y\} = 0$ for x, y in $\mathfrak{D}$. If this is the case, then T_1 is uniquely given by the formula $T_1 x = T^*x$, $x \in \mathfrak{D}$.*

Proof. If T_1 is a symmetric extension of T with domain $\mathfrak{D}$, then by Lemma 1 $T_1 \subseteq T^*$, and hence $\{x, y\} = i\{(T^*x, y) - (x, T^*y)\} = i\{(T_1 x, y) - (x, T_1 y)\} = 0$ for $x, y \in \mathfrak{D} = \mathfrak{D}(T_1)$. Clearly $T_1 x = T^* x$ for $x \in \mathfrak{D}$. Conversely, if we have $\{x, y\} = 0$ for $x \in \mathfrak{D}$ and define T_1 by $T_1 x = T^* x$ for $x \in \mathfrak{D}$, then $(T_1 x, y) - (x, T_1 y) = i\{x, y\} = 0$, so T_1 is symmetric. Q.E.D.

4 Definition. A subspace $\mathfrak{D}$ of $\mathfrak{D}(T^*)$ is called *symmetric* if

$\{x, y\} = 0$ for every x, y in $\mathfrak{D}$.

With this definition, Lemma 3 can be reformulated as follows.

8′ Lemma. *The most general symmetric extension T_1 of T is the restriction of T^* to a symmetric subspace $\mathfrak{D}$ of $\mathfrak{D}(T^*)$ which contains $\mathfrak{D}(T)$.*

To penetrate further into the problem we must make use of the topology introduced into $\mathfrak{D}(T^*)$ by the bilinear form $(x, y)^*$.

5 Lemma. (a) *With $(x, y)^*$ as inner product $\mathfrak{D}(T^*)$ becomes a complete Hilbert space.*

(b) *The bilinear form $\{x, y\}$ is continuous in the topology of $\mathfrak{D}(T^*)$ induced by the inner product $(x, y)^*$.*

(c) *A restriction T_1 of T^* is closed if and only if its domain $\mathfrak{D}(T_1)$ is a closed subspace of $\mathfrak{D}(T^*)$ in the topology induced by the inner product $(x, y)^*$.*

Proof. We have remarked that $(x, y)^*$ is simply the inner product which $\mathfrak{D}(T^*)$ inherits from $\mathfrak{H} \oplus \mathfrak{H}$ if we identify $\mathfrak{D}(T^*)$ with $\Gamma(T^*)$ by the mapping $x \leftrightarrow [x, T^*x]$. Since $\Gamma(T^*)$ is closed in $\mathfrak{H} \oplus \mathfrak{H}$ by Lemma I.6, (a) follows. Statement (c) follows in the same way. To prove (b), we have only to note that

$$
\begin{aligned}
|\{x, y\}| &\leq |(T^*x, y)| + |(x, T^*y)| \leq |T^*x| \cdot |y| + |x| \cdot |T^*y| \\
&= \sqrt{(T^*x, T^*x)}|y| + |x| \sqrt{(T^*y, T^*y)} \\
&\leq |y| \sqrt{(x, x) + (T^*x, T^*x)} + |x| \sqrt{(y, y) + (T^*y, T^*y)} \\
&\leq 2\{(y, y)^*(x, x)^*\}^{1/2}.
\end{aligned}
$$

Q.E.D.

Throughout the rest of this section, $\mathfrak{D}(T^*)$ will be understood to have the topology defined by the norm $|x|^* = \{(x, x)^*\}^{1/2}$ induced by the inner product $(x, y)^*$, unless the contrary is explicitly stated. When it is desired to emphasize this fact, a phrase like "the Hilbert space $\mathfrak{D}(T^*)$" will be used.

6 Lemma. (a) *The closure of a symmetric subspace of the Hilbert space $\mathfrak{D}(T^*)$ is a symmetric subspace.*

(b) *Any symmetric extension T_1 of T has a unique minimal closed extension, whose domain is simply the closure of $\mathfrak{D}(T_1)$ in the Hilbert space $\mathfrak{D}(T^*)$.*

Proof. Part (a) follows immediately from Lemma 5(b), and part (b) follows immediately from part (a) and Lemma 5(c). Q.E.D.

It follows from Lemma 6(b) that any symmetric operator with dense domain has a unique minimal closed symmetric extension. This fact leads us to make the following definition.

7 **Definition.** The minimal closed symmetric extension of a symmetric operator T with dense domain is called its *closure*, and written T.

8 **Lemma.** (a) *The closure T of T is the restriction of T^* to the closure of $\mathfrak{D}(T)$ in the Hilbert space $\mathfrak{D}(T^*)$.*

(b) *The operator T and its closure have the same closed extensions.*

(c) *The operator T and its closure have the same adjoint.*

(d) *An operator is closed if and only if T T.*

(e) *A self adjoint operator is closed.*

Proof. Part (d) is obvious from Definition 7. Part (e) follows immediately from Lemma 1.6.

To prove (c), we note that since $T \supseteq T$, $T^* \subseteq T^*$ by Lemma 1. On the other hand, if $x \in \mathfrak{D}(T^*)$ then (T^*x, y) (x, Ty) for all $[y, Ty]$ in $\Gamma(T)$, and hence for all $[y, Ty]$ in $\overline{\Gamma(T)}$. Since $\Gamma(T)$ $\overline{\Gamma(T)}$ by Lemma 6(b) and the remark following Definition 2(b), x is in $\mathfrak{D}(T^*)$.

Part (a) follows immediately from Lemma 6(b) and Lemma 3. Part (b) follows from this and from Lemma 5(c). Q.E.D

The significance of Lemma 8 is this: since we are searching for self adjoint extensions of T, it is enough, by 8(e), to search among the closed symmetric extensions of T. By Lemma 8(b), it is enough to search among the closed symmetric extensions of T. That is, we can confine ourselves to the consideration of closed operators, using Lemma 8 to replace any non-closed operator T_1 by its closure T_1, without losing anything essential.

Our next step is to make a further analysis of $\mathfrak{D}(T^*)$.

9 **Definition.** Let

$$\mathfrak{D}_+ \quad \{x \in \mathfrak{D}(T^*) | T^*x \quad ix\}; \qquad \mathfrak{D}_- \quad \{x \in \mathfrak{D}(T^*) | T^*x - \quad ix\}.$$

The spaces $\mathfrak{D}_+$ and $\mathfrak{D}_-$ are called the *positive* and *negative deficiency spaces of T* respectively. Their dimensions, (finite or infinite cardinal

numbers) denoted by n_+ and n_-, are called the *positive* and *negative deficiency indices of* T, respectively.

10 Lemma. (a) *$\mathfrak{D}(T)$, $\mathfrak{D}_+$, and $\mathfrak{D}_-$ are closed orthogonal subspaces of the Hilbert space $\mathfrak{D}(T^*)$.*

(b) $\mathfrak{D}(T^*) = \mathfrak{D}(T) \oplus \mathfrak{D}_+ \oplus \mathfrak{D}_-$.

Proof. By Lemma 8(a), $\mathfrak{D}(T)$ is closed. Suppose $\{x_n\}$ is a sequence of elements of $\mathfrak{D}_+$ converging to $x \in \mathfrak{D}(T^*)$, then $\{[x_n, T^*x_n]\} = \{[x_n, ix_n]\} \subset \Gamma(T^*)$ converges to $[x, ix] = [x, T^*x]$, since T^* is closed. Hence $T^*x = ix$, or $x \in \mathfrak{D}_+$. Hence $\mathfrak{D}_+$ is closed. Similarly, $\mathfrak{D}_-$ is closed. Since $\mathfrak{D}_+$ and $\mathfrak{D}_-$ are clearly linear subspaces of $\mathfrak{D}(T^*)$, it remains to show that the spaces $\mathfrak{D}(T)$, $\mathfrak{D}_+$, and $\mathfrak{D}_-$ are mutually orthogonal, and that their sum is $\mathfrak{D}(T^*)$.

Suppose $d \in \mathfrak{D}(T)$, $d_+ \in \mathfrak{D}_+$, and $d_- \in \mathfrak{D}_-$. We will show that $(d, d_+)^* = (d, d_-)^* = (d_-, d_+)^* = 0$. First, since $T^* \supseteq T$

$$\begin{aligned}(d, d_+)^* &= (d, d_+) + (T^*d, T^*d_+) = (d, d_+) + (Td, T^*d_+)\\ &= (d, d_+) + (Td, id_+) = (d, d_+) + (d, iT^*d_+)\\ &= (d, d_+) + (d, iT^*d_+) = (d, d_+) + (d, i^2d_+) = 0.\end{aligned}$$

Similarly, $(d, d_-)^* = 0$. Next,

$$\begin{aligned}(d_-, d_+)^* &= (d_-, d_+) + (T^*d_-, T^*d_+)\\ &= (d_-, d_+) + (-id_-, id_+) = 0.\end{aligned}$$

Hence the spaces $\mathfrak{D}(T)$, $\mathfrak{D}_+$, and $\mathfrak{D}_-$ are mutually orthogonal, and $\mathfrak{D}(T) \oplus \mathfrak{D}_+ \oplus \mathfrak{D}_-$ is contained in $\mathfrak{D}(T^*)$.

To show that $\mathfrak{D}(T^*) = \mathfrak{D}(T) \oplus \mathfrak{D}_+ \oplus \mathfrak{D}_-$, we will show that zero is the only vector orthogonal to the three subspaces involved. Suppose v is orthogonal to $\mathfrak{D}(T)$, $\mathfrak{D}_+$, and $\mathfrak{D}_-$. Then $0 = (d, v)^* = (d, v) + (T^*d, T^*v)$, for all d in $\mathfrak{D}(T)$. Hence $(d, v) = -(T^*d, T^*v)$. Since $(\cdot, v)$ is a continuous linear functional on the dense subset $\mathfrak{D}(T)$ of $\mathfrak{H}$, we see that T^*v is in $\mathfrak{D}(T^*)$ and $T^*(T^*v) = -v$. Hence $(I + T^*T^*)v = (I + iT^*)(I - iT^*)v = 0$. Hence $T^*[(I - iT^*)v] = i(I - iT^*)v$, or $(I - iT^*)v \in \mathfrak{D}_+$. Also, if $d_+ \in \mathfrak{D}_+$, then

$$\begin{aligned}0 &= (v, d_+)^* = (v, d_+) + (T^*v, T^*d_+) = (v, d_+) + (T^*v, id_+)\\ &= (v, d_+) - i(T^*v, d_+) = \big((I - iT^*)v, d_+\big).\end{aligned}$$

Since $(I-iT^*)v$ is in $\mathfrak{D}_+$, this implies that $(I-iT^*)v=0$. Hence $T^*v=-iv$, or $v \in \mathfrak{D}_-$. But $(\mathfrak{D}_-, v)^*=0$. Hence $v=0$. Therefore $\mathfrak{D}(T^*)=\mathfrak{D}(T) \oplus \mathfrak{D}_+ \oplus \mathfrak{D}_-$. Q.E.D.

11 LEMMA. *There is a one-to-one correspondence between closed symmetric subspaces $\mathfrak{S}$ of the Hilbert space $\mathfrak{D}(T^*)$ which contain $\mathfrak{D}(T)$ and closed symmetric subspaces $\mathfrak{S}^1$ of $\mathfrak{D}_+ \oplus \mathfrak{D}_-$, given by $\mathfrak{S}=\mathfrak{D}(T) \oplus \mathfrak{S}^1$.*

PROOF. We have $\{x, y\}=-i\{(Tx, y)-(x, T^*y)\}=0$ if $x \in \mathfrak{D}(T)$ and $y \in \mathfrak{D}(T^*)$; consequently $\{x, y\}=0$ if $x \in \mathfrak{D}(\bar{T})=\overline{\mathfrak{D}(T)}$ and $y \in \mathfrak{D}(T^*)$. Thus, if $\mathfrak{S}^1$ is symmetric, so is $\mathfrak{S}=\mathfrak{D}(T) \oplus \mathfrak{S}^1$. To see that $\mathfrak{S}$ is closed if $\mathfrak{S}^1$ is, we let $x_n=d_n+s_n \in \mathfrak{S}$, where $d_n \in \mathfrak{D}(T)$ and $s_n \in \mathfrak{S}^1$. If $x_n \rightarrow x$, then, since $\mathfrak{D}_+ \oplus \mathfrak{D}_-$ is orthogonal to $\mathfrak{D}(T)$, $(|x_n-x_m|^*)^2=(|d_n-d_m|^*)^2+(|s_n-s_m|^*)^2 \rightarrow 0$, so that both $\{d_n\}$ and $\{s_n\}$ are convergent sequences. If $d_n \rightarrow d$ and $s_n \rightarrow s$ then d is in $\mathfrak{D}(T)$ by Lemma 8(a), and $s \in \mathfrak{S}^1$ since $\mathfrak{S}^1$ is closed. Thus x is in $\mathfrak{D}(T) \oplus \mathfrak{S}^1=\mathfrak{S}$, and $\mathfrak{S}$ is closed.

Conversely, if $\mathfrak{S}$ is a closed symmetric subspace of $\mathfrak{D}(T^*)$ including $\mathfrak{D}(T)$, put $\mathfrak{S}^1=\mathfrak{S} \cap (\mathfrak{D}_+ \oplus \mathfrak{D}_-)$. Clearly, $\mathfrak{S}^1$ is closed and symmetric, and $\mathfrak{S} \supseteq \mathfrak{D}(T) \oplus \mathfrak{S}^1$. If $x \in \mathfrak{S}$, then, by Lemma 10, x can be written uniquely in the form $x=d+y$ with d in $\mathfrak{D}(T)$ and y in $\mathfrak{D}_+ \oplus \mathfrak{D}_-$. Since $\mathfrak{D}(T) \subseteq \mathfrak{S}$, $d \in \mathfrak{S}$; hence $y \in \mathfrak{S}$, from which we learn that y is in $\mathfrak{S} \cap (\mathfrak{D}_+ \oplus \mathfrak{D}_-)=\mathfrak{S}^1$, so that $\mathfrak{S} \subseteq \mathfrak{D}(T) \oplus \mathfrak{S}^1$. Q.E.D.

Lemma 11 shows that $\mathfrak{D}(T)$ plays a neutral role in the search for self adjoint extensions of T, and that the analysis need consider only the space $\mathfrak{D}_+ \oplus \mathfrak{D}_-$. The next theorem makes the decisive step. It should be observed that for elements d_+, e_+ in $\mathfrak{D}_+$ we have $(d_+, e_+)^*=2(d_+, e_+)$ and hence $|d_+|^*=\sqrt{2}|d_+|$. Similarly $|d_-|^*=\sqrt{2}|d_-|$ for d_- in $\mathfrak{D}_-$. These observations show that the transformation of part (a) in the following theorem is isometric in either norm.

12 THEOREM. *Let $\mathfrak{S}^1$ be a closed subspace of $\mathfrak{D}_+ \oplus \mathfrak{D}_-$, and $\mathfrak{S}=\mathfrak{D}(T) \oplus \mathfrak{S}^1$.*

(a) *The space $\mathfrak{S}$ is symmetric if and only if $\mathfrak{S}^1$ is the graph of an isometric transformation mapping a subspace of $\mathfrak{D}_+$ onto a subspace of $\mathfrak{D}_-$.*

(b) *The restriction of T^* to $\mathfrak{S}$ is self adjoint if and only if $\mathfrak{S}^1$ is the graph of an isometric transformation mapping $\mathfrak{D}_+$ onto all of $\mathfrak{D}_-$.*

PROOF. By Lemma 11, $\mathfrak{S}$ is symmetric if and only if $\mathfrak{S}^1$ is symmetric. Let two general elements s, t of $\mathfrak{S}^1$ be written $s = d_+ + d_-$, $t = e_+ + e_-$, where e_+, d_+ are in $\mathfrak{D}_+$, and e_-, d_- are in $\mathfrak{D}_-$. Then $\mathfrak{S}^1$ is symmetric if and only if

$$\begin{aligned}
\{s, t\} &= \quad i\{(T^*(d_+ + d_-), (e_+ + e_-)) - ((d_+ + d_-), T^*(e_+ + e_-))\} \\
&= -i\{((id_+ - id_-), (e_+ + e_-)) - ((d_+ + d_-), (ie_+ - ie_-))\} \\
&= -i\{2i(d_+, e_+) - 2i(d_-, e_-)\} \\
&= 2\{(d_+, e_+) - (d_-, e_-)\} = 0,
\end{aligned}$$

i.e., if and only if $(d_+, e_+) = (d_-, e_-)$. This clearly implies that $|d_+|^2 = |d_-|^2$ for all $s = d_+ + d_- \in \mathfrak{S}^1$; i.e., that $\mathfrak{S}^1$ is the graph of an isometric mapping of a subspace of $\mathfrak{D}_+$ onto a subspace of $\mathfrak{D}_-$.

On the other hand, if $|d_+|^2 = |d_-|^2$ for $s = d_+ + d_- \in \mathfrak{S}^1$, we have $|d_+ + e_+|^2 = |d_- + e_-|^2$ and $|d_+ - e_+|^2 = |d_- - e_-|^2$, so that

$$\begin{aligned}
\mathscr{R}(d_+, e_+) &= \frac{(|d_+ + e_+|^2 - |d_+ - e_+|^2)}{4} \\
&= \frac{(|d_- + e_-|^2 - |d_- - e_-|^2)}{4} \\
&= \mathscr{R}(d_-, e_-).
\end{aligned}$$

Applying the same argument to $it = ie_+ + ie_-$ we learn that $\mathscr{I}(d_+, e_+) = \mathscr{I}(d_-, e_-)$, so that $(d_+, e_+) = (d_-, e_-)$. It follows from the preceding paragraph that $\mathfrak{S}^1$ is symmetric. This proves (a).

To prove (b), we suppose first that $\mathfrak{S}^1$ is the graph of an isometric transformation A of a subspace of $\mathfrak{D}_+$ onto a subspace of $\mathfrak{D}_-$ and let T_1 be the restriction of T^* to $\mathfrak{S}$. If the domain $\mathfrak{D}(A)$ is a proper subset of $\mathfrak{D}_+$, then there is a non-zero vector y in $\mathfrak{D}_+$ such that $(\mathfrak{D}(A), y)^* = 0$. Now if x is in $\mathfrak{D}(T_1)$ then $x = x_1 + d_+ + d_-$ where $x_1 \in \mathfrak{D}(T)$, $d_+ \in \mathfrak{D}_+$ and $d_- \in \mathfrak{D}_-$. Hence $(x, y)^* = 0$, that is,

$$0 = (x, y) + (T^*x, T^*y) = (x, y) - i(T_1 x, y), \quad x \in \mathfrak{D}(T_1).$$

This proves that $(T_1 x, y) = -i(x, y)$ for all x in $\mathfrak{D}(T_1)$ which shows that y is in $\mathfrak{D}(T_1^*)$. Since y is not in $\mathfrak{D}(T_1)$ it is seen that if $\mathfrak{D}(A) \subset \mathfrak{D}_+$, then T_1 is not self adjoint. Similarly, if the range $\mathfrak{R}(A) \subset \mathfrak{D}_-$ there exists a non-zero vector z in $\mathfrak{D}_-$ such that $(\mathfrak{R}(A), z)^* = 0$; as before

this implies that $(x, z)^* = 0$ for all x in $\mathfrak{D}(T_1)$ from which it follows that z is in $\mathfrak{D}(T_1^*)$ and T_1 is not self adjoint.

To complete the proof of (b), suppose that A is an isometric transformation of $\mathfrak{D}_+$ onto all of $\mathfrak{D}_-$ and let T_1 be the restriction of T^* to the subspace $\mathfrak{S} = \mathfrak{D}(\bar{T}) \oplus \Gamma(A)$. By part (a), $\mathfrak{S}$ is symmetric and hence $T_1 \subseteq T_1^*$. It remains to show that $\mathfrak{D}(T_1^*) \subseteq \mathfrak{D}(T_1)$. Assuming that this is not the case, and applying Lemma 10(b) to the closed symmetric operator T_1, it is seen that there is a non-zero vector z in $\mathfrak{D}(T_1^*)$ such that either $T_1^* z = iz$ or $T_1^* z = -iz$. By Lemma 10(a), $(\mathfrak{D}(T_1), z)^* = 0$. Since $T_1^* \subseteq T^*$ it follows that z is either in $\mathfrak{D}_+$ or in $\mathfrak{D}_-$. If $z \in \mathfrak{D}_+$ then $z + Az$ is in $\mathfrak{D}(T_1)$ and hence

$$0 = (z + Az, z)^* = (z, z)^* + (Az, z)^*.$$

Since Az is in $\mathfrak{D}_-$ this shows that $0 = (z, z)^* = 2(z, z)$, whence $z = 0$. Similarly, if z is in $\mathfrak{D}_-$, then $A^{-1}(z) + z$ is in $\mathfrak{D}(T_1)$ and hence $0 = (A^{-1}(z) + z, z) = (A^{-1}(z), z)^* + (z, z)^*$. Since $A^{-1}(z)$ is in $\mathfrak{D}_+$, we again conclude that $z = 0$. Thus either case leads to a contradiction. Q.E.D.

13 Corollary. (a) *A symmetric operator T has self adjoint extensions if and only if its deficiency indices n_+ and n_- are equal.*

(b) *If $n_+ = n_- = 0$, the only self adjoint extension of T is its closure $\bar{T} = T^*$.*

Proof. This follows from Theorem IV.4.16 and Lemmas 8 and 10, Q.E.D.

With Theorem 12 and Corollary 13 we achieve our main aim and the rest of the section is devoted to reformulations and slight extensions of the results obtained up to this point.

14 Definition. The *adjoint* $\mathfrak{S}^*$ of a subspace $\mathfrak{S}$ of the Hilbert space $\mathfrak{D}(T^*)$ is the set of all x in $\mathfrak{D}(T^*)$ such that $\{x, \mathfrak{S}\} = 0$. A subspace $\mathfrak{S}$ of $\mathfrak{D}(T^*)$ is *self adjoint* if $\mathfrak{S} = \mathfrak{S}^*$.

15 Lemma. *If $T \subseteq T_1 \subseteq T^*$, then T_1^* is the restriction of T^* to the space $\mathfrak{D}(T_1)^*$, i.e., $\mathfrak{D}(T_1^*) = \mathfrak{D}(T_1)^*$.*

Proof. Suppose that y is in $\mathfrak{D}(T_1)^*$, then $\{x, y\} = 0$ for all x in $\mathfrak{D}(T_1)$. That is, $i\{(T^*x, y) - (x, T^*y)\} = 0$. Since $\mathfrak{D}(T_1)$ is dense in $\mathfrak{H}$, and $T^*x = T_1 x$, this implies that $y \in \mathfrak{D}(T_1^*)$. On the other

hand, if y is in $\mathfrak{D}(T_1^*)$, then $(T_1x, y) = (x, T_1^*y)$ for all x in $\mathfrak{D}(T_1)$. By Lemma 1(a), $T_1^* \subseteq T^*$. Hence $(T^*x, y) = (x, T^*y)$, so $\{x, y\} = 0$. This shows that $y \in \mathfrak{D}(T_1)^*$. Q.E.D.

16 LEMMA. (a) *A subspace $\mathfrak{S}$ of $\mathfrak{D}(T^*)$ is symmetric if and only if $\mathfrak{S} \subseteq \mathfrak{S}^*$.*

(b) *The restriction of T^* to a subspace $\mathfrak{S} \supseteq \mathfrak{D}(T)$ of $\mathfrak{D}(T^*)$ is a self adjoint extension of T if and only if $\mathfrak{S}$ is self adjoint.*

(c) *If $\mathfrak{S} \subseteq \mathfrak{D}(T^*)$ has the form $\mathfrak{S} = \mathfrak{D}(T) \oplus \mathfrak{S}^1$, where $\mathfrak{S}^1 \subseteq \mathfrak{D}_+ \oplus \mathfrak{D}_-$, then $\mathfrak{S}^* = (\mathfrak{S}^1)^*$.*

PROOF. Statement (a) follows immediately from Definition 14 and Definition 4. Statement (b) follows immediately from Definition 14 and Lemma 15. By Lemma 8(c) $\{x, y\} = 0$ for x in $\mathfrak{D}(T^*)$ and y in $\mathfrak{D}(T)$. It follows that $\{x, \mathfrak{D}(T) \oplus \mathfrak{S}^1\} = 0$ if and only if $\{x, \mathfrak{S}^1\} = 0$, proving (c). Q.E.D.

The next theorem describes a situation of common occurrence in which the symmetric operator T has equal deficiency indices, and thus has self adjoint extensions.

17 DEFINITION. A mapping U of $\mathfrak{H}$ into itself which satisfies $U(x+y) = Ux+Uy$, $U(\alpha x) = \bar{\alpha}Ux$, $(Ux, Uy) = (y, x)$, $x, y \in \mathfrak{H}$, and $U^2 = I$, is called a *conjugation*.

18 THEOREM. *Let T be a symmetric operator with dense domain which commutes with a conjugation U. Then T has equal deficiency indices.*

PROOF. Since $UT = TU$, $U\mathfrak{D}(T) \subseteq \mathfrak{D}(T)$. Since $U^2 = I$, it follows that $\mathfrak{D}(T) \subset U\mathfrak{D}(T)$, so that $\mathfrak{D}(T) = U\mathfrak{D}(T)$. By Lemma 1.6(d), $\mathfrak{D}_+ = \{(T+iI)\mathfrak{D}(T)\}^\perp$ and $\mathfrak{D}_- = \{(T-iI)\mathfrak{D}(T)\}^\perp$. If $x \in \mathfrak{D}_+$, then

$$0 = \big((T+iI)\mathfrak{D}(T), x\big) = \big(Ux, U(T+iI)\mathfrak{D}(T)\big) = \big(Ux, (T-iI)U\mathfrak{D}(T)\big)$$
$$= \big(Ux, (T-iI)\mathfrak{D}(T)\big).$$

Thus Ux is in $\mathfrak{D}_-$; so $U\mathfrak{D}_+ \subseteq \mathfrak{D}_-$. In the same way, $U\mathfrak{D}_- \subset \mathfrak{D}_+$. Using $U^2 = I$, we find $\mathfrak{D}_+ \subseteq U\mathfrak{D}_-$ and $\mathfrak{D}_- \subseteq U\mathfrak{D}_+$. Thus, $U\mathfrak{D}_- = \mathfrak{D}_+$, $U\mathfrak{D}_+ = \mathfrak{D}_-$. It follows from the properties of U that U maps any orthonormal basis for $\mathfrak{D}_+$ onto an orthonormal basis for $\mathfrak{D}_-$. Consequently, $\mathfrak{D}_+$ and $\mathfrak{D}_-$ have the same dimension. Q.E.D.

As an example of the application of Theorem 18, let $\mathfrak{H}$ be the space $L_2(0, \infty)$, and let q be a real continuous function defined on the interval $(0, \infty)$. Let T be the operator defined by:

(a) the domain $\mathfrak{D}(T)$ is the set of all functions which have at least two continuous derivatives and which vanish outside a compact subinterval of $(0, \infty)$.

(b) For f in $\mathfrak{D}(T)$ let $Tf = f'' + qf$.

Clearly T is an unbounded symmetric linear operator in $L_2(0, \infty)$. If for each f in $L_2(0, \infty)$ we let $(Uf)(t) = \overline{f(t)}$, the complex conjugate of $f(t)$, then U is a conjugation and $UT = TU$. Thus T has equal deficiency indices and consequently possesses self adjoint extensions.

The following result shows that the quantity

$$n_+ = \dim \{x | T^* x = ix\}$$

really pertains not to the complex number i but to the upper half plane, and provides at the same time a statement that will be useful in the theory of differential equations to be developed in the next chapter.

19 Theorem. *Let T be a symmetric operator, and for each complex number λ let $\mathfrak{M}_\lambda = \{x | T^* x = \lambda x\}$. If $\mathscr{I}(\lambda) > 0$, the dimension of $\mathfrak{M}_\lambda$ equals n_+, the positive deficiency index of T. Moreover, if n_+ is finite, there exists a family $\{\varphi_i(\lambda)\}$, $i = 1, \ldots, n_+$, of vector valued functions defined and analytic for $\mathscr{I}(\lambda) > 0$, with the property that for each λ, the vectors $\varphi_i(\lambda)$, $i = 1, \ldots, n_+$, form a basis for $\mathfrak{M}_\lambda$. Exactly similar results hold for λ in the lower half plane.*

Proof. Let T_1 be the restriction of T^* to the manifold $\mathfrak{D}(T_1) = \mathfrak{D}_- \oplus \mathfrak{D}(T)$. If $\lambda = \mu + i\nu$, where $\nu > 0$, and x is in $\mathfrak{D}(T_1)$ we have

$$\begin{aligned} |(T_1 - \lambda I)x|^2 &= |(T_1 - \mu I)x|^2 + \nu^2 |x|^2 \\ &\qquad - \big((T_1 - \mu I)x, i\nu x\big) - \big(i\nu x, (T_1 - \mu I)x\big) \\ &\geq \nu^2 |x|^2 + i\nu[(T_1 x, x) - (x, T_1 x)] \\ &= \nu^2 |x|^2 - \nu\{x, x\}. \end{aligned}$$

Writing $x = d + d_-$, $d \in \mathfrak{D}(T)$, $d_- \in \mathfrak{D}_-$, a straightforward calculation shows $\{x, x\} = \{d_-, d_-\} = -2|d_-|^2$. Thus $|(T_1 - \lambda I)x|^2 \geq \nu^2 |x|^2$, showing that $(T_1 - \lambda I)^{-1}$ is bounded and of bound at most $\nu^{-1} = |\mathscr{I}(\lambda)|^{-1}$. By Lemma 5(c) T_1 is closed. If $x_n \to x$ and $(T_1 - \lambda I)x_n \to y$, $T_1 x_n \to y + \lambda x$, so that $x \in \mathfrak{D}(T_1)$, and $T_1 x = y + \lambda x$, $(T_1 - \lambda I)x = y$.

Thus $(T_1-\lambda I)$ is closed, and hence by Lemma 1.2 $(T_1-\lambda I)^{-1}$ is closed. It follows from the boundedness of $(T_1-\lambda I)^{-1}$ and Lemma 1.2 that $\mathfrak{D}((T_1-\lambda I)^{-1})$ is closed.

We wish to show that $\mathfrak{D}((T_1-\lambda I)^{-1})$ is all of Hilbert space. Suppose that this is true for some given λ_0. Then $(T_1-\lambda_0 I)^{-1} = R(\lambda_0)$ is an everywhere defined, bounded operator of norm not more than $|\mathscr{I}(\lambda_0)|^{-1}$. Consequently, the series

$$[*] \qquad \sum_{n=0}^{\infty}(\lambda-\lambda_0)^n R(\lambda_0)^{n+1}$$

converges if $|\lambda-\lambda_0| < |\mathscr{I}(\lambda_0)|$. Since T_1 is closed, we have

$$\begin{aligned}(T_1-\lambda I) &\sum_{n=0}^{\infty} (\lambda-\lambda_0)^n R(\lambda_0)^{n+1} y \\ &= (T_1-\lambda_0 I-(\lambda-\lambda_0)I) \sum_{n=0}^{\infty} (\lambda-\lambda_0)^n R(\lambda_0)^{n+1} y = y\end{aligned}$$

for all y. Thus $(T_1-\lambda I)\mathfrak{D}(T_1) = \mathfrak{H}$ for $|\lambda-\lambda_0| < |\mathscr{I}(\lambda_0)|$. It follows by continuation that if $\mathfrak{D}((T_1-\lambda I)^{-1}) = \mathfrak{H}$ for one value λ_0, then this is true for every λ in the upper half plane. We shall show that $\mathfrak{D}((T_1-iI)^{-1}) = \mathfrak{H}$, thus proving that $\mathfrak{D}((T_1-\lambda I)^{-1}) = \mathfrak{H}$ for every λ in the upper half plane.

Since $\mathfrak{D}((T_1-iI)^{-1})$ is closed, it is sufficient to show that its orthocomplement consists only of the zero vector; or, using Lemma 1.6(d), that $(T_1^*+iI)x = 0$ implies $x = 0$. But $T_1 \supseteq T$, so $T_1^* \subseteq T^*$ by Lemma 1(a). Thus if $T_1^*x = -ix$, then x is in $\mathfrak{D}_- \subseteq \mathfrak{D}(T_1)$. Consequently,

$$\begin{aligned}0 = (x, (T_1^*+iI)x) &= ((T_1-iI)x, x) \\ &= ((T^*-iI)x, x) = -2i|x|^2,\end{aligned}$$

which shows that $x = 0$.

For convenience let $A(\lambda)$ denote the everywhere defined bounded operator $(T_1-\lambda I)^{-1}$ for $\mathscr{I}(\lambda) > 0$. Then $(T_1-\lambda I)A(\lambda) = (T^*-\lambda I)A(\lambda) = I$. Writing $K(\alpha, \beta) = (T_1-\alpha I)A(\beta)$, $\mathscr{I}(\alpha)$, $\mathscr{I}(\beta) > 0$, we have $K(\alpha, \alpha) = I$ and $K(\alpha, \beta)K(\beta, \gamma) = K(\alpha, \gamma)$. Thus $K(\alpha, \beta)$ is an everywhere defined bounded operator in $\mathfrak{H}$ with bounded inverse. Also $K(\alpha, \beta) = (T_1-\beta I)A(\beta)+(\beta-\alpha)A(\beta) = I+(\beta-\alpha)A(\beta)$. It follows that

$$\begin{aligned}(T^*-\lambda I)K(i,\lambda)u &= (T^*-\lambda I)u+(\lambda-i)(T^*-\lambda I)A(\lambda)u\\ &= (T^*-iI)u\end{aligned}$$

for each $u \in \mathfrak{D}(T^*)$, and thus $(T^*-iI)u = 0$ implies $(T^*-\lambda I)K(i,\lambda)u = 0$. Consequently $K(i,\lambda)$ defines a one-to-one map of $\mathfrak{M}_i$ into $\mathfrak{M}_\lambda$. By similar reasoning the inverse $K(\lambda, i)$ of $K(i,\lambda)$ defines a one-to-one map of $\mathfrak{M}_\lambda$ into $\mathfrak{M}_i$. Thus $K(i,\lambda)$ determines a continuous one-to-one map of $\mathfrak{M}_i$ onto $\mathfrak{M}_\lambda$ with bounded inverse, and it is now clear that $\mathfrak{M}_i$ and $\mathfrak{M}_\lambda$ have the same dimension.

From the form of the series [*] it is clear that $K(i,\lambda) = I+(\lambda-i)A(\lambda)$ is analytic in λ. If $\mathfrak{M}_i$ is finite dimensional, and $\{\varphi_j\}$, $j = 1, \ldots, n_+$, is a basis for $\mathfrak{M}_i$, then $\varphi_j(\lambda) = K(i,\lambda)\varphi_j$, $j = 1, \ldots, n_+$, defines a basis for $\mathfrak{M}_\lambda$. Q.E.D.

We turn finally to a detailed analysis of the case of frequent occurrence in which both deficiency indices n_+ and n_- of the operator T are finite.

We shall cast our abstract analysis in a form adapted to the theory of differential operators. Thus, abstract notions of boundary values, boundary conditions, etc., will be introduced. When in the next chapter the study of differential operators is taken up, concrete representations for the abstract notions will be obtained.

20 Definition. A *boundary value* for the operator T is a continuous linear functional on the Hilbert space $\mathfrak{D}(T^*)$ which vanishes on $\mathfrak{D}(T)$.

21 Lemma. *Let T be a symmetric operator whose deficiency indices n_+ and n_- are both finite. The space of boundary values for T is a Hilbert space of dimension $n_+ + n_-$. A set $A_1, \ldots, A_k$ of boundary values is linearly independent if and only if there exist elements $\varphi_1, \ldots, \varphi_k$ in $\mathfrak{D}(T^*)$ such that* $\det\left(A_i(\varphi_j)\right) \neq 0$; *or, if and only if for any set of complex numbers $\alpha_1, \ldots, \alpha_k$ there is an x in $\mathfrak{D}_+ \oplus \mathfrak{D}_-$ such that $A_i(x) = \alpha_i$, $i = 1, \ldots, k$; or, if and only if the matrix $\left(A_i(\psi_j)\right)$ is of rank k, where $\psi_1, \ldots, \psi_p$ is any basis for $\mathfrak{D}_+ \oplus \mathfrak{D}_-$.*

Proof. All these statements follow readily from Lemma 10, Lemma 8(a), and elementary facts concerning finite dimensional vector spaces. Q.E.D.

22 Definition. Let T be a symmetric operator whose deficiency

indices n_+ and n_- are both finite. A set of $n_+ + n_-$ linearly independent boundary values for T is called a *complete set of boundary values for* T.

23 LEMMA. *Let T be a symmetric operator whose deficiency indices n_+ and n_- are both finite. Let $A_1, \ldots, A_p$ be any complete set of boundary values for T. Then the bilinear form $\{x, y\}$ can be expressed uniquely in the form*

$$[*] \qquad \{x, y\} = \sum_{i,j=1}^{p} \alpha_{ij} A_i(x)\overline{A_j(y)}, \qquad x, y \in \mathfrak{D}(T^*),$$

the coefficients satisfying the equation $\alpha_{ji} = \overline{\alpha_{ij}}$.

PROOF. Using Lemma 21, let $\psi_1, \ldots, \psi_p$ be elements of $\mathfrak{D}_+ \oplus \mathfrak{D}_-$ such that $A_i(\psi_j) = \delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. It follows that the ψ_i are linearly independent and hence form a basis for $\mathfrak{D}_+ \oplus \mathfrak{D}_-$. Let $\alpha_{ij} = \{\psi_i, \psi_j\}$, then, since $\{x, y\} = \overline{\{y, x\}}$, it follows that $\alpha_{ij} = \overline{\alpha_{ji}}$. Clearly equation [*] is satisfied whenever both x and y are chosen from among the elements $\psi_1, \ldots, \psi_p$. Thus [*] holds for all x and y in $\mathfrak{D}_+ \oplus \mathfrak{D}_-$, since both sides of [*] are bilinear. But by Definition 20 and Definition 2(b) both sides of [*] vanish if either x or y is in $\mathfrak{D}(T)$. The desired representation now follows from Lemmas 10 and 8(a) and its uniqueness is clear. Q.E.D.

24 COROLLARY. *If $A_1, \ldots, A_p$ is a complete set of boundary values for T, and $\varphi_1, \ldots, \varphi_p$ is a set of elements of $\mathfrak{D}(T^*)$ such that* det $(A_i(\varphi_j)) \neq 0$, *then the α_{ij} of the previous lemma are uniquely determined by the set of equations*

$$\{\varphi_k, \varphi_l\} = \sum_{i,j=1}^{p} \alpha_{ij} A_i(\varphi_k)\overline{A_j(\varphi_l)}, \qquad 1 \leq k \leq l \leq p.$$

PROOF. Let (b_{ij}) be the matrix inverse to $(A_i(\varphi_j))$. Then an elementary calculation using formula [*] and the definition of the inverse matrix shows that

$$[**] \qquad \alpha_{ij} = \sum_{k,l=1}^{p} \{\varphi_k, \varphi_l\} b_{ki} \bar{b}_{lj}. \qquad \text{Q.E.D.}$$

25 DEFINITION. If B is a boundary value for T, then the equation $B(x) = 0$ is called a *boundary condition*. A set of boundary conditions $B_i(x) = 0$, $i = 1, \ldots, k$, is said to be *linearly independent* if the boundary values $B_1, \ldots, B_k$ are linearly independent. A set of

boundary conditions $C_j(x) = 0$, $j = 1, \ldots, m$, is said to be *stronger than* the set $B_i(x) = 0$, $i = 1, \ldots, k$, if the boundary values B_i are all linear combinations of the C_j. If each of two sets of boundary conditions is stronger than the other, then the sets are said to be *equivalent.* A set of boundary conditions $B_i(x) = 0$, $i = 1, \ldots, k$, is said to be *symmetric* if the equations $B_i(x) = B_i(y) = 0$, $i = 1, \ldots, k$, imply the equation $\{x, y\} = 0$.

26 Lemma. *Let T be an operator with finite deficiency indices. Every closed symmetric extension of T is the restriction of T^* to the subspace of $\mathfrak{D}(T^*)$ determined by a symmetric family of boundary conditions, $B_i(x) = 0$, $i = 1, \ldots, k$. Conversely, every such restriction T_1 of T^* is a closed symmetric extension of T.*

Proof. We shall prove the second statement first. As each B_i is a continuous linear functional on $\mathfrak{D}(T^*)$ vanishing on $\mathfrak{D}(T)$, it is clear from Lemma 5(c) that T_1 is a closed extension of T. Since the set of boundary conditions is symmetric, it follows from Definition 2(b) that $(T_1x, y) = (x, T_1y)$ for all x and y in $\mathfrak{D}(T_1)$, and hence T_1 is symmetric.

To prove the first assertion let T_1 be a closed symmetric extension of T, and let $\mathfrak{B}$ be the orthogonal complement of $\mathfrak{D}(T_1)$ in the Hilbert space $\mathfrak{D}(T^*)$. Then $\mathfrak{D}(T_1)$ is a closed subspace of $\mathfrak{D}(T^*)$ and $\mathfrak{B}$ is finite dimensional (cf. Lemmas 5(c) and 10). If $v_1, \ldots, v_k$ is a basis for $\mathfrak{B}$, we define the functions $B_i(x) = (x, v_i)^*$, for x in $\mathfrak{D}(T^*)$. Since $B_i(x) = 0$ for x in $\mathfrak{D}(T_1) \supseteq \mathfrak{D}(T)$, it is clear that the condition $B_i(x) = 0$ is a boundary condition. Moreover, since $\mathfrak{D}(T_1)$ is the orthocomplement of its orthocomplement, it follows that the family $B_i(x) = 0$, $i = 1, \ldots, k$, of boundary conditions determines the subspace $\mathfrak{D}(T_1)$ of $\mathfrak{D}(T^*)$. It follows from Lemma 8′ that $\mathfrak{D}(T_1)$ is a symmetric subspace of $\mathfrak{D}(T^*)$, and hence the set of boundary conditions $B_i(x) = 0$ is symmetric. Q.E.D.

27 Definition. Let T be a symmetric operator with finite deficiency indices. Let $A_1, \ldots, A_p$ be a complete set of boundary values for T, and let $\{x, y\} = \sum_{i,j=1}^{p} \alpha_{ij} A_i(x)\overline{A_j(y)}$ as in Lemma 23. Let

$$[*] \qquad B_j = \sum_{i=1}^{p} \beta_{ij} A_i \qquad j = 1, \ldots, s$$

be an arbitrary set of boundary values for T, and let $\xi_i^{(m)}$, $m = 1, \ldots, t$ be a basis for the set of solutions of the linear system $\sum_{i=1}^{p} \beta_{ij} \xi_i = 0$, $j = 1, \ldots, s$. Then the set of boundary conditions

$$\sum_{i,j=1}^{p} \alpha_{ij} \overline{\xi_j^{(m)}} A_i(y) = 0, \qquad m = 1, \ldots, t,$$

is called an *adjoint set of boundary conditions* to the conditions [*].

REMARK. It is important to observe that there are as many adjoint sets of boundary conditions as there are bases for the solutions of the equations $\sum_{i=1}^{p} \beta_{ij} \xi_i = 0$, $j = 1, \ldots, s$. However, any pair of these various adjoint sets of boundary conditions is clearly equivalent.

28 THEOREM. *Let T be a symmetric operator with finite deficiency indices, and let T_1 be the restriction of T^* to the subspace of $\mathfrak{D}(T^*)$ determined by a finite set of boundary conditions $B_j(x) = 0$, $j = 1, \ldots, s$. Then T_1^* is the restriction of T^* to the subspace of $\mathfrak{D}(T^*)$ determined by an adjoint set of boundary conditions.*

PROOF. Let $B_j = \sum_{i=1}^{p} \beta_{ij} A_i$, $j = 1, \ldots, s$, and let T_2 be the restriction of T^* to the subspace of $\mathfrak{D}(T^*)$ determined by an adjoint set for the boundary conditions $B_j(x) = 0$, $j = 1, \ldots, s$. If x is in $\mathfrak{D}(T_1)$, then x satisfies the equations $B_j(x) = \sum_{i=1}^{p} \beta_{ij} A_i(x) = 0$, $j = 1, \ldots, s$, and thus $A_i(x)$ is a linear combination of $\xi_i^{(1)}, \ldots, \xi_i^{(t)}$ for $i = 1, \ldots, p$. Hence if x is in $\mathfrak{D}(T_1)$ and y is in $\mathfrak{D}(T_2)$, we conclude, from Lemma 23, that $\{y, x\} = \sum_{i,j=1}^{p} \alpha_{ij} A_i(y) \overline{A_j(x)} = 0$. Consequently,

$$(T_2 y, x) - (y, T_1 x) = i\{y, x\} = 0,$$

and so $(T_1 x, y) = (x, T_2 y)$ for all x in $\mathfrak{D}(T_1)$ and all y in $\mathfrak{D}(T_2)$, showing that $T_2 \subseteq T_1^*$.

Suppose there is an element y in $\mathfrak{D}(T^*) \subseteq \mathfrak{D}(T_1^*)$ but not in $\mathfrak{D}(T_2)$. Then by Definition 27 there exists a solution $[\xi_1, \ldots, \xi_p]$ of the equations $\sum_{i=1}^{p} \beta_{ij} \xi_i = 0$, $j = 1, \ldots, s$, such that $\sum_{i,j=1}^{p} \alpha_{ij} \bar{\xi}_j A_i(y) \neq 0$. By Lemma 21 there exist elements y_j in $\mathfrak{D}(T^*)$ such that $A_i(y_j) = \delta_{ij}$, $i, j = 1, \ldots, p$. Putting $x_0 = \sum_{k=1}^{p} \xi_k y_k$, we have $x_0 \in \mathfrak{D}(T^*)$ and $A_i(x_0) = \xi_i$, $i = 1, \ldots, p$. Hence $B_j(x_0) = 0$, $j = 1, \ldots, s$ and so x_0 is in $\mathfrak{D}(T_1)$. However,

$$-i[(T^*y, x_0)-(y, T_1x_0)] = \{y, x_0\} = \sum_{i,j=1}^{p} \alpha_{ij} A_i(y)\xi_j \neq 0,$$

which is impossible. Q.E.D.

29 LEMMA. *Let T be a symmetric operator with finite deficiency indices whose sum is p. Let $A_1, \ldots, A_p$ be a complete set of boundary values for T, and let $\sum_{i,j=1}^{p}\alpha_{ij}A_i\bar{A}_j$ be the bilinear form of Lemma 23. A set of boundary conditions $\sum_{i=1}^{p}\beta_{ij}A_i(x) = 0$, $j = 1, \ldots, s$, is symmetric if and only if $\sum_{i=1}^{p}\beta_{ij}\xi_i = 0$, and $\sum_{i=1}^{p}\beta_{ij}\eta_i = 0$, $j = 1, \ldots, s$, together imply $\sum_{i,j=1}^{p}\alpha_{ij}\xi_i\bar{\eta}_j = 0$.*

PROOF. It is clear under the hypotheses that if, whenever x and y satisfy the equations $\sum_{i=1}^{p}\beta_{ij}A_i(x) = 0$ and $\sum_{i=1}^{p}\beta_{ij}A_i(y) = 0$, $j = 1, \ldots, s$, they also satisfy $\{x, y\} = 0$, then the set of boundary conditions is symmetric. Conversely, suppose there exists a pair $[\xi_1, \ldots, \xi_p]$ and $[\eta_1, \ldots, \eta_p]$ such that $\sum_{i=1}^{p}\beta_{ij}\xi_i = \sum_{i=1}^{p}\beta_{ij}\eta_i = 0$, $j = 1, \ldots, s$ while $\sum_{i,j=1}^{p}\beta_{ij}\xi_i\bar{\eta}_j \neq 0$. By Lemma 21, there exist elements u, v in $\mathfrak{D}(T^*)$ such that $A_i(u) = \xi_i$, $A_i(v) = \eta_i$, $i = 1, \ldots, p$. Thus both u and v satisfy the boundary conditions $\sum_{i=1}^{p}\beta_{ij}A_i(x) = 0$, $j = 1, \ldots, s$, while $\{u, v\} \neq 0$. Consequently, the set of boundary conditions is not symmetric. Q.E.D.

30 THEOREM. *Let T be a symmetric operator with equal finite deficiency indices $n = n_+ = n_-$. Then the restriction of T^* to the subspace of $\mathfrak{D}(T^*)$ determined by any symmetric family of n linearly independent boundary conditions is a self adjoint extension of T. Moreover, any self adjoint extension of T is of this form.*

PROOF. Let $B_i(x) = 0$, $i = 1, \ldots, n$, be a symmetric family of linearly independent boundary conditions, and let T_1 be the restriction of T^* to the subspace of $\mathfrak{D}(T^*)$ which they determine. Then by Lemma 10, $\mathfrak{D}(T_1) = \mathfrak{D}(T) \oplus \mathfrak{B}$ where $\mathfrak{B} \subseteq \mathfrak{D}_+ \oplus \mathfrak{D}_-$. Since the boundary values B_i vanish on $\mathfrak{D}(T)$, it follows that $\mathfrak{B}$ is n-dimensional. By Lemma 26, the operator T_1 is symmetric and thus Lemma 3′ shows that $\mathfrak{D}(T_1)$ is symmetric. Consequently, by Theorem 12(a), $\mathfrak{B}$ is the graph of an isometric map U of a subspace $\mathfrak{M}$ of $\mathfrak{D}_+$ onto a subspace $\mathfrak{N}$ of $\mathfrak{D}_-$. Since $n = \dim(\mathfrak{B}) \leq \dim(\mathfrak{M}) \leq n$, it follows that $\mathfrak{M}$ is n-dimensional, and hence $\mathfrak{M} = \mathfrak{D}_+$. Since U is one-to-one, $\mathfrak{N}$ is also n-dimensional, and hence $\mathfrak{N} = \mathfrak{D}_-$. Thus T_1 is self adjoint by Theorem 12(b).

Conversely, let T_1 be a self adjoint extension of T. Then by Lemma 26, T_1 is the restriction of T^* to a subspace $\mathfrak{W}$ of $\mathfrak{D}(T^*)$ determined by a symmetric family of linearly independent boundary conditions $B_i(x) = 0$, $i = 1, \ldots, k$, and we have only to show that $k = n$. By Lemma 11, $\mathfrak{W} = \mathfrak{D}(T) \oplus \mathfrak{V}$, and by Theorem 12(b) $\mathfrak{V}$ is n-dimensional. Since $\mathfrak{D}_+ \oplus \mathfrak{D}_-$ is $2n$-dimensional, and $\mathfrak{V}$ is the set of elements of $\mathfrak{D}_+ \oplus \mathfrak{D}_-$ satisfying k independent linear conditions, $\mathfrak{V}$ is $(2n-k)$-dimensional. Thus $2n-k = n$, so $k = n$. Q.E.D.

Finally, we give an explicit form for the most general self adjoint extension of T.

31 Theorem. *Let T be a symmetric operator with equal finite deficiency indices $n_- = n_+ = n$. Let $\varphi_1, \ldots, \varphi_n$ be an orthonormal basis for $\mathfrak{D}_+$, and let $\psi_1, \ldots, \psi_n$ be an orthonormal basis for $\mathfrak{D}_-$. Let $B_i(x) = (x, \varphi_i)^*$ and $C_i(x) = (x, \psi_i)^*$ for x in $\mathfrak{D}(T^*)$ and $1 \leq i \leq n$. Then any self adjoint extension of T is the restriction of T^* to the subspace of $\mathfrak{D}(T^*)$ determined by the boundary conditions*

$$B_i(x) - \sum_{j=1}^{n} \theta_{ij} C_j(x) = 0, \qquad i = 1, \ldots, n,$$

where (θ_{ij}) is any matrix satisfying $\sum_{j=1}^{n} \theta_{ij}\bar{\theta}_{kj} = \delta_{ik}$. Moreover, every such restriction of T^* is a self adjoint extension of T.

Proof. It is clear that B_i and C_i are boundary values. We now observe that the theorem may be stated equivalently as follows: An operator is a self adjoint extension of T if and only if it is the restriction of T^* to the subspace $\mathfrak{W}$ consisting of all x in $\mathfrak{D}(T^*)$ such that $(x, z)^* = (x, Uz)^*$, $z \in \mathfrak{D}_+$, where U is an isometric mapping of $\mathfrak{D}_+$ onto $\mathfrak{D}_-$. To prove this, consider the restriction of T^* to such a subspace $\mathfrak{W}$. Writing $x = d + d_+ + d_-$, $d \in \mathfrak{D}(T)$, $d_+ \in \mathfrak{D}_+$, $d_- \in \mathfrak{D}_-$, we see that x is in $\mathfrak{W}$ if and only if $(d_+, z)^* = (d_-, Uz)^*$ for all z in $\mathfrak{D}_+$, i.e., if and only if $d_- = Ud_+$. Thus $\mathfrak{W}$ is the direct sum of $\mathfrak{D}(T)$ and the graph of an isometric mapping of $\mathfrak{D}_+$ onto $\mathfrak{D}_-$ and hence the restriction of T^* to $\mathfrak{W}$ is a self adjoint extension of T. Conversely, if $\mathfrak{W}$ is the domain of a self adjoint extension of T, then by Theorem 12(b), $\mathfrak{W} = \mathfrak{D}(T) \oplus \Gamma(U)$, where U is an isometric map of $\mathfrak{D}_+$ onto $\mathfrak{D}_-$. An elementary calculation shows that $\mathfrak{W}$ consists of the set of x in $\mathfrak{D}(T^*)$ with $(x, z)^* = (x, Uz)^*$ for all z in $\mathfrak{D}_+$. Q.E.D.

5. Semi-bounded Symmetric Operators

In this section we study the self adjoint extensions of those operators in a class of symmetric operators which arise frequently from the boundary value problems of mathematical physics.

1 Definition. A symmetric operator T is *bounded above* (*bounded below*) if there is a real number c such that $(Tx, x) \leqq c(x, x)$ $\big((Tx, x) \geqq c(x, x)\big)$ for all x in $\mathfrak{D}(T)$. If T is bounded above or below we say that T is *semi-bounded*. The number c is called a *bound* for T and the smallest (largest) such c is called *the upper* (*lower*) *bound* for T. If $(Tx, x) \geqq 0$ for all x in $\mathfrak{D}(T)$, then T is *non-negative*.

This section is devoted to proving the following theorem of von Neumann and Friedrichs.

2 Theorem. *Every semi-bounded symmetric operator with dense domain has a semi-bounded self adjoint extension with the same bound.*

Proof. If T is semi-bounded, then, for some constant α, either $T+\alpha I$ or $-T+\alpha I$ is bounded below by one. It is clear from Lemma 1.6 we may suppose without loss of generality that $(Tx, x) \geqq (x, x)$ for x in $\mathfrak{D}(T)$.

For x, y in $\mathfrak{D}(T)$ define $(x, y)^+ = (Tx, y)$. Then $(x, x)^+ \geqq (x, x) \geqq 0$, and $\mathfrak{D}(T)$ is a linear space with a Hermitian bilinear scalar product $(x, y)^+$ satisfying properties (i), . . ., (v) of Definition IV.2.26 for an inner product in an abstract Hilbert space. Since the proof of the Schwarz inequality (Theorem IV.4.1) does not require the completeness axiom IV.2.26 (vi), we see that $|(x, y)^+| \leqq |x|^+|y|^+$ for x, y in $\mathfrak{D}(T)$. Thus $\mathfrak{D}(T)$ with the norm $|x|^+ = \big((x, x)^+\big)^{\frac{1}{2}}$ is a normed linear space which, in general, is not complete.

Now define $\mathfrak{D}_0$ to be the subspace of elements x of $\mathfrak{H}$ with the property that there exists a sequence $\{x_n\} \subseteq \mathfrak{D}(T)$ such that $x_n \to x$ in $\mathfrak{H}$ and $\lim_{m,n\to\infty} |x_m - x_n|^+ = 0$. It will be shown that the Hermitian bilinear form $(x, y)^+$ may be extended from $\mathfrak{D}(T)$ to $\mathfrak{D}_0$, making $\mathfrak{D}_0$ a Hilbert space (i.e., complete) under $(x, y)^+$. It will then be shown that the restriction T_1 of T^* to $\mathfrak{D}_0 \cap \mathfrak{D}(T^*)$ is a self adjoint extension of T with the same bound. If $\lim_{m,n\to\infty} |x_m - x_n|^+ = 0$ and $\lim_{n\to\infty} x_n = x_0$, then $\lim_{m\to\infty} |x_m|^+$ exists, since $\big||x_m|^+ - |x_n|^+\big| \leqq |x_m - x_n|^+$. If $\{y_m\} \subseteq \mathfrak{D}(T)$ and $\lim_{m\to\infty} y_m = x_0$ also, and $\lim_{m,n\to\infty} |y_m - y_n|^+ = 0$,

then, letting $z_n = x_n - y_n$, we have $\lim_{n\to\infty} z_n = 0$ and $\lim_{m,n\to\infty} |z_m - z_n|^+ = 0$. Consequently there is a number M such that $|z_m|^+ \leq M$, $m = 1, 2, \ldots$. Moreover, given $\varepsilon > 0$ there is an integer N such that if $m, n > N$, then $|z_m - z_n|^+ < \varepsilon$. Thus

$$(|z_n|^+)^2 \leq |(z_n, z_m)^+| + |(z_n, z_n - z_m)^+| \leq |(z_n, z_m)^+| + M\varepsilon.$$

Since $|(z_n, z_m)^+| = |(z_n, Tz_m)| \leq |z_n||Tz_m|$, $\lim_{n\to\infty} (z_n, z_m)^+ = 0$, $m = 1, 2, \ldots$. Consequently, $\lim_{n\to\infty} \sup (|z_n|^+)^2 \leq M\varepsilon$, and thus $\lim_{n\to\infty} |z_n|^+ = 0$. Since $||x_n|^+ - |y_n|^+| \leq |x_n - y_n|^+ = |z_n|^+$, we see $\lim |y_n|^+ = \lim |x_n|^+$. It follows that there is a well-defined extension of the norm $|y|^+$ from $\mathfrak{D}(T)$ to $\mathfrak{D}_0$ obtained by defining $|x|^+ = \lim_{n\to\infty} |x_n|^+$ whenever $\lim_{n\to\infty} x_n = x$ and $\lim_{m,n\to\infty} |x_m - x_n|^+ = 0$.

It will next be shown that the conditions $x_n \to x$, $x_n \in \mathfrak{D}(T)$, and $\lim_{m,n\to\infty} |x_n - x_m|^+ = 0$ imply that $\lim_{n\to\infty} |x_n - x|^+ = 0$. We observe first that it follows from the preceding paragraph that $\lim_{m\to\infty} |x_m - x_n|^+ = |x - x_n|^+$ and $\lim_{m\to\infty} |x_m + x_n|^+ = |x + x_n|^+$ for each n. Also the identity

$$(|x_m - x_n|^+)^2 + (|x_m + x_n|^+)^2 = 2[(|x_m|^+)^2 + (|x_n|^+)^2]$$

shows that $\lim_{m,n\to\infty} |x_m + x_n|^+ = 2|x|^+$. Consequently,

$$\lim_{n\to\infty} |x + x_n|^+ = \lim_{n\to\infty} \lim_{m\to\infty} |x_m + x_n|^+ = 2|x|^+.$$

However, by letting $m \to \infty$ in the identity above, it is seen that

$$(|x - x_n|^+)^2 + (|x + x_n|^+)^2 = 2[(|x|^+)^2 + (|x_n|^+)^2].$$

It follows that $\lim_{n\to\infty} |x - x_n|^+ = 0$. From this fact a familiar limiting argument establishes the relations $|x| \leq |x|^+$, $|x + y|^+ \leq |x|^+ + |y^+|$, and $|\alpha x|^+ = |\alpha|\,|x|^+$ for all $x, y \in \mathfrak{D}_0$ and every scalar α. Thus $\mathfrak{D}_0$ is a normed linear space under the norm $|x|^+$ and $\mathfrak{D}(T)$ is dense in $\mathfrak{D}_0$.

Moreover, since $|(x, y)^+| \leq |x|^+|y|^+$ on the dense subset $\mathfrak{D}(T) \times \mathfrak{D}(T)$ of $\mathfrak{D}_0 \times \mathfrak{D}_0$, the form $(x, y)^+$ may be extended by continuity (cf. I.6.17) to yield a bilinear Hermitian form defined on $\mathfrak{D}_0$.

If it can be shown that $\mathfrak{D}_0$ is complete, then $\mathfrak{D}_0$ will be a Hilbert space under the inner product $(x, y)^+$. However, if $\{x_n\}$ is a Cauchy sequence in $\mathfrak{D}_0$, we can find a sequence $\{y_n\}$ in the dense subset $\mathfrak{D}(T) \subseteq \mathfrak{D}_0$ such that $|x_n - y_n|^+ \leq 1/n$. Then $\lim_{m,n\to\infty} |y_m - y_n|^+ = 0$,

and hence $\lim_{m,n\to\infty}|y_m - y_n| = 0$. If x is an element of $\mathfrak{H}$ such that $|y_n - x| \to 0$, then by definition, x is an element of $\mathfrak{D}_0$. However, we have shown under these conditions that $\lim_{n\to\infty}|y_n - x|^+ = 0$. Consequently $\lim_{n\to\infty}|x_n - x|^+ = 0$ and $\mathfrak{D}_0$ is complete.

Now define the extension T_1 of T by letting $\mathfrak{D}(T_1) = \mathfrak{D}_0 \cap \mathfrak{D}(T^*)$, $T_1 x = T^* x$ for x in $\mathfrak{D}(T_1)$. Clearly $T \subseteq T_1 \subseteq T^*$ and T_1 is a linear operator with dense domain. Since

$$(x, y)^+ = (x, Ty), \qquad x, y \in \mathfrak{D}(T),$$

it follows by continuity that this same equation holds for x in $\mathfrak{D}(T_1)$ and y in $\mathfrak{D}(T)$. However, since $(x, y)^+ = (T^*x, y)$, for x in $\mathfrak{D}(T_1)$ and y in $\mathfrak{D}(T)$, it follows by continuity that

$$(x, y)^+ = (T^*x, y) = (T_1 x, y), \qquad x, y \in \mathfrak{D}(T_1).$$

From the equation $(x, y)^+ = \overline{(y, x)^+}$ for $x, y \in \mathfrak{D}_0$, it is seen that T_1 is a symmetric operator. The relation $(T_1 x, x) = (x, x)^+ \geqq (x, x)$ shows T_1 has the same lower bound as T. In order to show T_1 is self adjoint, we shall need the fact that $T_1\mathfrak{D}(T_1)$ is the whole Hilbert space $\mathfrak{H}$. If x is an arbitrary element of $\mathfrak{H}$, the function $(\cdot, x)$ is continuous on the Hilbert space $\mathfrak{D}_0$, since if $|y_n|^+ \to 0$ then $|y_n| \to 0$, and thus $\lim_n(y_n, x) = 0$. It follows from Theorem IV.4.5 that there is a vector z in $\mathfrak{D}_0$ such that $(y, x) = (y, z)^+$ for $y \in \mathfrak{D}_0$. Since $(y, z)^+ = (Ty, z)$ for y in $\mathfrak{D}(T)$, it follows that z is in $\mathfrak{D}(T^*)$ and that $T^*z = x$. Consequently, $z \in \mathfrak{D}_0 \cap \mathfrak{D}(T^*) = \mathfrak{D}(T_1)$ and $T_1 z = x$. Thus $T_1\mathfrak{D}(T_1) = \mathfrak{H}$. It follows from Lemma 1.6(d) that T_1^* is one-to-one. On the other hand since T_1 is symmetric we have $T_1 \subseteq T_1^*$. Thus T_1, T_1^* both have range $\mathfrak{H}$ and, since T_1^* is one-to-one, we must have $\mathfrak{D}(T_1) = \mathfrak{D}(T_1^*)$. This shows that $T_1 = T_1^*$, i.e., T_1 is a self adjoint extension of T. Q.E.D.

3 COROLLARY. *If T is a semi-bounded symmetric operator, let $\mathfrak{D}_0$ be the set of vectors x in $\mathfrak{H}$ for which there is a sequence $\{x_n\} \subseteq \mathfrak{D}(T)$ with $x_n \to x$ and $(T(x_m - x_n), (x_m - x_n)) \to 0$. Then the restriction of T^* to $\mathfrak{D}_0 \cap \mathfrak{D}(T^*)$ is a self adjoint extension of T with the same bound.*

6. Unitary Semi-groups

In the first section of Chapter VIII we have investigated the theory of strongly continuous semi-groups of bounded operators

(cf. Definition VIII.1.1). The aim of this section is to determine the form of the infinitesimal generator (Definition VIII.1.6) when the elements of the semi-group are unitary operators on Hilbert space.

1 THEOREM. (*Stone*) *If* $\{U(t),\ t \geqq 0\}$ *is a strongly continuous semi-group of unitary operators in Hilbert space, then there is a unique (possibly unbounded) self adjoint operator* B *such that*

$$U(t) = e^{itB}, \qquad t \geqq 0.$$

PROOF. Let A be the infinitesimal generator of the semi-group $\{U(t)\}$. According to VIII.1.8 and VIII.1.11, A is a closed operator with dense domain whose resolvent set includes the whole half plane $\mathscr{R}(\lambda) > 0$. Moreover, we have

$$[*] \qquad R(\lambda; A)x = \int_0^\infty e^{-\lambda t} U(t) x dt, \qquad \mathscr{R}(\lambda) > 0.$$

Put $S(t) = U(t)^* = \{U(t)\}^{-1}$ for $t \geqq 0$. It is clear that $S(t_1)S(t_2) = S(t_1+t_2)$. For $t_1 \geqq t_2$ it follows that $|S(t_1)x - S(t_2)x| = |x - U(t_1)S(t_2)x| = |x - U(t_1 - t_2)x|$, and hence $|S(t_1)x - S(t_2)x| = |x - U(|t_1 - t_2|)x|$ for all $t_1, t_2 \geqq 0$. Thus $\{S(t)\}$ is also a strongly continuous semi-group. It is clear that $S(t)^* = \{S(t)\}^{-1} = U(t)$ for $t \geqq 0$. If $x \in \mathfrak{D}(A)$, then

$$\lim_{t\to 0} \frac{S(t)x - x}{t} = \lim_{t \to 0} S(t) \left(\frac{x - U(t)x}{t}\right) = -Ax.$$

Thus, if A_1 is the infinitesimal generator of $\{S(t)\}$, we find that $A_1 \supseteq -A$. In the same way it is seen that $-A \subseteq A_1$. Hence $A_1 = -A$, and consequently

$$[**] \qquad R(\lambda; -A)x = \int_0^\infty e^{-\lambda t} U(t)^* x dt, \qquad \mathscr{R}(\lambda) > 0.$$

We now define $B = -iA$. Using the equation $R(\lambda; \alpha B) = (\lambda I - \alpha B)^{-1} = \alpha^{-1} R(\lambda\alpha^{-1}; B)$ it is seen from [*] that

$$R(-i\lambda; B)x = i \int_0^\infty e^{-\lambda t} U(t) x dt, \qquad \mathscr{R}(\lambda) > 0,$$

and, putting $-i\lambda = \mu$, that

$$R(\mu; B)x = i \int_0^\infty e^{-i\mu t} U(t) x dt, \qquad \mathscr{I}(\mu) < 0.$$

In the same way, it follows from [**] that

$$R(\mu; B)x = -i \int_0^\infty e^{i\mu t} U(t)^* x dt, \qquad \mathscr{I}(\mu) > 0.$$

Thus, if $\mathscr{I}(\mu) > 0$,

$$\begin{aligned}(R(\mu; B)x, y) &= -i \int_0^\infty e^{i\mu t} (U(t)^* x, y) dt \\ &= -i \int_0^\infty e^{i\mu t} (x, U(t)y) dt \\ &= -i \int_0^\infty (x, e^{-i\bar{\mu} t} U(t)y) dt \\ &= \left(x, -i \int_0^\infty e^{-i\bar{\mu} t} U(t)y dt\right) \\ &= (x, R(\bar{\mu}; B)y).\end{aligned}$$

Consequently, $R(\mu; B)^* = R(\bar{\mu}; B)$. By using Lemma 1.6(b) and 1.6(c) it is seen that this implies that $\bar{\mu}I - B^* = \bar{\mu}I - B$, so that $B^* = B$, and B is self adjoint.

If E is the resolution of the identity for B and if $V(t) = e^{itB}$ then, by Theorem 2.6,

$$(V(t)x, y) = \int_{-\infty}^{+\infty} e^{it\lambda} (E(d\lambda)x, y).$$

Hence, for $\mathscr{R}(\mu) > 0$, an application of Fubini's theorem and Theorem 2.6(e) shows that

$$\begin{aligned}\int_0^\infty e^{-\mu t} (V(t)x, y) dt &= \int_0^\infty \int_{-\infty}^{+\infty} e^{-(\mu - i\lambda)t} (E(d\lambda)x, y) dt \\ &= \int_{-\infty}^{+\infty} \frac{1}{\mu - i\lambda} (E(d\lambda)x, y) \\ &= (R(\mu; iB)x, y).\end{aligned}$$

Since $R(\mu; iB) = R(\mu; A)$, it follows from [*] that

$$\int_0^\infty e^{-\mu t} (V(t)x, y) dt = \int_0^\infty e^{-\mu t} (U(t)x, y) dt, \qquad \mathscr{R}(\mu) > 0.$$

By Lemma VIII.1.15, $e^{-\varepsilon t}(V(t)x, y) = e^{-\varepsilon t}(U(t)x, y)$ for $t \geqq 0$ and arbitrary $\varepsilon > 0$. Hence $U(t) = V(t)$ for $t \geqq 0$, and the theorem is proved. Q.E.D.

7. The Canonical Factorization

In this section we shall prove that each closed operator T with dense domain in Hilbert space has a unique factorization $T = PA$, where A is a positive (i.e., $(Ax, x) \geqq 0$, $x \in \mathfrak{D}(A)$) self adjoint transformation, and P is a partial isometry (cf. Definition 4 below). This result may be regarded as a far-reaching generalization of the fact that each complex number α has a unique representation $\alpha = re^{i\theta}$, where $r \geqq 0$, and $|e^{i\theta}| = 1$. By analogy with the fact that $r = (\bar{\alpha}\alpha)^{\frac{1}{2}}$, we shall first seek to obtain the self adjoint operator A from the operator T^*T.

1 Lemma. *Let T be a closed linear operator with dense domain. Then*

(a) *$\mathfrak{D}(T^*)$ is dense and $T^{**} = T$;*

(b) *$(I+T^*T)^{-1}$ exists and is a bounded self adjoint transformation;*

(c) *T^*T is self adjoint and positive;*

(d) *if T' is the restriction of T to $\mathfrak{D}(T^*T)$, then $\overline{\Gamma(T')} = \Gamma(T)$.*

Proof. By Lemma 1.5, $\Gamma(T^*) = [A_2\Gamma(T)]^{\perp}$, where A_2 is the isometric isomorphism $[x, y] \to [y, -x]$ of $\mathfrak{H} \oplus \mathfrak{H}$ onto itself. Thus if x is in $[\mathfrak{D}(T^*)]^{\perp}$, then

$$[x, 0] \in [\Gamma(T^*)]^{\perp} = [A_2\Gamma(T)]^{\perp\perp} = \overline{A_2\Gamma(T)} = A_2\Gamma(T).$$

Here we have used the fact that T is closed. Hence $[0, -x]$ is in $\Gamma(T)$, and thus $x = 0$. This shows that $\mathfrak{D}(T^*)$ is dense in $\mathfrak{H}$. To prove that $T^{**} = T$ we shall show that $\Gamma(T^{**}) = \Gamma(T)$. By Lemma 1.5

$$\begin{aligned}\Gamma(T^{**}) &= [A_2\Gamma(T^*)]^{\perp} = [A_2[A_2\Gamma(T)]^{\perp}]^{\perp} \\ &= [A_2^2\Gamma(T)]^{\perp\perp} = [-\Gamma(T)]^{\perp\perp} = \Gamma(T).\end{aligned}$$

This completes the proof of (a).

Since T is closed, the manifold $\mathfrak{D}(T)$ with the inner product $(x, y)_1 = (x, y) + (Tx, Ty)$ is a Hilbert space, which we shall denote by the symbol $\mathfrak{H}_1$. Moreover, in $\mathfrak{H}_1$ the function (x, y) is a Hermitian bilinear form. The inequalities $|(x, y)| \leq |x|\,|y| \leq |x|_1|y|_1$ show that it is continuous. It follows from Lemma X.2.2 that there is a bounded self adjoint operator A on $\mathfrak{H}_1$ such that $(x, y) = (Ax, y)_1$, $x, y \in \mathfrak{D}(T)$.

We may also regard A as a mapping from the dense subspace $\mathfrak{D}(T)$ of $\mathfrak{H}$ into the space $\mathfrak{H}_1$. In this case A is still continuous, for

$$|Ax|_1^2 = (Ax, Ax)_1 = (A^2x, x)_1 = (Ax, x), \qquad x \in \mathfrak{D}(T),$$

and, by the inequalities above,

$$(Ax, x) \leq |Ax|\,|x| \leq |Ax|_1|x|,$$

showing that $|Ax|_1 \leq |x|$. By Theorem I.6.17, A may be extended uniquely to a continuous mapping B of $\mathfrak{H}$ into $\mathfrak{H}_1$. A continuity argument shows that $(Bx, y)_1 = (x, y)$ for $x \in \mathfrak{H}$, $y \in \mathfrak{D}(T)$, and that $|Bx| \leq |Bx|_1 \leq |x|$ for $x \in \mathfrak{H}$. Thus regarding B as a mapping of $\mathfrak{H}$ into a dense subset of itself, we see B is an operator in $\mathfrak{H}$ of norm at most one. Since for x and y in the dense set $\mathfrak{D}(T)$,

$$(Bx, y) = (Ax, y) = (A^2x, y)_1 = (Ax, Ay)_1 = (x, Ay) = (x, By),$$

it follows that B is self adjoint. Now for $x \in \mathfrak{H}$, $y \in \mathfrak{D}(T)$,

$$\begin{aligned} (x, y) = (Bx, y)_1 &= (Bx, y) + (TBx, Ty) \\ &= (Bx, y) + (T^*TBx, y) \\ &= \big((I + T^*T)Bx, y\big). \end{aligned}$$

Thus, since $\mathfrak{D}(T)$ is dense, $(I + T^*T)Bx = x$ for $x \in \mathfrak{H}$.

On the other hand, if x is in $\mathfrak{D}(T^*T)$ and y is in $\mathfrak{H}$, then By is in $\mathfrak{D}(T)$ and

$$\begin{aligned} (B(I + T^*T)x, y) &= \big((I + T^*T)x, By\big) \\ &= (x, By) + (Tx, TBy) \\ &= (x, By)_1 = (Bx, y)_1 = (x, y). \end{aligned}$$

Thus $B(I + T^*T)x = x$ for x in $\mathfrak{D}(T^*T) = \mathfrak{D}(I + T^*T)$. This concludes the proof of (b).

By Lemma 1.6, $(I + T^*T)^* = (B^{-1})^* = (B^*)^{-1} = B^{-1} = I + T^*T$, so $I + T^*T$ is self adjoint. But then T^*T is self adjoint since $(T^*T)^* = (I + T^*T)^* - I = T^*T$. The relation

$$(T^*Tx, x) = (Tx, Tx) \geq 0, \qquad x \in \mathfrak{D}(T^*T)$$

shows T^*T is positive, proving (c).

Now let T' be the restriction of T to $\mathfrak{D}(T^*T)$ and let $\mathfrak{R}$ denote

the orthogonal complement of $\Gamma(T')$ in the closed manifold $\Gamma(T)$ of $\mathfrak{H} \oplus \mathfrak{H}$. Then $[x, Tx] \in \mathfrak{K}$ if and only if $(x, y)_1 = ([x, Tx], [y, Ty]) = 0$ for all $y \in \mathfrak{D}(T') = \mathfrak{D}(T^*T)$. To prove (d) we shall prove that if $[x, Tx] \in \mathfrak{K}$, then $x = 0$. Since $B\mathfrak{H} = \mathfrak{D}(I+T^*T) = \mathfrak{D}(T^*T)$, the relation $(x, y)_1 = 0$, $y \in \mathfrak{D}(T^*T)$, implies $(x, Bz)_1 = (Bx, z)_1 = 0$ for all $z \in \mathfrak{H}$. Since $(Bx, z)_1 = (x, z)$, it follows that $x = 0$. Q.E.D.

Next we shall require some information on positive self adjoint transformations and their square roots.

2 Lemma. *A self adjoint transformation T is positive if and only if $\sigma(T)$ is a subset of the interval $[0, \infty)$.*

Proof. Let E be the resolution of the identity for T so that by Theorem 2.3 we have

$$T_n = TE([-n, n]) = \int_{-n}^{n} \lambda E(d\lambda) = E([-n, n])TE([-n, n]).$$

Then, also by Theorem 2.3, $T_n x \to x$ for every x in $\mathfrak{D}(T)$. By Theorem 2.9(b) we have

$$[*] \qquad \sigma(T) \cup \{0\} \supseteq \bigcup_{n=1}^{\infty} \sigma(T_n) \supseteq \sigma(T).$$

Thus, if $\sigma(T) \subseteq [0, \infty)$, it follows from Theorem X.4.2 that $T_n \geqq 0$. Hence $(Tx, x) = \lim_n (T_n x, x) \geqq 0$ and T is positive. Conversely, if $T \geqq 0$, then $(T_n x, x) = (TE([-n, n])x, E([-n, n])x) \geqq 0$ which shows that $T_n \geqq 0$. Hence, by Theorem X.4.2, $\sigma(T_n) \subseteq [0, \infty)$. Thus $[*]$ shows that $\sigma(T) \subset [0, \infty)$. Q.E.D.

The next lemma shows that a positive self adjoint transformation has a unique positive "square root".

3 Lemma. *If T is a positive self adjoint transformation, there is a unique positive self adjoint transformation A such that $A^2 = T$.*

Proof. By Lemma 2, $\sigma(T) \subseteq [0, \infty)$ and, by Theorem 2.6(d), the positive function $f(\lambda) = \lambda^{\frac{1}{2}}$ on $\sigma(T)$ defines the self adjoint operator $A = f(T)$. By Corollary 2.7(c), $A^2 \subseteq T$ and $\mathfrak{D}(A^2) = \mathfrak{D}(T) \cap \mathfrak{D}(A)$. However, it follows from Theorem 2.6(a) that $\mathfrak{D}(T) \subseteq \mathfrak{D}(A)$ and so $\mathfrak{D}(A^2) = \mathfrak{D}(T)$. Thus $A^2 = T$. The positivity of A follows from Theorem 2.9(b) and Lemma 2. To see that A is unique, suppose that $B^2 = T$ where B is also self adjoint and positive.

Let δ be a Borel set in $[0, \infty)$, let $\delta_1 = \{\lambda | \lambda \in [0, \infty), \lambda^{\frac{1}{2}} \in \delta\}$, and let $E(\cdot; A)$, $E(\cdot; B)$, and $E(\cdot; T)$ be the resolution of the identity for A, B, T, respectively. Then, by Theorem 2.9(c), $E(\delta; A) = E(\delta_1; T) = E(B^2; \delta_1) = E(B; \delta)$. Thus $E(\cdot; A) = E(\cdot; B)$ and so, by Theorem 2.3, $A = B$. Q.E.D.

Next we introduce the class of operators which will correspond in our analysis to the factors $e^{i\theta}$ in the scalar case. At first sight, one might suppose that the unitary operators were the appropriate class. However, a moment's reflection shows that such a simple outcome is not to be expected. Consider, for instance, two-dimensional Hilbert space, and the operator whose matrix is

$$\begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}.$$

It is clear that the appropriate factorization should be

$$\begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix},$$

where $\alpha = re^{i\theta}$ is the corresponding factorization of α. However, the first matrix on the right is not an isometry, but only a partial isometry in the precise sense given in the following definition.

4 Definition. A bounded linear operator P in Hilbert space $\mathfrak{H}$ is called a *partial isometry* if there is a closed subspace $\mathfrak{M}$ such that $|Px| = |x|$ for x in $\mathfrak{M}$ and $P(\mathfrak{M}^\perp) = \{0\}$. The subspace $\mathfrak{M}$ is called the *initial domain* of P and $P\mathfrak{M}(= P\mathfrak{H})$ is called the *final domain* of P.

5 Lemma. *A bounded linear operator P in Hilbert space is a partial isometry if and only if P^*P is a projection. In this case PP^* is also a projection and the ranges of P^*P and PP^* are the initial and final domains, respectively, of P.*

Proof. If P^*P is a projection let $\mathfrak{M} = P^*P\mathfrak{H}$. Then for x in $\mathfrak{M}$ and y in $\mathfrak{M}^\perp$, we have

$$\begin{aligned} |x|^2 &= (x, x) = (P^*Px, x) = (Px, Px) = |Px|^2, \\ 0 &= (P^*Py, y) = (Py, Py) = |Py|^2, \end{aligned}$$

and so P is a partial isometry whose initial domain is $\mathfrak{M}$. To see that PP^* is a projection whose range is the final domain $\mathfrak{N} = P\mathfrak{M}$, let Q

be the operator which equals P^{-1} on $\mathfrak{N}$ and zero on $\mathfrak{N}^{\perp}$. If $x \in \mathfrak{N}$ then $Qx \in \mathfrak{M}$ and thus $Qx = P^*PQx = P^*x$ and therefore $x = PP^*x$. Also, if $y \perp \mathfrak{N} = P\mathfrak{M} = P\mathfrak{H}$ then $P^*y = 0$ and $PP^*y = 0$. Thus PP^* is a projection whose range is $\mathfrak{N} = P\mathfrak{M}$, the final domain of P.

To complete the proof it will suffice to show that P^*P is a projection if P is a partial isometry. Let $x, v \in \mathfrak{M}$, the initial domain of P. Then the identity $|x+v|^2 = |Px+Pv|^2$ shows that $(x, v)+(v, x) = (Px, Pv)+(Pv, Px)$. Hence $\mathscr{R}(x, v) = \mathscr{R}(Px, Pv)$. Since $\mathscr{I}(x, v) = \mathscr{R}(x, iv)$, it follows that $(x, v) = (Px, Pv)$ if $x, v \in \mathfrak{M}$. If $x \in \mathfrak{M}$ and $w \in \mathfrak{M}^{\perp}$ then $Pw = 0$ and hence $0 = (x, w) = (Px, Pw)$. Thus for every vector y in Hilbert space we have $(x, y) = (Px, Py)$ if $x \in \mathfrak{M}$. This means that $(x, y) = (P^*Px, y)$ for each y and hence that $P^*Px = x$ for x in $\mathfrak{M}$. Since P vanishes on $\mathfrak{M}^{\perp}$ so does P^*P. Thus P^*P is the orthogonal projection onto $\mathfrak{M}$. Q.E.D.

6 Corollary. *P is a partial isometry if and only if P^* is a partial isometry.*

7 Theorem. *If T is a closed transformation whose domain is dense, then T can be written in one and only one way as a product $T = PA$, where P is a partial isometry whose initial domain is $\overline{\mathfrak{R}(T^*)}$, and A is a positive self adjoint transformation such that $\overline{\mathfrak{R}(A)} = \overline{\mathfrak{R}(T^*)}$.*

Proof. Let A be the positive square root of T^*T whose existence follows from Lemma 3 and Lemma 1(c). Then $(Ax, Ax) = (A^2x, x)$ $(T^*Tx, x) = (Tx, Tx)$ for $x \in \mathfrak{D}(T^*T)$. If we let $P_0Ax = Tx$ for $x \in \mathfrak{D}(T^*T)$, it follows that P_0 is a well-defined isometry whose domain is $\mathfrak{R}(A')$, where A' is the restriction of A to $\mathfrak{D}(T^*T)$.

Since $T^*T = A^*A$, Lemma 1(d) shows that $\Gamma(A')$ is dense in $\Gamma(A)$. Thus if $[x, y]$ is in $\Gamma(A)$, there is a sequence $\{[x_n, y_n]\} \subset \Gamma(A')$ such that $[x_n, y_n] \to [x, y]$. Consequently $y_n \to y$, proving that $\mathfrak{R}(A')$ is dense in $\mathfrak{R}(A)$. Let P_1 be the isometric extension of P_0 to $\overline{\mathfrak{R}(A)}$ and let E be the perpendicular projection of $\mathfrak{H}$ onto $\overline{\mathfrak{R}(A)}$. If we define $P = P_1E$, then P is a partial isometry whose initial domain is $\overline{\mathfrak{R}(A)}$. Moreover, $PAx = Tx$ for $x \in \mathfrak{D}(T^*T)$.

Let $\{y_n\}$ be a sequence of elements of $\mathfrak{D}(A)$ such that $y_n \to y$ and $PAy_n \to z$. Since P is an isometric mapping of $\overline{\mathfrak{R}(A)}$, Ay_n converges to a certain element u. Since A is closed, $y \in \mathfrak{D}(A)$ and $Ay = u$. Thus

$y \in \mathfrak{D}(PA)$ and $PAy = z$. Consequently PA is closed.

We now verify the formula $T = PA$. Let T' be the restriction of T to $\mathfrak{D}(T^*T)$. Then $\Gamma(T')$ is dense in $\Gamma(T)$ by Lemma 1(d). If x is in $\mathfrak{D}(T)$, then there is a sequence $\{[x_n, Tx_n]\} \subset \Gamma(T')$ which converges to $[x, Tx]$. Since $PAx_n = Tx_n$ and PA is closed, $PAx = \lim PAx_n = \lim Tx_n = Tx$. Thus $T \subset PA$. On the other hand, $\Gamma(A')$ is dense in $\Gamma(A)$. Thus if $x \in \mathfrak{D}(A)$, there is a sequence $\{[x_n, Ax_n]\}$ of elements of $\Gamma(A')$ which converges to $[x, Ax]$. Since $PAx_n = Tx_n$, and T is closed, $PAx = \lim PAx_n = \lim Tx_n = Tx$. Thus $PA \subset T$, and consequently $PA = T$.

The fact that $\overline{\mathfrak{R}(A)} = \overline{\mathfrak{R}(T^*)}$ is a consequence of Lemmas 1.6(d) and 1(a), since $A^*x = Ax = 0$ if and only if $T^{**}x = Tx = PAx = 0$.

Finally we show that the decomposition $T = PA$ of the theorem is unique. By Lemma 1.6(c), $AP^* = T^*$. Hence $T^*T = AP^*PA$. Since, by Lemma 5, P^*P is a projection onto $\overline{\mathfrak{R}(A)}$, it follows that $T^*T = A^2$. The uniqueness of A now follows from Lemma 3. Since A is unique, P is uniquely determined on $\mathfrak{R}(A)$ by the equation of $P(Ax) = Tx$. Further the extension of P by continuity from $\mathfrak{R}(A)$ to $\overline{\mathfrak{R}(A)}$ is unique. Since P is zero on $\overline{\mathfrak{R}(A)}^{\perp}$ it follows that P is uniquely determined by T. Q.E.D.

8. Moment Theorems

The term *moment* in the title of this section was introduced into mathematical terminology by Stieltjes who, in his famous memoir *Recherches sur les fractions continues* published in 1894 stated and solved what he called the *problem of moments*. The terminology, borrowed from theoretical mechanics, refers to the following situation: if $\mu(\delta)$ is the mass distributed over a set δ on a line then the integrals $\int t\mu(dt)$, $\int t^2\mu(dt)$ give the statical moment and the moment of inertia with respect to the origin on the line. In general, $\int t^n\mu(dt)$ is called the nth moment with respect to the origin and the problem proposed and solved by Stieltjes is to find a mass distribution μ on $[0, \infty)$ which has prescribed moments. The Hamburger moment problem is similar to that of Stieltjes and differs from it by using the whole real axis $(-\infty, \infty)$ instead of $[0, \infty)$. The Hausdorff moment problem

is again similar but refers to a finite interval of reals.

In this section we shall show how the spectral theorem for self adjoint operators may be applied to prove a number of results from the theory of moments and shall begin our discussion by giving a solution to the Hamburger moment problem.

1 THEOREM. *Let m_n, $n = 0, 1, 2, \ldots$ be a sequence of real numbers. A necessary and sufficient condition that there exist a non-negative measure μ defined on the Borel sets of the real line such that $\int_{-\infty}^{\infty}|t|^n\mu(dt) < \infty$ and*

$$m_n = \int_{-\infty}^{\infty} t^n\mu(dt), \qquad n = 0, 1, 2, \ldots,$$

is that

$$\sum_{i,j=0}^{n} m_{i+j}\alpha_i\bar{\alpha}_j \geq 0$$

for every finite set $\alpha_0, \ldots, \alpha_n$ of complex numbers.

PROOF. We observe first that the condition is necessary. For if $\{m_n\}$ has such a representation, and $\alpha_0, \ldots, \alpha_n$ is any finite set of complex numbers

$$\begin{aligned}\sum_{i,j=0}^{n} m_{i+j}\alpha_i\bar{\alpha}_j &= \int_{-\infty}^{\infty}\left(\sum_{i,j=0}^{n} t^{i+j}\alpha_i\bar{\alpha}_j\right)\mu(dt)\\ &= \int_{-\infty}^{\infty}\left(\sum_{i=0}^{n} t^i\alpha_i\right)\overline{\left(\sum_{j=0}^{n} t^j\alpha_j\right)}\mu(dt)\\ &= \int_{-\infty}^{\infty}\left|\sum_{j=0}^{n} t^j\alpha_j\right|^2\mu(dt) \geq 0.\end{aligned}$$

To prove the sufficiency, let $\mathfrak{A}$ be the linear space of all sequences α_n, $n = 0, 1, 2, \ldots$ of complex numbers whose elements are all zero for sufficiently large n. If $\xi = [\alpha_n]$ and $\eta = [\beta_n]$ are in $\mathfrak{A}$ define

$$(\xi, \eta) = \sum_{i,j=0}^{\infty} m_{i+j}\alpha_i\bar{\beta}_j.$$

Then (ξ, η) is a Hermitian bilinear form defined on $\mathfrak{A}$ and, by assumption, $(\xi, \xi) \geq 0$, $\xi \in \mathfrak{A}$. By defining $|\xi| = \sqrt{(\xi, \xi)}$, it follows from these facts (cf. the proof of Theorem IV.4.1) that the Schwarz

inequality $|(\xi, \eta)| \leq |\xi|\,|\eta|$ is valid, and $|\xi+\eta| \leq |\xi|+|\eta|$ for all $\xi, \eta \in \mathfrak{A}$.

Now let $\mathfrak{A}_0$ denote the subspace of $\mathfrak{A}$ consisting of all sequences ξ for which $|\xi| = 0$. If $\eta = \xi + \xi_0$ where $\xi \in \mathfrak{A}$, $\xi_0 \in \mathfrak{A}_0$, it follows from the Schwarz inequality that $(\xi, \mathfrak{A}_0) = 0$, and thus $|\eta| = |\xi|$. Thus, if $\mathfrak{B}$ denotes the factor space $\mathfrak{A}/\mathfrak{A}_0$ of elements $x = \xi + \mathfrak{A}_0$, $\xi \in \mathfrak{A}$, we may define $|x| = |\xi|$. Clearly $\mathfrak{B}$ is a normed linear space under this norm. Let $\mathfrak{H}$ be the closure in $\mathfrak{B}^{**}$ of $\kappa(\mathfrak{B})$, where κ is the natural isometric imbedding of $\mathfrak{B}$ in $\mathfrak{B}^{**}$ (cf. II.3.19). It is seen from Lemma I.6.7 that $\mathfrak{H}$ is complete. We shall show $\mathfrak{H}$ is a Hilbert space. Since $(\xi, \mathfrak{A}_0) = 0$, $\xi \in \mathfrak{A}$, it follows that $(\xi, \eta) = (\xi_1, \eta_1)$ if $\xi - \xi_1$ and $\eta - \eta_1$ are in $\mathfrak{A}_0$. Thus we may define a bilinear form on $\mathfrak{B}$ by setting $(x, y) = (\xi, \eta)$ if $x = \xi + \mathfrak{A}_0$, $y = \eta + \mathfrak{A}_0$. Clearly if (x, y) is extended by continuity from $\mathfrak{B}\times\mathfrak{B}$ to $\mathfrak{H}\times\mathfrak{H}$ (cf. I.6.17), $\mathfrak{H}$ is a Hilbert space.

Now for $\xi = [\alpha_i]$ in $\mathfrak{A}$, define $T\xi = [\beta_i]$ where $\beta_0 = 0$, $\beta_{i+1} = \alpha_i$, $i \geq 1$. If $\eta = [\gamma_i]$ is in $\mathfrak{A}$, then

$$\begin{aligned}(T\xi, \eta) &= \sum_{i,j=0}^{\infty} m_{i+j}\beta_i\bar{\gamma}_j \\ &= \sum_{i=1,j=0}^{\infty} m_{i+j}\alpha_{i-1}\bar{\gamma}_j \\ &= \sum_{i,j=0}^{\infty} m_{i+j+1}\alpha_i\bar{\gamma}_j = (\xi, T\eta).\end{aligned}$$

Thus $|T\xi|^2 = (T^2\xi, \xi) = 0$ if $\xi \in \mathfrak{A}_0$, i.e., $T\mathfrak{A}_0 \subseteq \mathfrak{A}_0$. Defining $Sx = T\xi$ if $x = \xi + \mathfrak{A}_0$, we obtain a linear mapping of $\mathfrak{B}$ into itself satisfying $(Sx, y) = (x, Sy)$, $x, y \in \mathfrak{B}$. Since $\mathfrak{B}$ is dense in $\mathfrak{H}$, S is a symmetric operator in $\mathfrak{H}$.

If V is the map of $\mathfrak{A}$ into itself defined by $V[\alpha_i] = [\bar{\alpha}_i]$, then evidently $V(\xi+\eta) = V\xi + V\eta$, $V(\alpha\xi) = \bar{\alpha}V\xi$, $(V\xi, V\eta) = \overline{(\xi, \eta)}$, and $V^2 = I$. Consequently $V\mathfrak{A}_0 \subseteq \mathfrak{A}_0$, and the map $U : \mathfrak{B} \to \mathfrak{B}$, defined by setting $Ux = V\xi$ if $x = \xi + \mathfrak{A}_0$, may be extended by continuity to a map, which we shall also denote by U, of $\mathfrak{H}$ into itself which is a conjugation (cf. Definition 4.17). Since $SUx = USx$ for x in $\mathfrak{B}$, it follows from Theorem 4.18 that S has equal deficiency indices. Thus by Corollary 4.13, S has a self adjoint extension S_1.

Let ν be the sequence $[1, 0, 0, \ldots]$ in $\mathfrak{A}$ and $u = \nu + \mathfrak{A}_0 \in \mathfrak{B}$.

Since $\mathfrak{B} = \mathfrak{D}(S)$ and $S\mathfrak{B} \subseteq \mathfrak{B}$, it follows that $u \in \mathfrak{D}(S^n)$ for each n. Hence, by Theorem 2.6.

$$\int_{-\infty}^{\infty} t^{2n}(E(dt)u, u) < \infty, \qquad n \geq 0,$$

where $E(\cdot)$ denotes the resolution of the identity for S_1. Moreover,

$$(S^n u, u) = \int_{-\infty}^{\infty} t^n(E(dt)u, u), \qquad n \geq 0,$$

However, $(S^n u, u) = (T^n v, v) = m_n$, $n = 0, 1, 2, \ldots$. Thus

$$m_n = \int_{-\infty}^{\infty} t^n \mu(dt), \qquad n \geq 0,$$

where $\mu(e) = (E(e)u, u)$. Q.E.D.

With a little extra effort, the method of the preceding proof may be employed to solve the Stieltjes moment problem.

2 THEOREM. *Let m_n, $n = 0, 1, 2, \ldots$, be a sequence of real numbers. A necessary and sufficient condition that there exist a non-negative measure μ defined on the Borel sets of the interval $[0, \infty)$ such that*

$$m_n = \int_0^{\infty} t^n \mu(dt), \qquad n = 0, 1, 2, \ldots,$$

is that

[*]
$$\sum_{i,j=0}^{n} m_{i+j}\alpha_i\bar{\alpha}_j \geq 0$$

and

[**]
$$\sum_{i,j=0}^{n} m_{i+j+1}\alpha_i\bar{\alpha}_j \geq 0$$

for every finite set $\alpha_0, \ldots, \alpha_n$ of complex numbers.

PROOF. It follows as in Theorem 1 that [*] is necessary. To prove the necessity of [**], note that if $m_n = \int_0^\infty t^n \mu(dt)$, then

$$\begin{aligned}\sum_{i,j=0}^{n} m_{i+j+1}\alpha_i\bar{\alpha}_j &= \int_0^{\infty}\left(\sum_{i,j=0}^{n} t^{i+j+1}\alpha_i\bar{\alpha}_j\right)\mu(dt) \\ &= \int_0^{\infty} t\left|\sum_{i=0}^{n} t^i\alpha_i\right|^2 \mu(dt) \geq 0.\end{aligned}$$

To prove the converse, note that under hypothesis [**] the

operator T of the proof of Theorem 1 satisfies

$$(T\xi, \xi) = \sum_{i,j=0}^{\infty} m_{i+j+1}\alpha_i\bar{\alpha}_j \geqq 0$$

for each $\xi = [\alpha_i]$ in $\mathfrak{A}$. Thus $(Sx, x) \geqq 0$ for $x \in \mathfrak{B}$, so S is a non-negative symmetric operator in $\mathfrak{H}$. By Theorem 5.2 it has a non-negative self adjoint extension S_1. If $E(\cdot)$ denotes the resolution of the identity of S_1, it follows from the proof of Theorem 1 that

$$m_n = \int_{-\infty}^{\infty} t^n \mu(dt), \qquad n \geqq 0,$$

where $\mu(e) = (E(e)u, u)$. However, $E((-\infty, 0)) = 0$ by Theorem 2.9 and Lemma 7.2. Thus $\mu((-\infty, 0)) = 0$, and hence

$$m_n = \int_{0}^{\infty} t^n \mu(dt), \qquad n \geqq 0. \quad \text{Q.E.D.}$$

A modification of the same method may be used to prove the following moment theorem of Bochner.

3 THEOREM. *Let m be a complex valued function of the real variable t. In order that there exist a finite non-negative measure μ defined on the Borel sets of the real line such that*

$$m(t) = \int_{-\infty}^{\infty} e^{its}\mu(ds)$$

it is necessary and sufficient that

(a) *m is continuous;*

(b) *for each set $\alpha_1, \ldots, \alpha_n$ of complex numbers, and each set $t_1, \ldots, t_n$ of real numbers,*

$$\sum_{i,j=1}^{n} m(t_i - t_j)\alpha_i\bar{\alpha}_j \geqq 0.$$

PROOF. If m has the representation $m(t) = \int_{-\infty}^{\infty} e^{its}\mu(ds)$, the continuity of m follows from the dominated convergence theorem (III.6.16). Moreover,

$$\sum_{j,k=1}^{n} \alpha_j\bar{\alpha}_k \int_{-\infty}^{\infty} e^{i(t_j-t_k)s}\mu(ds) = \int_{-\infty}^{+\infty}\left(\sum_{j=1}^{n}\alpha_j e^{it_js}\right)\left(\sum_{k=1}^{n}\bar{\alpha}_k e^{-it_ks}\right)\mu(ds)$$

$$= \int_{-\infty}^{+\infty}\left|\sum_{j=1}^{n}\alpha_j e^{it_js}\right|^2\mu(ds) \geqq 0.$$

Thus the necessity of conditions (a) and (b) is established.

To prove the sufficiency, let $\mathfrak{A}$ be the set of all complex valued functions f of the real variable t for which $f(t) = 0$ except for a finite number of values of t. If f, g are in $\mathfrak{A}$, let $(f, g) = \sum_{s,t} m(t-s)f(t)\overline{g(s)}$; since this sum has only finitely many non-zero terms it is well defined. Then (f, g) is a Hermitian bilinear form on $\mathfrak{A}$ which, by assumption, satisfies $(f, f) \geqq 0$. Let $|f| = (f, f)^{\frac{1}{2}}$ for $f \in \mathfrak{A}$, and

$$\mathfrak{A}_0 = \{f | f \in \mathfrak{A}, |f| = 0\}.$$

It follows, just as in the proof of Theorem 1, that $\mathfrak{A}_0$ is a linear subspace of $\mathfrak{A}$ and that $(\mathfrak{A}_0, f) = (f, \mathfrak{A}_0) = 0$ for f in $\mathfrak{A}$. Defining $(x, y) = (f, g)$ for elements $x = f+\mathfrak{A}_0$ and $y = g+\mathfrak{A}_0$ of $\mathfrak{B} = \mathfrak{A}/\mathfrak{A}_0$, we obtain a Hermitian bilinear form on $\mathfrak{B}$. As in the case of Theorem 1, $\mathfrak{B}$ is a dense subspace of a Hilbert space $\mathfrak{H}$, and the inner product in $\mathfrak{H}$ is the continuous extension of the function (x, y) in $\mathfrak{B}$.

For each t let $V(t)$ be the mapping of $\mathfrak{A}$ into itself defined by $(V(t)f)(s) = f(s-t)$. Then $V(s)V(t) = V(s+t)$ for all s and t and

$$\begin{aligned}(V(t)f, V(t)g) &= \sum m(s_1-s_2)f(s_1-t)\overline{g(s_2-t)}\\ &= \sum_{s_1, s_2} m(s_1+t-s_2-t)f(s_1)\overline{g(s_2)}\\ &= (f, g),\end{aligned}$$

it follows from the Schwarz inequality that $V(t)\mathfrak{A}_0 \subseteq \mathfrak{A}_0$. Consequently, we can define a mapping $U(t) : \mathfrak{B} \to \mathfrak{B}$ by letting $U(t)x = V(t)f+\mathfrak{A}_0$, if $x = f+\mathfrak{A}_0$ is in $\mathfrak{B}$. Then clearly

$$[*] \qquad U(s+t) = U(s)U(t), \qquad (U(t)x, U(t)y) = (x, y)$$

for $-\infty < t < \infty$, $x, y \in \mathfrak{B}$. Since, in particular, $|U(t)x| = |x|$, the mapping $U(t)$ may be extended by continuity from $\mathfrak{B}$ to the space $\mathfrak{H}$ in such a way that equations [*] hold for all t and $x, y \in \mathfrak{H}$. (We shall also denote this extension by the symbol $U(t)$.) Thus the family $\{U(t)\}$, $-\infty < t < \infty$, is a group of unitary operators on $\mathfrak{H}$.

It will be shown next that the group $\{U(\cdot)\}$ is strongly continuous. Let $x = f+\mathfrak{A}_0$ be in $\mathfrak{B}$, where f is in $\mathfrak{A}$. Then

$$\begin{aligned}|U(t_1)x-U(t_2)x|^2 &= |U(|t_1-t_2|)x-x|^2\\ &= |U(|t_1-t_2|)x|^2+|x|^2-2\mathscr{R}(U(|t_1-t_2|)x, x)\\ &= 2\mathscr{R}\{|x|^2-(U(|t_1-t_2|)x, x)\}\end{aligned}$$

$$= 2\mathscr{R}\Big(\sum_{s_1, s_2} m(s_1-s_2) f(s_1)\overline{f(s_2)} - \sum_{s_1, s_2} m(s_1-s_2) f(s_1-|t_1-t_2|)\overline{f(s_2)}\Big)$$
$$= 2\mathscr{R}\Big(\sum_{s_1, s_2} [m(s_1-s_2)-m(s_1-s_2+|t_1-t_2|)] f(s_1)\overline{f(s_2)}\Big).$$

Since we are assuming $m(t)$ to be continuous,

$$|U(t_1)x-U(t_2)x| \to 0$$

as $|t_1-t_2| \to 0$ for each $x \in \mathfrak{B}$. By Theorem II.1.18, this holds not only for $x \in \mathfrak{B}$, but also for $x \in \mathfrak{H}$. Thus $U(t)$ is a strongly continuous group of unitary operators.

It now follows from Theorem 6.1 that there exists an unbounded (or possibly bounded) self adjoint operator T such that $U(t) = e^{itT}$. Let E be the spectral resolution of T, and let v be the function defined by $v(t) = 0$ for $t \neq 0$, $v(0) = 1$. If u is the element $v + \mathfrak{A}_0$ of $\mathfrak{B}$, then by Theorem 2.3,

$$\begin{aligned} m(t) = \big(V(t)v, v\big) &= \big(U(t)u, u\big) \\ &= \int_{-\infty}^{+\infty} e^{it\lambda}\big(E(d\lambda)u, u\big). \end{aligned}$$

Thus, defining $\mu(e) = (E(e)u, u)$, we have $m(t) = \int_{-\infty}^{+\infty} e^{it\lambda}\mu(d\lambda)$, and the sufficiency of conditions (a) and (b) is established. Q.E.D.

The reader will have little difficulty applying the method of the previous theorem to prove the following theorem of the same type for the interval $[0, 2\pi)$.

4 THEOREM. *Let m_n, $n = 0, \pm 1, \pm 2, \ldots$, be a sequence of complex numbers. In order that there exist a finite non-negative measure defined on the Borel sets of $[0, 2\pi)$ such that*

$$m_n = \int_0^{2\pi} e^{ins}\mu(ds), \qquad n = 0, \pm 1, \pm 2, \ldots,$$

it is necessary and sufficient that

$$\sum_{i,j=-n}^{n} m_{i-j}\alpha_i\bar{\alpha}_j \geq 0$$

for every finite sequence α_i, $-n \leq i \leq n$, of complex numbers.

9. Exercises

1 A symmetric transformation whose range is all of Hilbert space is self adjoint.

2 If T and T^* are everywhere defined, T is bounded.

3 Let T be self adjoint and B bounded. Then $BT \subseteq TB$ if and only if B commutes with the resolution of the identity for T. If $BT \subseteq TB$ then for every Borel measurable function f, $Bf(T)$ has a closure and $\overline{Bf(T)} = f(T)B$.

4 The point spectrum of a symmetric operator in a separable space is a denumerable subset of the real axis.

5 If T is a densely defined symmetric transformation,

$$[\dagger] \qquad V = (T-iI)(T+iI)^{-1}$$

is an isometric transformation (which is not necessarily everywhere defined). Show the following: The operator T is closed if and only if V is closed. The operator $I-V$ is one-to-one, has a dense range, and

$$[\dagger\dagger] \qquad T = i(I+V)(I-V)^{-1}.$$

Conversely, if V is an isometric operator such that $I-V$ is one-to-one and has a dense range, equation [††] defines a symmetric operator T in terms of which [†] holds. Any isometric extension V_1 of V is such that $I-V_1$ is one-to-one, and T is maximal among symmetric transformations if and only if V is maximal among isometric transformations.

The operator T is self adjoint if and only if V is unitary. Show finally how this construction, due to von Neumann, can be used to prove Corollary 4.13.

6 Let a maximal symmetric operator T in a Hilbert space $\mathfrak{H}$ be given. Show that $\mathfrak{H}$ may be decomposed into an orthogonal direct sum

$$\mathfrak{H} = \mathfrak{H}_0 \oplus \textstyle\sum_\alpha \mathfrak{H}_\alpha$$

of subspaces invariant under T and T^*, such that:

(a) The restriction T_0 of T to $\mathfrak{H}_0$, defined by the equations $\mathfrak{D}(T_0) = \mathfrak{D}(T) \cap \mathfrak{H}_0$, $T_0x = Tx$, $x \in \mathfrak{D}(T_0)$, is self adjoint.

(b) For each α, there exists an isometry U_α of $\mathfrak{H}_\alpha$ onto $L_2(0, \infty)$ such that if T_α denotes the restriction of T to $\mathfrak{H}_\alpha$ (defined by $\mathfrak{D}(T_\alpha) =$

$\mathfrak{D}(T) \cap \mathfrak{H}_\alpha$, $T_\alpha x = Tx$ for $x \in \mathfrak{D}(T_\alpha)$), the operator $U_\alpha T_\alpha U_\alpha^{-1}$ is the operator $\pm i\Delta$, where Δ is defined by the equations

$$\begin{aligned} \mathfrak{D}(\Delta) = \{f \in L_2(0, \infty) \mid f &\text{ is absolutely continuous} \\ f' &\in L_2(0, \infty),\ f(0) = 0\}, \\ \Delta f &= f', \qquad f \in \mathfrak{D}(\Delta). \end{aligned}$$

We have $+i\Delta$ for all α if the positive deficiency index of T is zero, $-i\Delta$ for all α if the negative deficiency index of T is zero. (Hint: See Theorem XIII.2.10 and Corollary XIII.2.12. Use the construction of Exercise 5 and decompose the maximal isometric operator V.)

7 If $\{T(t)\}_{t \geq 0}$ is a strongly continuous semi-group of operators in Hilbert space with infinitesimal generator A, then $\{T^*(t)\}_{t \geq 0}$ is a strongly continuous semi-group with infinitesimal generator A^*.

8 (Cooper) The infinitesimal generator of a strongly continuous semi-group $\{V(t)\}$ of partially isometric operators whose initial domains are $\mathfrak{H}$ is of the form iT, where T is a maximal symmetric operator with positive deficiency index zero. Conversely, any such maximal symmetric operator is the infinitesimal generator of a strongly continuous semi-group of partially isometric operators with initial domain $\mathfrak{H}$.

Hilbert space $\mathfrak{H}$ may be decomposed into an orthogonal direct sum $\mathfrak{H} = \mathfrak{H}_0 \oplus \sum_\alpha \mathfrak{H}_\alpha$ in such a way that:

(a) The restriction of $V(t)$ to $\mathfrak{H}_0$ is unitary, $t \geq 0$.

(b) For each α there exists an isometry U_α of $\mathfrak{H}_\alpha$ onto $L_2(0, \infty)$ such that

$$\begin{aligned} (U_\alpha(V(t)|\mathfrak{H}_\alpha)U_\alpha^{-1})f(x) &= 0, \qquad 0 \leq x < t, \\ &= f(x-t), \qquad x \geq t. \end{aligned}$$

Establish the corresponding result for a strongly continuous semi-group of partially isometric operators whose final domains are $\mathfrak{H}$.

9 An unbounded transformation T in $\mathfrak{H}$ is *normal* if T is closed, densely defined, and $TT^* = T^*T$. Show that

(a) if T is normal, T^* is normal;

(b) a transformation T is normal if and only if $\mathfrak{D}(T) = \mathfrak{D}(T^*)$ and $|Tx| = |T^*x|$ for each x in $\mathfrak{D}(T)$;

(c) a normal transformation has no proper normal extensions.

(cf. Sz.-Nagy, *Spektraldarstellung linearer Transformationen des Hilbertschen Raumes*, pp. 33, 34.)

10 If $T = UH \subseteq HU$ where H is self adjoint and U unitary, then T is normal and $UH = HU$. Conversely, if T is normal, then $T = UH = HU$ where U is unitary and H is self adjoint and positive.

11 A closed and densely defined operator T is normal if and only if $\mathfrak{D}(T) = \mathfrak{D}(T^*)$ and $\overline{T+T^*}$ and $i(\overline{T-T^*})$ are self adjoint and have commuting resolutions of the identity. An operator T is normal if and only if $T = A+iB$ where A and B are self adjoint and have commuting resolutions of the identity.

12 T is normal if and only if there exists a resolution of the identity E in the complex plane P such that

$$\mathfrak{D}(T) = \left\{x \,\Big|\, \lim_{n\to\infty}\int_{|\lambda|\leq n} \lambda E(d\lambda)x \text{ exists}\right\}$$

and

$$Tx = \lim_{n\to\infty}\int_{|\lambda|\leq n} \lambda E(d\lambda)x, \qquad x\in\mathfrak{D}(T).$$

13 A normal transformation has no residual spectrum.

14 A strongly continuous semi-group $\{N_t\}$ of bounded normal operators can be written in the form

$$[*] \qquad N_t = \int_P e^{zt}E(dz)$$

where $E(\cdot)$ is a spectral measure defined on the Borel sets of the complex plane P. If $\{N_t\}$ consists of self adjoint operators then $E(\cdot)$ may be chosen so that the integral in [*] need be carried out only over the real axis, and then $\{N_t\}$ is continuous in the uniform topology for $t > 0$.

15 Let T be closed. The operators T^*T and TT^* are unitarily equivalent if and only if

$$\dim\{x|Tx = 0\} = \dim\{x|T^*x = 0\}.$$

More generally, let

$$\mathfrak{H}_1 = \{x|Tx = 0\}^\perp, \qquad \mathfrak{H}_2 = \{x|T^*x = 0\}^\perp.$$

Then there exists an isometric mapping U of $\mathfrak{H}_1$ onto $\mathfrak{H}_2$, such that $UT^*TU^{-1}x = TT^*x$, $x\in\mathfrak{H}_2$.

16 Show that the equation in Exercise 15

$$\dim\{x|Tx = 0\} = \dim\{x|T^*x = 0\},$$

does not always hold.

17 (Schmidt) The non-zero eigenvalues of T^*T are the same as the non-zero eigenvalues of TT^*, even as to multiplicity (the positive square roots of these eigenvalues are sometimes called the *characteristic numbers* of T).

18 Let T be a bounded operator. Then T is compact if and only if T^*T is compact and T^*T is compact if and only TT^* is compact. If T is compact and $\{\lambda_1, \lambda_2, \ldots\}$ is the sequence of non-zero characteristic numbers of T, arranged in decreasing order and each repeated a number of times equal to the multiplicity of its square as an eigenvalue of TT^*, then

$$\lambda_n = \min_{\varphi_1, \ldots, \varphi_{n-1}} \max_{\substack{(\psi, \varphi_i)=0 \\ i=1, \ldots, n-1}} \frac{|T\psi|}{|\psi|}.$$

19 The positive real number λ is a characteristic number of T if and only if there exist non-zero vectors φ and ψ such that

$$T\varphi = \lambda\psi, \qquad T^*\psi = \lambda\varphi.$$

20 Let T be compact and $\{\lambda_i\}$ the sequence of non-zero characteristic numbers of T arranged in decreasing order, each repeated a number of times equal to the multiplicity of its square as an eigenvalue of TT^*. Let $\{\varphi_i\}$ be a corresponding sequence of orthonormal eigenvectors of TT^*. Then there exists a corresponding sequence $\{\psi_i\}$ of orthonormal eigenvectors of T^*T such that for each f,

$$T^*f = \sum_{i=1}^{\infty} \lambda_i (f, \varphi_i)\psi_i,$$

$$Tf = \sum_{i=1}^{\infty} \lambda_i (f, \psi_i)\varphi_i,$$

the series converging in the strong topology.

21 (Naĭmark) There exists a closed, densely defined symmetric operator T such that $\mathfrak{D}(T^2) = \{0\}$.

22 Let $T = PA$ be the canonical factorization of T. Then T is

one-to-one if and only if A is positive definite, in which case P is an isometry. The operator T^* is also one-to-one if and only if P is unitary.

28 If an operator T has a closed linear extension there exists a unique closed linear extension $\bar{T}$ such that if T_1 is any closed linear extension of T then $\bar{T} \subseteq T_1$. $\bar{T}$ is called the *closure* of T.

(a) There exists a densely defined operator with no closed linear extension.

(b) An operator T with dense domain has a closed linear extension if and only if its adjoint is densely defined, in which case $(\bar{T})^* = T^*$.

24 Give examples of closed symmetric operators having a given pair (m, n) of cardinal numbers as deficiency indices.

25 (Naĭmark) Let T be a closed symmetric operator in a Hilbert space $\mathfrak{H}$. There exists a Hilbert space $\mathfrak{H}_1 \supseteq \mathfrak{H}$, and a self adjoint operator T_1 in $\mathfrak{H}_1$ such that

$$\mathfrak{D}(T) = \mathfrak{D}(T_1) \cap \mathfrak{H}, \qquad T_1 x = Tx, \qquad x \in \mathfrak{D}(T).$$

(Hint: Let $\mathfrak{H}_1$ be the direct sum of $\mathfrak{H}$ and its "complex conjugate"!)

26 (Naĭmark) Let T be as in Exercise 25. Then there exists a set function $F(\cdot)$ defined for the Borel subsets of the real axis, having values which are positive bounded Hermitian operators, such that

(a) $F(\cdot)x$ is countably additive for each $x \in \mathfrak{H}$,

(b) $\mathfrak{D}(T) = \{x \mid \int_{-\infty}^{\infty} |\lambda|^2 (F(d\lambda)x, x) < \infty\}$,

(c) $Tx = \int_{-\infty}^{\infty} \lambda F(d\lambda)x, \qquad x \in \mathfrak{D}(T)$,

the infinite integral converging as a proper value in the strong topology.

27 (Naĭmark) Let $F(\cdot)$ be a set function defined on a σ-field Σ of subsets of a set S, having values which are positive bounded Hermitian operators of norm less than or equal to one, in a Hilbert space $\mathfrak{H}$, such that $F(\cdot)x$ is countably additive for each $x \in \mathfrak{H}$. Show that there exists a Hilbert space $\mathfrak{H}_1$ containing $\mathfrak{H}$, and a countably additive set function $E(\cdot)$ defined on Σ whose values are orthogonal projections in $\mathfrak{H}_1$ such that

$$F(e)x = PE(e)x, \qquad e \in \Sigma, \qquad x \in \mathfrak{H},$$

P denoting the orthogonal projection of $\mathfrak{H}_1$ on $\mathfrak{H}$. (Hint: First consider

the case in which $\mathfrak{H}$ is one-dimensional, and then generalize the solution given in this case.)

28 Let a self adjoint operator A in a Hilbert space $\mathfrak{H}$ with $0 \leq A \leq I$ be given. Then there exists a Hilbert space $\mathfrak{H}_1 \supseteq \mathfrak{H}$, and an orthogonal projection Q in $\mathfrak{H}_1$ such that

$$Ax = PQx, \qquad x \in \mathfrak{H},$$

P denoting the orthogonal projection of $\mathfrak{H}_1$ on $\mathfrak{H}$.

29 Let $\{T_n\}$ be a sequence of bounded operators in Hilbert space $\mathfrak{H}$. Then there exists a Hilbert space $\mathfrak{H}_1 \supseteq \mathfrak{H}$, and a sequence $\{N_n\}$ of commuting normal operators in $\mathfrak{H}_1$ such that

$$T_n x = PN_n x, \qquad x \in \mathfrak{H},$$

P denoting the orthogonal projection of $\mathfrak{H}_1$ onto $\mathfrak{H}$. (Hint: Consider real and imaginary parts separately and use Exercise 27).

30 Let $\{A_n\}$ be a sequence of bounded self adjoint transformations in Hilbert space $\mathfrak{H}$. Suppose that there exists a constant M, $0 < M < \infty$, such that

$$a_0 I + a_1 A_1 + \ldots + a_n A_n \geqq 0$$

whenever the real polynomial

$$a_0 + a_1 \lambda + \ldots + a_n \lambda^n$$

is non-negative in the interval $[-M, M]$. Then there exists a Hilbert space $\mathfrak{H}_1 \supseteq \mathfrak{H}$, and a Hermitian operator B in $\mathfrak{H}_1$ such that

$$A_n x = PB^n x, \qquad x \in \mathfrak{H},$$

P denoting the orthogonal projection of $\mathfrak{H}_1$ on $\mathfrak{H}$. (Hint: Represent A_n as $\int_{-M}^{M} \lambda^n F(d\lambda)$ and use Exercise 27.)

31 (Sz.-Nagy) Let $\{A_n\}$, $-\infty < n < +\infty$, be a uniformly bounded sequence of operators in Hilbert space $\mathfrak{H}$ such that

$$\sum_{n=-\infty}^{+\infty} r^{|n|} e^{in\theta} A_n \geq 0, \qquad 0 \leqq \theta \leqq 2\pi, \qquad 0 \leqq r < 1.$$

Then there exists a Hilbert space $\mathfrak{H}_1 \supseteq \mathfrak{H}$, and a unitary operator U in $\mathfrak{H}_1$, such that

$$A_n x = cPU^n x, \qquad x \in \mathfrak{H}, \qquad -\infty < n < +\infty,$$

where P denotes the orthogonal projection of $\mathfrak{H}_1$ on $\mathfrak{H}$ and c is a positive constant. (Hint: Modify the argument of Exercise 30.)

32 (Sz. Nagy) Let T be an operator in $\mathfrak{H}$ such that $|T| \leq 1$. Then there exists a Hilbert space $\mathfrak{H}_1 \supseteq \mathfrak{H}$, and a unitary operator U in $\mathfrak{H}_1$, such that

$$T^n x = PU^n x, \qquad x \in \mathfrak{H}, \qquad 0 \leq n < \infty,$$

P denoting the orthogonal projection of $\mathfrak{H}_1$ on $\mathfrak{H}$.

33 (von Neumann) Let $u(z)$ be an analytic function defined for $|z| < r$, where $r > 1$, and such that $|u(z)| \leq 1$ if $|z| \leq 1$. Let T be an operator in Hilbert space such that $|T| \leq 1$. Then $|u(T)| \leq 1$. Similarly, if $\mathscr{R}u(z) \geq 0$ for $|z| \leq 1$, then $u(T) + (u(T))^* \geq 0$. (Hint: Use Exercise 32.)

34 There exist self adjoint operators A and B such that $\mathfrak{D}(A) \cap \mathfrak{D}(B) = \{0\}$. Thus, neither $A+B$ nor $AB+BA$ need to be self adjoint if A and B are. Even AB can happen to be defined only for (0).

35 The set $\{(Tx, x) | |x| = 1, x \in \mathfrak{D}(T)\}$ is convex.

36 (Sz. Nagy) Let Γ be the graph of the closed operator T, and E_Γ the orthogonal projection of $\mathfrak{H} \oplus \mathfrak{H}$ on Γ. Then

$$E_\Gamma[x, 0] = [(I+T^*T)^{-1}x, T(I+T^*T)^{-1}x].$$

37 (Sz.-Nagy) If A_n is self adjoint for all $n \geq 1$, if $A_n x \to A_\infty x$ for all x in $\mathfrak{D}(A_\infty)$, and if the closure A of A_∞ is self adjoint, then

$$(I+A_n^2)^{-1} \to (I+A^2)^{-1} \qquad \text{strongly},$$
$$A_n(I+A_n^2)^{-1} \to A(I+A^2)^{-1} \qquad \text{strongly}.$$

If we have

$$\lim_{n\to\infty} \sup_{x \in \mathfrak{D}(A_\infty)} \frac{|A_n x - A_\infty x|^2}{|x|^2 + |A_\infty x|^2} = 0$$

then

$$(I+A_n^2)^{-1} \to (I+A^2)^{-1} \qquad \text{uniformly},$$
$$A_n(I+A_n^2)^{-1} \to A(I+A^2)^{-1} \qquad \text{uniformly}.$$

38 (Rellich) Let A_n be self adjoint for all $n \geq 1$, and let $A_n x \to A_\infty x$ for all x in $\mathfrak{D}(A_\infty)$. Suppose that the closure A of A_∞

is self adjoint and has resolution of the identity $E(\cdot)$. Then $f(A_n) \to f(A)$ strongly for each bounded function f of a real variable which is continuous except on a closed set C such that $E(C) = 0$. If

$$\lim_{n\to\infty} \sup_{x \in \mathfrak{D}(A_\infty)} \frac{|A_n x - A_\infty x|^2}{|x|^2 + |A_\infty x|^2} = 0,$$

$g(A_n) \to g(A)$ uniformly for each bounded continuous function g of a real variable.

39 There do not exist two bounded Hermitian operators A and B such that $AB - BA = iI$. But unbounded Hermitian operators such that $ABx - BAx = ix$ for all x in a dense subset of Hilbert space do exist.

40 (Uncertainty Principle, Heisenberg) Let A and B be self adjoint operators in Hilbert space such that $\mathfrak{D}_0 = \mathfrak{D}(AB) \cap \mathfrak{D}(BA)$ is dense. Let x be in $\mathfrak{D}_0$ and write

$$E(C) = (Cx, x), \qquad \sigma^2(C) = |(C - E(C)I)x|^2$$

for each operator C for which Cx is defined. Then

$$\sigma^2(A)\sigma^2(B) \geq \tfrac{1}{4}|(E(AB - BA))|^2.$$

41 (Bodiou) With the hypothesis and notations of the preceding exercise,

$$\sigma^2(A)\sigma^2(B) \geq \tfrac{1}{4}|(E(AB - BA))|^2 + \tfrac{1}{4}|E(D)|^2,$$

where

$$D = (A - E(A)I)(B - E(B)I) + (B - E(B)I)(A - E(A)I).$$

Generalizations of a Theorem of Paley and Wiener

42 Let $\{x_i\}$ and $\{y_i\}$ be sequences of elements in a B-space. If there exists a number θ with $0 \leq \theta < 1$ and such that

$$\left|\sum_{i=1}^{n} \alpha_i(x_i - y_i)\right| \leq \theta \left|\sum_{i=1}^{n} \alpha_i x_i\right|$$

for all finite sequences of scalars $\alpha_1, \ldots, \alpha_n$, and if $\{x_i\}$ is fundamental, then $\{y_i\}$ is fundamental. If $\{x_i\}$ is a basis, then $\{y_i\}$ is a basis.

43 Let $\{x_i\}$ and $\{y_i\}$ be sequences of elements in a B-space. If there exists a θ such that $0 \leq \theta < \frac{1}{3}$, and

$$|\sum_{i=1}^{n}\alpha_i(x_i - y_i)| < \theta\{|\sum_{i=1}^{n}\alpha_i x_i| + |\sum_{i=1}^{n}\alpha_i y_i|\},$$

for each sequence of scalars $\alpha_1, \ldots, \alpha_n$, then (x_i) is fundamental if and only if (y_i) is fundamental, and (x_i) is a basis if and only if (y_i) is a basis.

44 (Pollard-Sz.-Nagy) Let (x_i) and (y_i) be sequences of elements in Hilbert space. If there exist $\theta_1, \theta_2, \theta_3$ such that $0 \leq \theta_1 < 1$, $0 \leqq \theta_3 < 1$, $0 \leqq \theta_2^2 < (1-\theta_1)(1-\theta_3)$, and such that

$$\begin{aligned}|\sum_{i=1}^{n}\alpha_i(x_i - y_i)|^2 \leq \theta_1 & |\sum_{i=1}^{n}\alpha_i x_i|^2 \\ & + 2\theta_2|\sum_{i=1}^{n}\alpha_i x_i|\,|\sum_{i=1}^{n}\alpha_i y_i| \\ & + \theta_3|\sum_{i=1}^{n}\alpha_i y_i|^2\end{aligned}$$

for all finite sequences of scalars $\alpha_1, \ldots, \alpha_n$, then (x_i) is fundamental if and only if (y_i) is fundamental, and (x_i) is a basis if and only if (y_i) is a basis.

45 (Pollard) Let (x_i) and (y_i) be orthonormal sequences of elements in Hilbert space. Suppose that there exists a $\mu > 0$ such that

$$\sum_{i \neq j=1}^{n}\alpha_i\alpha_j(x_i, y_j) \geq \mu\sum_{i=1}^{n}|\alpha_i|^2$$

for all finite sequences $\alpha_1, \ldots, \alpha_n$ of scalars. Then (x_i) is fundamental if and only if (y_i) is fundamental. (Hint: Use Exercise 44).

46 (Duffin-Eachus) Let (x_n) and (y_n) be sequences of elements in Hilbert space, and suppose that (x_n) is complete and orthonormal. Suppose that

$$|\sum_{n=1}^{\infty}\alpha_n(x_n - y_n)|^2 < \sum_{n=1}^{\infty}|\alpha_n|^2$$

for all sequences (α_n) of scalars for which $\sum_{n=1}^{\infty}|\alpha_n|^2 < \infty$. Then (y_n) is fundamental.

47 (Duffin-Eachus) Let (x_n) be a complete orthonormal set in Hilbert space. Let (T_n) be a sequence of bounded operators, and let α_{nk} be a double sequence of scalars such that $|\alpha_{nk}| \leq c_k$, $n, k \geq 1$. Let $\sum_{k=1}^{\infty}c_k|T_k| < 1$. Then

$$y_n = x_n + \sum_{k=1}^{\infty} \alpha_{nk} T_k x_n$$

defines a fundamental sequence (y_n).

48 Let (y_i) be a sequence of elements in Hilbert space and let θ be a number such that $0 \leq \theta < 1$. Suppose that

$$(1-\theta)^2 \sum_{i=1}^{n} |\alpha_i|^2 \leq |\sum_{i=1}^{n} \alpha_i y_i|^2 \leq (1+\theta)^2 \sum_{i=1}^{n} |\alpha_i|^2$$

for each finite sequence $\alpha_1, \ldots, \alpha_n$ of scalars. Then there exists an orthonormal set (x_i) of vectors such that

$$|\sum_{i=1}^{n} \alpha_i (y_i - x_i)| \leq \theta (\sum_{i=1}^{n} |\alpha_i|^2)^{\frac{1}{2}}$$

for each finite sequence $\alpha_1, \ldots, \alpha_n$ of scalars. (Hint: Apply the canonical factorization theorem to a suitable map).

49 Let (λ_n) be a sequence of complex numbers such that $|\lambda_n - n| \leq \pi^{-1} \log 2$, $-\infty < n < +\infty$. Then $(e^{i\lambda_n x})$ is fundamental in $L_2(0, 2\pi)$. (Hint: Write $e^{i\lambda_n x} = e^{inx}(e^{i(\lambda_n - n)x})$, and expand the second factor as a Taylor series). (It is not known to the present authors if $\pi^{-1} \log 2$ is the best possible constant. Levinson [1], p. 48, Theorem 29, shows that the constant cannot be greater than or equal to $\frac{1}{4}$).

50 (Walsh-Boas) Let $g_n(z)$ be analytic in $|z| < 1$, and suppose that

$$\sum_{n=0}^{\infty} |g_n(z) - z^n|^2 \leq \theta < 1$$

for each z such that $|z| < 1$. Then each function f analytic for $|z| < 1$ and continuous for $|z| \leq 1$ has a unique expansion

$$f(z) = \sum_{n=0}^{\infty} a_n g_n(z),$$

the series converging uniformly in each interior circle.

51 Let (S, Σ, μ) be a measure space, and T a self adjoint operator in $L_2(S, \Sigma, \mu)$. Let $e \in \Sigma$ be a set of finite measure such that each function f in the range of T is essentially bounded on e. Show that for each Borel subset σ of the real axis which is at a positive distance

from the point $\lambda = 0$, there exists a $\mu\times\mu$ measurable function $E(\sigma; s, t)$ of the variable $[s, t]$, defined in $e\times S$, satisfying the condition

$$\int |E(\sigma; s, t)|^2\mu(dt) < \infty, \qquad s\in e,$$

and such that

$$\int E(\sigma; s, t)f(t)\mu(dt) = (E(\sigma; T)f)(s)$$

for almost all $s\in e$.

(Hint: Use the methods of Theorem XII.8.11).

The mapping $f\to f(T)$ where f is analytic on the spectrum of T has, in certain cases, been extended to give a definition of $f(T)$, f being a real harmonic function. Such an extension has been made by Ciprian Foias, (cf. C. Foias, La mesure harmonique-spectrale et la théorie spectrale des opérateurs généraux d'un espace de Hilbert. *Bull. Soc. Math. France*, 85, 263–282 (1957)). The following two exercises present a special case of the Foias extension.

52 (Foias) Let T be an operator in the Hilbert space $\mathfrak{H}$ with $|T|\leq 1$, let $\mathscr{H}_0$ be the real algebra of real functions u defined and harmonic on a domain $D(u)$ containing the closed unit disc $(\lambda | |\lambda|\leq 1)$, and let $\mathscr{H}$ be the real algebra of all real functions continuous on this disc and harmonic in its interior. Let both algebras $\mathscr{H}_0$ and $\mathscr{H}$ be ordered by defining $u\geq v$ to mean that $u(\lambda)\geq v(\lambda)$ for $|\lambda|\leq 1$. Let $\mathscr{A}(\mathfrak{H})$ be the real ordered algebra of all bounded self adjoint operators on $\mathfrak{H}$. For an operator S in $\mathfrak{H}$ let $\mathscr{R}(S) = (S+S^*)/2$. Show that for a function f, analytic on a domain including the unit disc, the operator $\mathscr{R}(f(T))$ depends only upon the function $\mathscr{R}f(\lambda)$. Thus for u in $\mathscr{H}_0$ we may define $u(T) = \mathscr{R}(f(T))$.

(a) Show that the map $u\to u(T)$ is an order-preserving linear map of the real algebra $\mathscr{H}_0$ into the real algebra $\mathscr{A}(\mathfrak{H})$, and that

$$|u(T)|\leq \max_{|\lambda|\leq 1} |u(\lambda)|.$$

(Hint: Use Exercise 32.)

(b) Extend the homomorphism of part (a) to an order-preserving linear map of the real algebra $\mathscr{H}$ into $\mathscr{A}(\mathfrak{H})$ which has the property that

$$|u(T)|\leq \max_{|\lambda|\leq 1} |u(\lambda)|, \qquad u\in\mathscr{H}.$$

(Hint: Use the fact that every u in $\mathscr{H}$ is the uniform limit on the unit disc of a sequence in $\mathscr{H}_0$. Cf. M. Brélot, Sur l'approximation et la convergence dans la théorie des functions harmoniques ou holomorphes, *Bull. Soc. Math. France*, 73, 71–73 (1945)).

53 (Foias) For the homomorphism $\mathscr{H} \to \mathscr{A}(\mathfrak{H})$ of Exercise 52 show that

(a) $\inf_{|\lambda|\leq 1} u(\lambda) \leq \inf_{|x|\leq 1} (u(T)x, x) \leq \sup_{|x|=1} (u(T)x, x) \leq \sup_{|\lambda|\leq 1} u(\lambda).$

(b) Let λ be a complex number and (x_n) a sequence in $\mathfrak{H}$ with $|x_n| = 1$ and $(\lambda I - T)x_n \to 0$. Then

$$(u(T)x_n, x_n) \to u(\lambda).$$

(c) If λ is in $\sigma_p(T)$ and x_λ is a vector with $|x_\lambda| = 1$ and $Tx_\lambda = \lambda x_\lambda$ then

$$(u(T)x_\lambda, x_\lambda) = u(\lambda).$$

10. Notes and Remarks

The spectral theorem. The spectral theory of bounded self adjoint operators in Hilbert space was essentially created by Hilbert [1; IV], although, as we have noted in Section X.9, he expressed his results in terms of quadratic forms. Somewhat closer in spirit and terminology to the present work than that of Hilbert is the book of F. Riesz [6]. The first significant advance towards the analysis of unbounded symmetric operators was made, in 1923, by Carleman [1] in his study of singular integral equations. However, it was several years later that von Neumann [8], in 1927, evidently motivated by the development of quantum mechanics, turned his attention to the decomposition of unbounded self adjoint operators. His fundamental paper (von Neumann [7]) develops the theory of unbounded operators in a very complete and systematic fashion — this paper is truly a classic. Von Neumann was soon joined in his work by Stone [8, 10]. F. Riesz [14], in 1930, gave an elegant elementary proof of the spectral theorem for unbounded self adjoint operators. Since that time a number of different proofs and discussions of the spectral theorem have been given. The reader should consult Section X. 9 for references to these papers.

It should be noted that the work of Carleman showed that symmetry alone is not a sufficient condition to obtain a complete extension of the spectral theorem to unbounded operators. According to von Neumann [7; p. 72], the notion of an unbounded self adjoint operator is due to Erhard Schmidt, who observed (von Neumann [7; p. 62]) that it is necessary to restrict one's attention to such operators in order to obtain a spectral resolution. The reader should be warned that von Neumann and other writers use the terms Hermitian and hypermaximal Hermitian for what we call symmetric and self adjoint, respectively.

Consideration of the graph of an operator is due to von Neumann [16], although he had already obtained the results of Section 1 directly. The notion of the graph of an operator from one Hilbert space to another was also used systematically by Murray [4]. The fact that an everywhere defined symmetric operator is bounded and self adjoint is essentially due to Hellinger and Toeplitz [1]; see also Stone [3; p. 59], Stone and Tamarkin [1], and von Neumann [7; p. 107].

Lemmas 2.1 and 2.2 were given explicitly by Stone [3; p. 142, 145–6]. For Theorem 2.3, see von Neumann [7; p. 92], F. Riesz [14; p. 51] and Stone [3; p. 180], and other references cited earlier. The operational calculus, described in Theorems 2.6 and 2.7 is due to Stone [3; Chap. VI, Sec. 2] and von Neumann [16]. Theorem 2.10 goes back to the work of Stieltjes and was employed by Hellinger [1] for bounded operators and by Stone [3; p. 168, 183] for unbounded operators.

Spectral representation. The problem of determining when two self adjoint operators are unitarily equivalent is closely related to the theories of spectral representation and spectral multiplicity. The first part of Section 3 has close contact with Stone [3; Chap. VII], which extends to unbounded operators the theory of Hellinger [1] and Hahn [5] on bounded forms. For a related development, see Ahiezer and Glazman [1; Secs. 69—73]. Additional references are Halmos [6], Nakano [10, 11], Plessner and Rohlin [1], Segal [5] and Wecken [2].

The analytical representation as given in Section 3 is due to Bade and Schwartz [1]. It is a slight improvement on a result of Mautner [1] on abstract eigenfunction expansions. Mautner's result has also been elucidated and applied by Gårding [1] and F. Browder

[1]. These theories will be discussed in the course of the following chapters on differential equations.

Extensions of symmetric operators. The problem of determining whether a given symmetric operator has a self adjoint extension is of crucial importance in determining whether the spectral theorem may be employed. If the answer to this problem is affirmative, it is important to know what the self adjoint extensions look like and how they are related to the original operator. These rather intricate problems are treated in Section 4. The principal theorems of the first half of this section are essentially due to von Neumann [7] and are also discussed in Stone [3; Chap. IX]. The method used here, however, is much closer in spirit to that of Calkin [1]. This method has the advantage that it provides an abstract setting for the selection of self adjoint extensions by suitable restrictions (i.e., imposing "boundary conditions") on the domain of the adjoint operator. The reader will see, however, that the actual development presented here is substantially different in detail from that in Calkin [1].

In Section 4 we considered a symmetric operator T with domain dense in $\mathfrak{H}$ and have been concerned with finding extensions of T which are self adjoint operators in $\mathfrak{H}$. There are at least two ways in which this problem can be generalized. The first way is to drop the requirement that the domain of T is dense, and to suppose that T is symmetric in the sense that $(Tx, y) = (x, Ty)$ for all x and y in $\mathfrak{D}(T)$. This problem has been considered for bounded operators by Kreĭn [9] and for unbounded operators by Krasnosel'skiĭ [1, 2, 4]. Another possible extension is to search for self adjoint extensions but to allow the extended operator to act in a Hilbert space containing the original one. In Section X.9 we discussed some related problems, considered by Naĭmark [8], Sz.-Nagy [11] and others. It can be shown that any symmetric operator with arbitrary deficiency indices has a self adjoint extension in some larger Hilbert space (see Naĭmark [7, 8] or Sz.-Nagy [11; Sec. 2]). This implies that a symmetric operator can be decomposed in a form resembling the spectral theorem, but the analogy, of course, is not complete. For an extended discussion of these matters the reader should consult the papers cited and Appendix I in Ahiezer and Glazman [1].

Extension by the Cayley transform. Since it is an important and

frequently used device, it is appropriate that we give a brief sketch indicating how the Cayley transform can be used to determine when a symmetric operator has a self adjoint extension. Let T be a symmetric operator with domain $\mathfrak{D}(T)$ dense in $\mathfrak{H}$. Then if x is in $\mathfrak{D}(T)$, we have

$$|(T \pm iI)x|^2 = (Tx, Tx) \mp i(x, Tx) \pm i(Tx, x) + (x, x)$$
$$= |Tx|^2 + |x|^2 \geqq |x|^2.$$

This shows that if $(T \pm iI)x = 0$, then $x = 0$ and so the operators $T \pm iI$ have inverses. Let V be the operator with domain $\mathfrak{D}(V) = (T+iI)\mathfrak{D}(T)$ and satisfying

$$Vy = (T-iI)(T+iI)^{-1}y, \qquad y \in \mathfrak{D}(V).$$

The operator V is called the *Cayley transform* of T. If y is in $\mathfrak{D}(V)$, then let x be the element in $\mathfrak{D}(T)$ such that $y = (T+iI)x$; hence $x = (T+iI)^{-1}y$. Since $|(T+iI)x|^2 = |(T-iI)x|^2$ for all x in $\mathfrak{D}(T)$, we conclude that

$$|y|^2 = |(T+iI)x|^2 = |(T-iI)x|^2 = |Vy|^2$$

for all y in $\mathfrak{D}(V)$. This shows that the Cayley transform of a symmetric operator is *isometric*, but not generally everywhere defined or invertible. Conversely, if W is any isometric operator for which $(I-W)\mathfrak{D}(W)$ is dense, and if S is defined by

$$Sx = i(I+W)(I-W)^{-1}x, \qquad x \in (I-W)\mathfrak{D}(W),$$

then S is a symmetric operator with domain $(I-W)\mathfrak{D}(W)$. Further, if this last relation is applied when $W = V$, the Cayley transform of T, then $S = T$. Hence there is a one-to-one correspondence between symmetric operators T with dense domains and isometric operators V with $(I-V)\mathfrak{D}(V)$ dense in $\mathfrak{H}$. The operator T is closed if and only if V is closed; also, if $T_1 \subseteq T_2$, then their Cayley transforms are related by $V_1 \subseteq V_2$, and conversely. Finally, T is self adjoint if and only if V is unitary.

It is seen, therefore, that the problem of finding self adjoint extensions of a symmetric operator T may be treated by finding unitary extensions of the Cayley transform of T. For convenience, suppose that T is closed, then $\mathfrak{D}(V)$ and $\mathfrak{R}(V)$ are closed subspaces.

Let $\mathfrak{D}_+ = \mathfrak{H} \ominus \mathfrak{D}(V)$ and $\mathfrak{D}_- = \mathfrak{H} \ominus \mathfrak{R}(V)$, and denote the dimensions of these subspaces by d_+ and d_-, respectively. It may be proved that $\mathfrak{D}_+$ and $\mathfrak{D}_-$ are the same manifolds introduced in Definition 4.9, and that an isometric operator V is unitary if and only if $\mathfrak{D}(V) = \mathfrak{H} = \mathfrak{R}(V)$, i.e., if $d_+ = 0 = d_-$. Also it is clear that a closed isometric operator has a unitary extension if and only if there is an isometric mapping of $\mathfrak{D}_+$ onto $\mathfrak{D}_-$, and that this can happen if and only if $d_+ = d_-$.

This elegant extension procedure is due to von Neumann [7], and has also been used by Stone [3; Chap. IX], Riesz and Sz.-Nagy [1; Sec. 123] and Ahiezer and Glazman [1; Secs. 78—80].

Maximal symmetric operators. If T is a symmetric operator with dense domain, then it has proper symmetric extensions provided both of its deficiency indices are different from zero. A *maximal symmetric operator* is one which has no proper symmetric extensions; hence, a closed symmetric operator is maximal if at least one of its deficiency indices is zero. If both are zero, then it is self adjoint, while if $d_+ \neq 0$, $d_- = 0$, or if $d_+ = 0$, $d_- \neq 0$, then the operator is not self adjoint and has no self adjoint extensions (within the same Hilbert space). It is an interesting fact, due to von Neumann [7; p. 98], (see also Stone [8; p. 851] and Ahiezer and Glazman [1; Sec. 82]) that the maximal symmetric operators may be put into a standard form, which we shall now indicate.

Let $\mathfrak{H}$ be a separable Hilbert space, let $\{x_1, x_2, \ldots\}$ be a complete orthonormal system in $\mathfrak{H}$, and let V_1 be the shift operation defined in $\mathfrak{H}$ by $V_1(\sum_{k=1}^{\infty} c_k x_k) = \sum_{k=1}^{\infty} c_k x_{k+1}$. It may be seen that V_1 is isometric and $\mathfrak{D}(I - V_1)$ is dense in $\mathfrak{H}$ so that V_1 is the Cayley transform of a symmetric operator T_1 whose deficiency indices are $d_+ = 0$, $d_- = 1$. The operator T_1 is called an *elementary symmetric operator*. It may be proved that if T is maximal symmetric with indices $d_+ = 0$, $d_- = n$ (where n is any cardinal number), then $\mathfrak{H}$ may be broken into a direct sum of pairwise orthogonal subspaces $\mathfrak{H}_0, \mathfrak{H}_1, \ldots, \mathfrak{H}_n$ such that the operator T is self adjoint in $\mathfrak{H}_0$ and is an elementary symmetric operator in the spaces $\mathfrak{H}_1, \ldots, \mathfrak{H}_n$. The case that $d_+ = n$, $d_- = 0$, may be handled by considering the operator $-T$ which has indices 0, n, or by considering the operator shifting in the opposite direction.

Deficiency indices. Theorem 4.19 shows that the notion of the

deficiency indices is not dependent on the numbers $\pm i$ used in their definition. Weyl [5] showed this to be the case for differential operators. If T is a linear operator with dense domain, let $\gamma(T)$ be the set of all complex numbers λ such that the inverse operator $(T-\lambda I)^{-1}$ exists and is bounded on its domain. The set $\gamma(T)$ is called the *domain of regularity* of T (or the set of *points of regular type*) and contains the resolvent set $\rho(T)$ and perhaps some of the residual spectrum. It may be shown to be an open set, and by a continuation argument it is proved (see Ahiezer and Glazman [1; Sec. 78]) that the dimension of the deficiency spaces $\mathfrak{H} \ominus (T-\lambda I)\mathfrak{H}$ is constant on each connected component of $\gamma(T)$. (This result is due to Kreĭn and Krasnosel'skiĭ [2]. It shows that if T is a symmetric operator with at least one real number in $\gamma(T)$, then the deficiency indices are equal. This latter result was established by Calkin [8].)

Semi-bounded operators. Von Neumann [7; p. 103] proved that a semi-bounded symmetric operator can be extended to a self adjoint operator with arbitrarily small change in the bound. He conjectured that no increase in the bound was actually necessary and this conjecture was established by Stone [8; p. 388] and Friedrichs [3; I]. The simplification, given by Freudenthal [8], of Friedrichs' proof is the one presented in the text. For another proof of the theorem, see Calkin [3] and Eberlein [2; p. 699], and for applications to partial differential equations, consult Friedrichs [8].

The Friedrichs extension determines a specific extension of a symmetric semi-bounded operator. Kreĭn [9] has made a systematic study of all the extensions of a semi-bounded operator, and also given applications to differential equations. Kreĭn's approach is similar to the Cayley transform method and is briefly discussed in Riesz and Sz.-Nagy [1; Sec. 125].

Unitary semi-groups (see also the remarks in Section VIII.10). Theorem 6.1 was announced, in 1930, by Stone [10; III] in the case of a group of unitary operators. Many proofs of this celebrated and important theorem have been given; for example, Stone [16], von Neumann [10], Bochner [7], F. Riesz [22], Sz.-Nagy [14], Nakano [17] and Cooper [8]. See also Riesz and Sz.-Nagy [1; Secs. 137–140] where two proofs are given, including that of Bochner (which is based on his well-known moment theorem) and Ahiezer and Glazman

[1; Sec. 62]. For additional references, see Hille [1; Chap. IX]. For application of Stone's theorem to ergodic theory and quantum mechanics, consult von Neumann [20] and Maeda [1]. Plessner [1] has obtained an analogue of Stone's theorem for a semi-group of operators satisfying $U^*U = I$, but not necessarily $UU^* = I$.

Theorems giving representations for groups and semi-groups of self adjoint, normal, and isometric operators may be found in Hille [1; Chap. XIX], Riesz and Sz.-Nagy [1; Sec. 141] and Sz.-Nagy [3; pp. 73—76] [14].

Stone's theorem has been extended to unitary representations of locally compact Abelian groups by Naimark [1], Ambrose [4], Godement [5], Arnous [1], and Phillips [5]. (See also Loomis [1; p. 147]).

The canonical factorization. The results and techniques of Section 7 are due to von Neumann [16]. The reader may also consult Riesz and Sz.-Nagy [1; Sec. 110] for the bounded and Stone [3; pp. 329—333] and Sz.-Nagy [3; pp. 52—53] for the unbounded case. We remark that Lemma 7.1 shows that if T is a closed operator with dense domain, then T^*T has a dense domain. This is to be contrasted with an example of Naïmark [6] of a closed symmetric operator T with dense domain but such that the domain of T^2 is $\{0\}$.

Moment theorems. For a detailed study of various moment theorems, historical remarks, and many references, the reader should consult the excellent book of Shohat and Tamarkin [1]. A briefer discussion is given in Widder [1; Chap. III]. For relations between the Riesz representation theorem for $C[0,1]$ and the moment theorem, see Hildebrandt [8] and Hildebrandt and Schoenberg [1].

Theorem 8.3 is due to Bochner [6; Sec. 20] and is of considerable importance in harmonic analysis. It is also proved in E. Hopf [1; Sec. 4] and Ahiezer and Glazman [1; Sec. 60]. Nakano [18] proved this result using Stone's theorem. The closely related Theorem 8.4 is due to Herglotz [1]; Riesz and Sz. Nagy [1; Sec. 53] have given an elegant proof of this result using the lemma of Fejér and F. Riesz. Generalizations of Bochner's theorem to locally compact groups have been given in Weil [1; Sec. 30], Raikov [3], Cartan and Godement [1], Godement [2], and Loomis [1; p. 142].

Jacobi Matrices and the Moment Problem

The investigations of the moment problem made in Section 8 can be carried considerably farther by applying the theory of unbounded symmetric operators in Hilbert space to *Jacobi matrices.* An infinite matrix $\{a_{jk}\}$, $j, k \geqq 0$, is said to be a Jacobi matrix if

$$\text{(i)} \qquad a_{pq} = \bar{a}_{qp}, \qquad \text{all } p, q,$$

$$\text{(ii)} \qquad a_{pq} = 0, \qquad |p-q| > 1.$$

Such a matrix defines an unbounded operator A in the sequential Hilbert space l_2 as follows.

$$\text{(a)} \quad \mathfrak{D}(A) = \{x = [x_i] \in l_2 | \sum_{p=1}^{\infty} |a_{p,p-1}x_{p-1} + a_{p,p}x_p + a_{p,p+1}x_{p+1}|^2 < \infty\},$$

$$\text{(b)} \qquad Ax = [a_{p,p-1}x_{p-1} + a_{p,p}x_p + a_{p,p+1}x_{p+1}].$$

In (a) and (b) we have put $a_{0,-1} = 0$. Then it may be shown that A is closed, that its adjoint A^* is symmetric, and that the deficiency indices of A^* are either (1,1) or (0,0). The sequence of polynomials P_n defined inductively by the formulae

$$P_{-1}(t) = 0, \;\; P_0(t) = 1,$$
$$a_{k,k+1}P_{k+1}(t) = -a_{k,k-1}P_{k-1}(t) + (t - a_{k,k})P_k(t)$$

is called the sequence of polynomials associated with the matrix $\{a_{jk}\}$. Then it may be shown that the deficiency indices of A^* are (1,1) if and only if

$$\sum_{n=0}^{\infty} |P_n(z)|^2 < \infty$$

for each non-real z, and are (0, 0) if and only if

$$\sum_{n=0}^{\infty} |P_n(z)|^2 = \infty$$

for each non-real z.

Suppose now that a sequence of constants c_j, $j \geqq 0$ is given,

and that this sequence can be represented in the form

$$[*] \qquad c_j = \int_{-\infty}^{\infty} t^j \mu(dt), \qquad g \geqq 0$$

where μ is a positive Borel measure on the real axis such that all the integrals

$$\int_{-\infty}^{+\infty} |t|^{2j} \mu(dt), \qquad g \geqq 0,$$

converge. Is the measure μ unique? This fundamental problem of the theory of moments may be answered as follows. Let $\{P_n(t)\}$ be the sequence of polynomials determined by orthonormalizing the sequence $1, t, t^2, \ldots$ of elementary polynomials with respect to μ. That is, let P_n be the sequence of polynomials determined by the conditions

(i) P_n is a polynomial of order n with a positive leading coefficient;

(ii)
$$\int_{-\infty}^{+\infty} P_n(t)\overline{P_m(t)}\mu(dt) = 1 \qquad n = m$$
$$= 0 \qquad n \neq m.$$

Then it is clear that the matrix $\{a_{jk}\}$ determined by the formula

$$a_{jk} = \int_{-\infty}^{+\infty} tP_j(t)\overline{P_k(t)}\mu(dt), \qquad j, k, \geqq 0,$$

is a Jacobi matrix, and it is easy to see that $\{P_j\}$ is the sequence of polynomials associated with this Jacobi matrix. It may then be shown that the equations [$*$] determine the positive measure μ uniquely if and only if A^* has deficiency indices (0,0). If A^* has deficiency indices (1,1), the family of all the positive measures μ satisfying [$*$] may be constructed from the set of all self adjoint extensions of A^*.

For a very concise and clear exposition of the theory sketched here, see Ahiezer [1]. A more comprehensive and extended treatment is found in Stone [3; p. 530—614]. Additional results, generalizations, connections with the theory of continued fractions, etc. are to be found in the monograph of Shohat and Tamarkin [1].

Miscellaneous Remarks

A number of special results are available for Hermitian operators defined by kernels $K(x, y)$ satisfying the inequality

$$\int_S |K(x, y)|^2 \mu(dy) < \infty$$

for μ-almost all x. For an account of this theory, due to Carleman, see Carleman [1], Stone [8] p. 397–424. Cf. also Exercise 9.51.

An operator K in Hilbert space is called *symmetrizable* relative to a Hermitian operator H if HK is self adjoint. A number of the properties of symmetric transformations may be extended to symmetrizable transformations. See, in particular, Zaanen [5], especially p. 370–391.

CHAPTER XIII

Ordinary Differential Operators

1. Introduction: Elementary Properties of Formal Differential Operators

From the point of view of applications, the most important single class of operators are the differential operators. The study of these operators is complicated by the fact that they are necessarily unbounded. Consequently, the problem of choosing a domain for a differential operator is by no means trivial; the study of symmetric unbounded operators in Section XII.4 indicates that for unbounded operators, the choice of domains can be quite crucial.

In more detail, the situation is this. We are given, to begin with, a "formal differential operator," i.e., an expression of the form

$$\tau = a_n(t)\left(\frac{d}{dt}\right)^n + a_{n-1}(t)\left(\frac{d}{dt}\right)^{n-1} + \cdots + a_0(t).$$

There is an obvious sense in which such an expression can be "applied" to a function f, if, say, f belongs to C^n. Thus, we can define an operator whose domain is C^n (but whose range is not in C^n, only in C). We might, however, have "applied" the formal differential operator to a larger or a smaller class of functions to begin with, and defined operators with larger or smaller domains. How can we tell which definition is the most advantageous?

A number of guiding principles suggest themselves. First, in order to be able to apply the abstract Hilbert-space theory of Chapter XII, it is necessary to relate all operators to a Hilbert-space of square-integrable functions. Once this is done, Chapter XII indicates that the question of adjoints can be expected to assume paramount importance. That is, having settled on any preliminary domain of functions to which the formal differential operator τ can be applied, we must attempt to discover the adjoint of the resulting unbounded

operator in Hilbert space. It will be seen below that all these problems permit satisfactory solutions.

One more word about another peculiarity of the theory of this chapter. In a number of cases the essence of a proof will not be its analytic but its formal side. For this reason we will occasionally give the formal details of the proof, and leave the details of the analysis to the reader. The details omitted should not be hard to supply.

In this whole chapter, the letter I will denote an interval of the real axis. The interval I can be open, half-open, or closed. The interval $[a, \infty)$ is considered to be half-open; the interval $(-\infty, +\infty)$ to be open. Thus a closed interval is a compact set. An end point t of I that is not in I is called a *free end point* of I. In this definition we permit t to be $\pm\infty$. Thus, the real axis has two free end points, the interval $[0,1)$ has one free end point. An end point of I which belongs to I is called a *fixed end point*.

The spaces $C^n(J)$, where J is a compact interval, have been defined in Chapter IV. If f is in $C^n(J)$, let us agree to write the norm of f as $|f|^{(n)}$ whenever it is desired to emphasize the integer n. The space

$$C^\infty(J) = \bigcap_{n=0}^{\infty} C^n(J)$$

of functions differentiable to an arbitrarily high order will play a role in the considerations below. If we put

$$|f|^{(\infty)} = \sum_{n=0}^{\infty} 2^{-n} \frac{|f|^{(n)}}{1+|f|^{(n)}},$$

$C^\infty(J)$ becomes an F-space. If I is an interval which is not compact, we shall define the space $C^n(I)$, where $n = \infty$ is permissible, as the collection of all functions f defined on I whose restriction $f|J$ to any compact subinterval J of I belongs to $C^n(J)$. If $\{J_k\}$ is an increasing subsequence of compact subintervals of I whose union is I, then we put

$$|f|^{(n)} = \sum_{k=0}^{\infty} 2^{-k} \frac{|(f|J_k)|^{(n)}}{1+|(f|J_k)|^{(n)}}.$$

With this definition, $C^n(I)$ is an F-space in general, and a B-space in case $n < \infty$ and I is compact. The norm just introduced defines

a topology for $C^n(I)$, which will normally be referred to as *the* topology for $C^n(I)$. It is nearly evident that the topology for $C^n(I)$ is independent of the particular sequence $\{J_k\}$ of compact subintervals of I used in its definition.

1 DEFINITION. *A formal differential operator of order* n on the interval I is an expression

$$\tau = a_n(t)\left(\frac{d}{dt}\right)^n + a_{n-1}(t)\left(\frac{d}{dt}\right)^{n-1} + \cdots + a_0(t),$$
$$= \sum_{i=0}^{n} a_i(t)\left(\frac{d}{dt}\right)^i,$$

such that the complex-valued functions a_i, called the *coefficient functions*, belong to $C^\infty(I)$, and such that the function a_n, called the *leading coefficient*, is not zero at any point of I.

If the coefficients of τ are in $C^\infty(I)$, but the leading coefficient a_n is allowed to vanish at some point in I, τ will be called an *irregular formal differential operator*. If it is desired to emphasize the distinction between the case in which a_n is allowed to vanish and the opposite case, a formal differential operator may sometimes be referred to as a *regular formal differential operator*.

The reason that we require $a_n(t) \neq 0$ in the definition of a (regular) formal differential operator will be made clearer in the proof of Theorem 8 below.

2 DEFINITION. By $A^n(I)$ we denote the space of all functions f which have $(n-1)$ continuous derivatives in I, and for which $f^{(n-1)}$ is not only continuous but also absolutely continuous over each compact subinterval of I. Thus $f^{(n)}$ exists almost everywhere, and is integrable over any compact subinterval of I. If a point c is a fixed end point of I, the statement that f has a continuous derivative in I, or (in case I is compact) belongs to $C^1(I)$, means that f' is continuous from the left (or right) at c.

It should be remarked that if τ is a formal differential operator of order n on the interval I, and $f \in A^n(I)$, then the expression τf has a definite meaning in a very natural sense: we put

$$(\tau f)(t) = a_n(t)\left(\frac{d}{dt}\right)^n f(t) + \ldots + a_0(t)f(t).$$

The function τf is defined almost everywhere, and integrable over every closed subinterval of I.

When the operator τ is written in terms of a classical notation such as d^n/dt^n we shall follow custom and write $(d^n/dt^n)f(t)$ or $(d/dt)^n f(t)$ rather than $(d^n f/dt^n)(t)$. Frequently the symbol $f^{(n)}(t)$ will be used in place of $(d^n/dt^n)f(t)$.

Basic analytic information on the existence and uniqueness of solutions of differential equations is stated in the following theorem.

8 THEOREM. *Let τ be a formal differential operator of order n on the interval I. Suppose that g is a measurable complex-valued function integrable over every compact subinterval of I. Let $t_0 \in I$, and let $c_0, c_1, \ldots, c_{n-1}$ be an arbitrary set of n complex numbers. Then there exists a unique $f \in A^n(I)$ such that*

(a) $$\tau f = g,$$

(b) $$\left(\frac{d}{dt}\right)^i f(t_0) = c_i, \qquad i = 0, \ldots, n-1.$$

PROOF. We shall give the formal part of the proof, leaving various analytic gaps which are to be filled in by the reader. First of all, consider the case in which I is closed. Introduce the space of functions $F(t) = [f_0(t), f_1(t), \ldots, f_{n-1}(t)]$ with values in n-dimensional complex Euclidean space E^n. Then replace equation (a) by the system of equations

$$\begin{aligned}
&\frac{d}{dt} f_0(t) - f_1(t) = 0\\
&\frac{d}{dt} f_1(t) - f_2(t) = 0\\
\text{(a')}\quad &\cdot \qquad\quad \cdot\ \cdot\ \cdot\\
&\frac{d}{dt} f_{n-1}(t) + \frac{1}{a_n(t)}\{a_{n-1}(t)f_{n-1}(t) + \cdots + a_0(t)f_0(t)\}\\
&\qquad\qquad = \frac{1}{a_n(t)}\cdot g(t).
\end{aligned}$$

A moment's consideration will reveal that there is a one-to-one correspondence between solutions of the nth order equation (a) and solutions of the system of equations (a'). Now, the system (a') can be written in more condensed form as

$$\frac{d}{dt}F(t)+A(t)F(t) = G(t),$$

where $F(t) = [f_0(t), \ldots, f_{n-1}(t)]$, $G(t) = [0, \ldots, 0, [a_n(t)]^{-1}g(t)]$ and $A(t)$ is the linear transformation in E^n defined by the matrix $A_{ij}(t)$, where

$$\begin{aligned} A_{ij}(t) &= \delta_{i+1,j} && 0 \leqq i < n-1, \quad 0 \leqq j \leqq n-1, \\ A_{n-1,j}(t) &= a_n(t)^{-1}a_j(t), && 0 \leqq j \leqq n-1. \end{aligned}$$

In the same way, the boundary condition (b) is equivalent to the condition

$$\text{(b')} \qquad F(t_0) = C,$$

where C is the vector $C = [c_0, c_1, \ldots, c_{n-1}]$. Now (a') and (b') are together equivalent to the integral relation

$$\text{(e)} \qquad F(t) + \int_{t_0}^{t} A(s)F(s)ds = C + \int_{t_0}^{t} G(s)ds.$$

If we put $C + \int_{t_0}^{t} G(s)ds = H(t)$, and introduce the operator Φ in the space $\{L_1(I)\}^n$ of all vector-valued functions $Y(t) = [y_0(t), \ldots, y_{n-1}(t)]$ whose components $y_i(t)$ are integrable over I by the definition $(\Phi Y)(t) = \int_{t_0}^{t} A(s)Y(s)ds$, then (e) can be written in the form

$$\text{(e')} \qquad (I+\Phi)F = H.$$

We have, inductively,

$$\begin{aligned} |(\Phi^k Y)(t)| &= \left|\int_{t_0}^{t} A(s)(\Phi^{k-1}Y)(s)ds\right| \\ &\leqq \frac{K^{k-1}|Y|}{(k-1)!}\int_{t_0}^{t} |A(s)|\,|s-t_0|^{k-1}ds \\ &\leqq \frac{K^k|Y|}{k!}|t-t_0|^k, \end{aligned}$$

where we have taken $|v|$ to denote the norm of a vector v in Euclidean n-space, and $|A|$ to denote the norm of an operator A in Euclidean n-space, and where $K = \max_{t \in I} |A(t)|$. Thus, if we use the norm

$$|Y| = \int_I |Y(t)| dt$$

in the B-space $\{L_1(I)\}^n$, we have $|\Phi^k| \leq K_1^n/k!$, where $K_1 = K \cdot \max_{t \in I} |t-t_0|$. Thus, equation (e′) has the unique solution (cf. Lemma VII.3.4)

$$F = (I+\Phi)^{-1}H = \sum_{j=0}^{\infty} (-1)^j \Phi^j H.$$

Since all the terms in equation (e) but the first are absolutely continuous, it follows that F is absolutely continuous. Thus Theorem 1 is proved for the special case in which I is a closed interval.

Now suppose that I is not known to be closed. It still follows from the proof given above that if J is any closed subinterval of I containing the point t_0, there exists a unique function $f_J \in A^n(J)$ such that

(a) $\quad f_J(t) = g(t) \quad$ almost everywhere in J,

(b) $\quad \left(\dfrac{d}{dt}\right)^i f_J(t_0) = c_i, \quad i = 0, \ldots, n-1.$

From the uniqueness of f_J it is obvious that $f_{J_1}(t) = f_{J_2}(t)$ for $t \in J_1 \cap J_2$. Thus, by putting

$$f(t) = f_J(t), \qquad t \in J.$$

for J an arbitrary closed subinterval of I, we define a single valued function $f \in A^n(I)$ satisfying (a) and (b). That f is unique follows immediately from the corresponding result for closed subintervals J of I. Q.E.D.

4 Corollary. *If g has k continuous derivatives in I, then f has $n+k$ continuous derivatives in I.*

Proof. We may clearly suppose without loss of generality that I is bounded and closed. Then

$$\left(\frac{d}{dt}\right)^n f(t) = a_n(t)^{-1} \sum_{i=0}^{n-1} a_i(t) \left(\frac{d}{dt}\right)^i f(t) + g(t).$$

Since $f \in A^n(I)$, all the terms on the right belong to $C(I)$, and it follows that $f \in C^n(I)$. Hence all the terms on the right belong to $C^1(I)$, so that $f \in C^{n+1}(I)$. It is clear that we can continue this inductive argument so as to obtain the final conclusion $f \in C^{n+k}(I)$. Q.E.D.

5 COROLLARY. *Suppose that the coefficient functions a_i of the formal differential operator τ, the function g, and the initial values c_i all depend continuously (or analytically) on one or more additional parameters:*

$$a_i = a_i(t, \lambda), \qquad g = g(t, \lambda), \qquad c_i = c_i(\lambda).$$

Then the solution $f = f(t, \lambda)$ and its first $n-1$ derivatives will depend continuously (or analytically) on the parameters λ uniformly for t in any compact subinterval of I.

PROOF. We will consider only the case in which I is closed, leaving the other case to the reader. We use the notations of Theorem 3. It follows readily from our hypothesis that both the element $H = H(\lambda)$ of $\{L_1(I)\}^n$ and the operator $\Phi = \Phi(\lambda)$ depend continuously (analytically) on the parameters λ. Hence the vector $F(\lambda) \in \{L_1(I)\}^n$ which is defined by the equation

$$\begin{aligned} F(\lambda) &= \big(I+\Phi(\lambda)\big)^{-1}H(\lambda) \\ &= \sum_{j=0}^{\infty} (-1)^j \big(\Phi(\lambda)\big)^j H(\lambda) \end{aligned}$$

is also a continuous (analytic) function of λ. Since

$$F(t, \lambda) = -\int_{t_0}^{t} A(s, \lambda)F(s, \lambda)ds + H(t, \lambda),$$

it follows that $F(t, \lambda)$ depends continuously (analytically) on λ, uniformly for $t \in I$. Let $F(t, \lambda) = [f_0(t, \lambda), \ldots, f_{n-1}(t, \lambda)]$. Since $f_i(t, \lambda) = (d^i/dt^i)f(t, \lambda)$, $i = 0, \ldots, n-1$, it follows that both $f(t, \lambda)$ and its first $n-1$ derivatives are continuous (analytic) functions of λ, uniformly for $t \in I$. Q.E.D.

If we study the *homogeneous* equation $\tau f = 0$, then it is clear that the one-to-one correspondence between the n-tuple $[c_0, \ldots, c_{n-1}]$ of complex numbers and the solution of $\tau f = 0$ such that $f^{(i)}(t_0) = c_i$,

$i = 0, \ldots, n-1$, is linear. Thus we see: *the set of solutions of an nth order homogeneous linear differential equation form an n-dimensional linear vector space.*

2. Adjoints and Boundary Values of Differential Operators

Throughout this section τ will be a formal differential operator of order n on an interval I. Unless the contrary is explicitly stated, we presume τ to be regular. Our objective will be to define linear operators in the space $L_2(I)$ which correspond to τ and to study their adjoints and extensions. However, before this can be done, it will be necessary to formulate the notion of the adjoint τ^* of the (regular or irregular) formal differential operator τ and to prove a fundamental formula, known as Green's formula, which relates τ and τ^*.

For the sake of simplicity let us suppose first that I is a finite closed interval $[a, b]$ and that f and g are functions in $C^n(I)$. Consider the integral

$$\int_a^b (\tau f)(t)\overline{g(t)}dt = \sum_{k=0}^n \int_a^b a_k(t) \left[\left(\frac{d}{dt}\right)^k f(t)\right] \overline{g(t)}dt.$$

Integrating the kth term on the right by parts k-times we see that

$$\begin{aligned}
\int_a^b \left[\left(\frac{d}{dt}\right)^k f(t)\right] a_k(t)\overline{g(t)}dt
&= -\int_a^b \left[\left(\frac{d}{dt}\right)^{k-1} f(t)\right]\left[\left(\frac{d}{dt}\right) \{a_k(t)\overline{g(t)}\}\right] dt \\
&\quad + a_k(t)\overline{g(t)} \left(\frac{d}{dt}\right)^{k-1} f(t)\Big|_a^b = \ldots \\
&= (-1)^k \int_a^b f(t) \left(\frac{d}{dt}\right)^k \{a_k(t)\overline{g(t)}\}dt \\
&\quad + \sum_{i=1}^k (-1)^{i-1} \left[\left(\frac{d}{dt}\right)^{i-1} \{a_k(t)\overline{g(t)}\}\right] \left(\frac{d}{dt}\right)^{k-i} f(t)\Big|_a^b.
\end{aligned}$$

Thus

$$[*] \qquad \int_a^b (\tau f)(t)\overline{g(t)}dt = \int_a^b f(t) \sum_{k=0}^n (-1)^k \left(\frac{d}{dt}\right)^k \{a_k(t)\overline{g(t)}\}dt + F_b(f, g) - F_a(f, g)$$

where

$$F_t(f, g) = \sum_{k=1}^{n} \sum_{i=1}^{k} (-1)^{i-1} \left[\left(\frac{d}{dt}\right)^{i-1} \{a_k(t)\overline{g(t)}\} \left(\frac{d}{dt}\right)^{k-i} f(t) \right].$$

However, by using Leibniz' rule

$$\left(\frac{d}{dt}\right)^{j} \{a_k(t)\overline{g(t)}\} = \sum_{l=0}^{j} \binom{j}{l} \left[\left(\frac{d}{dt}\right)^{j-l} a_k(t) \right] \left[\left(\frac{d}{dt}\right)^{l} \overline{g(t)} \right],$$

it is seen that the integral on the right in [$*$] may be written in the form $\int_a^b f(t)\overline{\tau^* g(t)}dt$ where τ^* is the operator

$$\tau^* = \sum_{j=0}^{n} b_j(t) \left(\frac{d}{dt}\right)^{j}$$

and

$$b_j(t) = \sum_{k=j}^{n} (-1)^k \binom{k}{j} \left(\frac{d}{dt}\right)^{k-j} \overline{a_k(t)}.$$

Observe that if τ is a (regular) formal differential operator, then $b_n(t) = (-1)^n \overline{a_n(t)} \neq 0$, so τ^* is a (regular) formal differential operator.

Applying Leibniz' rule to the boundary term we obtain the formula

$$\begin{aligned} &F_t(f, g) \\ &= \sum_{0 \leq j < i \leq k \leq n} (-1)^{i-1} \binom{i-1}{j} \left[\left(\frac{d}{dt}\right)^{k-i} f(t) \right] \left[\left(\frac{d}{dt}\right)^{j} \overline{g(t)} \right] \left[\left(\frac{d}{dt}\right)^{i-j-1} a_k(t) \right] \\ &= \sum_{\substack{0 \leq j < i \leq n \\ 0 \leq l \leq n-i}} (-1)^{i-1} \binom{i-1}{j} \left[\left(\frac{d}{dt}\right)^{i-j-1} a_{l+i}(t) \right] f^{(l)}(t)\overline{g^{(j)}(t)} \\ &= \sum_{j=0}^{n-1} \sum_{l=0}^{n-j-1} \sum_{i=j}^{n-l-1} (-1)^i \binom{i}{j} \left[\left(\frac{d}{dt}\right)^{i-j} a_{l+i+1}(t) \right] f^{(l)}(t)\overline{g^{(j)}(t)}. \end{aligned}$$

If the $n \times n$ square matrix $\{F_t^{lj}(\tau)\}$ is defined for $0 \leqq l, j \leqq n-1$ by the equations

$$F_t^{lj}(\tau) = \sum_{i=j}^{n-l-1} (-1)^i \binom{i}{j} \left(\frac{d}{dt}\right)^{i-j} a_{l+i+1}(t), \qquad j+l \leqq n-1,$$

$$F_t^{lj}(\tau) = 0, \qquad j+l > n-1,$$

then the boundary term may be written as

$$F_t(f,g) = \sum_{j,l=0}^{n-1} F_t^{lj}(\tau) f^{(l)}(t)\overline{g^{(j)}(t)}.$$

Having made these preliminary remarks, we are in a position to make certain basic definitions.

1 Definition. Let τ be a (regular or irregular) formal differential operator on an interval I (which is not necessarily closed). The matrix $F_t^{lj}(\tau)$ is called the *boundary matrix* for τ at the point $t \in I$. The bilinear expression

$$F_t(f,g) = \sum_{j,l=0}^{n-1} F_t^{lj}(\tau) f^{(l)}(t)\overline{g^{(j)}(t)}$$

is called the *boundary form* for τ at the point t. The (regular or irregular) formal differential operator

$$\tau^* = \sum_{j=0}^{n} b_j(t)\left(\frac{d}{dt}\right)^j$$

where

$$b_j(t) = \sum_{k=j}^{n} (-1)^k \binom{k}{j}\left(\frac{d}{dt}\right)^{k-j}\overline{a_k(t)},$$

is called the *formal adjoint* of τ. If $\tau = \tau^*$ then τ is said to be *formally self adjoint* or *formally symmetric*. If all the *coefficients* a_j of τ are real, τ is said to be *real*.

2 Lemma. *If τ is a (regular) formal differential operator, the boundary matrix for τ is non-singular.*

Proof. This follows from the fact that

$$F_t^{lj}(\tau) = 0, \qquad j+l > n-1$$

$$F_t^{lj}(\tau) = (-1)^j a_n(t), \qquad j+l = n-1,$$

showing that the determinant of $\{F_t^{lj}(\tau)\}$ is $(\pm 1)\{a_n(t)\}^n$, and hence that it never vanishes. Q.E.D.

3 Definition. Let τ be a regular or irregular formal differential operator of order n on the interval I. Let $H_\tau^n(I)$ denote the set of all functions f in $A^n(I)$ such that f and τf belong to $L_2(I)$, and let $H^n(I)$ denote those functions f in $A^n(I)$ such that f and $f^{(n)}$ are in $L_2(I)$.

If I is a closed interval and $a \in C^\infty(I)$, $f \in L_2(I)$ then $af \in L_2$. Hence if $f \in H^n(I)$ then $\tau f = \sum_{i=0}^n a_i f^{(i)} \in L_2$, that is, $H^n(I) \subset H_\tau^n(I)$. If τ is regular and f is in $H_\tau^n(I)$ then $f^{(n)} = a_n^{-1}(\tau f - \sum_{i=0}^{n-1} a_i f^{(i)})$ is in L_2, and therefore $H_\tau^n(I)$ and $H^n(I)$ coincide.

4 THEOREM. (*Green's formula*) *Let τ be a regular or irregular formal differential operator of order n on the finite closed interval $I = [a, b]$. If $f, g \in H_\tau^n(I)$, then*

$$\int_a^b (\tau f)(t)\overline{g(t)}dt = \int_a^b f(t)\overline{(\tau^* g)(t)}dt + F_b(f, g) - F_a(f, g).$$

PROOF. In the discussion above, this formula was established for the case $f, g \in C^n(I)$. However, the arguments are equally valid for $f, g \in H_\tau^n(I)$. Q.E.D.

It will be convenient for what follows to record other situations in which Green's formula is valid but where I is not assumed to be closed.

5 COROLLARY. *If I is an arbitrary interval, Green's formula is valid for each pair of functions $f, g \in H_\tau^n(I)$ $\big($or even $f \in H^n(I), g \in A^n(I)\big)$, provided that either f or g vanishes outside a compact subinterval of I.*

PROOF. As in the case of Theorem 4, the proof depends merely on the possibility of integrating by parts, which is assured under the stated hypotheses. Q.E.D.

We next observe an important property of the formal adjoint of a (regular or irregular) formal differential operator τ.

6 LEMMA. *Let τ be a (regular or irregular) formal differential operator on the interval I. Then $\tau = (\tau^*)^*$.*

PROOF. If τ is of order n, the coefficient of $(d/dt)^i$ in $(\tau^*)^*$ is

$$\begin{aligned} c_i(t) &= \sum_{k=i}^n (-1)^k \binom{k}{i} \left(\frac{d}{dt}\right)^{k-i} \left(\sum_{j=k}^n (-1)^j \binom{j}{k} \left(\frac{d}{dt}\right)^{j-k} a_j(t) \right) \\ &= \sum_{\substack{j,k \\ i \leq k \leq j \leq n}} (-1)^j(-1)^k \binom{k}{i} \binom{j}{k} \left(\frac{d}{dt}\right)^{j-i} a_j(t) \\ &= \sum_{j=i}^n \left\{ \sum_{k=i}^j (-1)^k \binom{j}{k} \binom{k}{i} \right\} (-1)^j \left(\frac{d}{dt}\right)^{j-i} a_j(t). \end{aligned}$$

By expanding $x^j = (1-(1-x))^j$ with two applications of the binomial theorem it is seen that

$$\sum_{k=i}^{j} (-1)^k \binom{j}{k}\binom{k}{i} = \begin{cases} 0, & j \neq i \\ (-1)^i, & j = i. \end{cases}$$

Thus $c_i(t) = a_i(t)$ which shows that $(\tau^*)^* = \tau$. Q.E.D.

A number of other principles belonging to the formal algebra of formal differential operators are worth mentioning although we shall not make much use of them; for this reason they will not be numbered as theorems and lemmas. They will be valid for both regular and irregular formal differential operators.

If $\tau_1 = \sum_{i=0}^{n} a_i(t)(d/dt)^i$ and $\tau_2 = \sum_{i=0}^{n} b_i(t)(d/dt)^i$ are two formal differential operators, we can define their sum as the formal differential operator $\tau_1+\tau_2 = \sum_{i=0}^{n}\big(a_i(t)+b_i(t)\big)(d/dt)^i$. Since we wish to define the product of τ_1 and τ_2 in such a way that $\tau_1(\tau_2 f) = (\tau_1\tau_2)f$, we are led by Leibniz' rule to put

$$\tau_1\tau_2 = \sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{k=0}^{i}\binom{i}{k} a_i(t) b_j^{(i-k)}(t)\left(\frac{d}{dt}\right)^{j+k}.$$

It may readily be verified that this multiplication is associative and distributive, but not commutative. In addition, we have (as may readily be verified) the two laws

$$(\tau_1+\tau_2)^* = \tau_1^*+\tau_2^*; \qquad (\tau_1\tau_2)^* = \tau_2^*\tau_1^*.$$

Thus, since $(d/dt)^* = -(d/dt)$, we have $\big((d/dt)^i\big)^* = (-1)^i(d/dt)^i$. Since the adjoint of the zero order formal differential operator $\tau_0 = a(t)$ of order zero is $\tau_0^* = \overline{a(t)}$, we have consequently

$$\tau_1^* = \left(\sum_{i=0}^{n} a_i(t)\left(\frac{d}{dt}\right)^i\right)^* = \sum_{i=0}^{n}(-1)^i\left(\frac{d}{dt}\right)^i\overline{a_i(t)}.$$

The formula given in Definition 1 for τ_1^* is evidently the expanded form of the right hand term of this equation obtained by Leibniz' rule.

Making use of the concept of product and sum for formal differential operators, we may write formal differential operators in such forms as $(d/dt)p(t)(d/dt)+q(t)$, $\sum_{i=0}^{n}(-1)^i(d/dt)^i p_i(t)(d/dt)^i$, etc. Note

that since $\tau_1^* \tau_2^* = (\tau_2\tau_1)^*$, the operator

$$\sum_{i=0}^{n} (-1)^i \left(\frac{d}{dt}\right)^i p_i(t) \left(\frac{d}{dt}\right)^i$$

is formally self adjoint provided only that the coefficients p_i are real. In the same way, the formal differential operator $(i/2)(d/dt)^n\{p(t)(d/dt)+(d/dt)p(t)\}(d/dt)^n$ is formally self adjoint provided that $p(t)$ is a real function. If we use these observations inductively, we can give a closed form for the most general formally symmetric formal differential operator of order n. Indeed, let τ be such an operator, and let its leading coefficient be a_n. Then the leading coefficient in τ^* is $(-1)^n\overline{a_n}$; thus, if n is even, a_n is real, while if n is odd, a_n is pure imaginary. If n is even, then $\tau_1 = (d/dt)^{n/2}a_n(t)$ $(d/dt)^{n/2}$ is a formally self adjoint differential operator with the same leading coefficient as τ. If n is odd, then $\tau_2 = (i/2)(d/dt)^{(n-1)/2}$ $\big((d/dt)a_n(t)+a_n(t)(d/dt)\big)(d/dt)^{(n-1)/2}$ is a formally self adjoint differential operator with the same leading coefficient as τ. Thus, either $\tau-\tau_1$ or $\tau-\tau_2$ is a formally self adjoint differential operator of order $n-1$. Continuing this reduction process inductively, we find the following result:

Any formally self adjoint formal differential operator τ of order n can be written in the form

$$\tau = \sum_{j=0}^{[n/2]} (-1)^j \left(\frac{d}{dt}\right)^j a_j(t) \left(\frac{d}{dt}\right)^j + i \sum_{j=0}^{[(n-1)/2]} \left(\frac{d}{dt}\right)^j \left[\left(\frac{d}{dt}\right) b_j(t) + b_j(t)\left(\frac{d}{dt}\right)\right] \left(\frac{d}{dt}\right)^j$$

where the coefficients a_j and b_j are real.

It may be shown that this representation is unique. Conversely, every such formal operator is formally self adjoint.

It is easily seen that if the formal differential operator τ is real, then the complex second term of the above operator vanishes. Thus a real self adjoint formal differential operator is of even order $n = 2m$, and the general form of such an operator is

$$\tau = \sum_{j=0}^{m} (-1)^j \left(\frac{d}{dt}\right)^j a_j(t) \left(\frac{d}{dt}\right)^j.$$

the coefficients a_j being real. If $n = 2$, we have the so-called *Sturm-Liouville* operator

$$= -\left(\frac{d}{dt}\right) p(t) \left(\frac{d}{dt}\right) + q(t),$$

where the coefficients p and q are real.

Our next objective will be to define linear operators in $L_2(I)$ corresponding to the formal differential operator τ and investigate their adjoints and extensions. Later sections will deal with the spectral properties of these operators.

7 DEFINITION. Let $H_0^n(I)$ denote the set of all functions in $H^n(I)$ each of which vanishes outside some compact subset of the interior of I. (The compact subset may vary with the function).

8 DEFINITION. If τ is a (regular or irregular) differential operator of order n, we define the operators $T_0(\tau)$ and $T_1(\tau)$ in $L_2(I)$ by the formulas

(a) $\mathfrak{D}(T_0(\tau)) = H_0^n(I)$. $\quad T_0(\tau)f = \tau f$, $\quad f \in \mathfrak{D}(T_0(\tau))$,

(b) $\mathfrak{D}(T_1(\tau)) = H_\tau^n(I)$, $\quad T_1(\tau)f = \tau f$, $\quad f \in \mathfrak{D}(T_1(\tau))$.

Observe that $T_0(\tau)$ and $T_1(\tau)$ are both unbounded densely defined operators in $L_2(I)$ and that $T_0(\tau) \subset T_1(\tau)$. Our next task is to prove that if τ is regular $T_1(\tau) = T_0(\tau^*)^*$. In case τ is formally self adjoint it will follow that $T_0(\tau) \subseteq T_1(\tau) = T_0(\tau)^*$, showing that $T_0(\tau)$ is symmetric, and (cf. Lemma XII.4.3′) that every self adjoint extension of $T_0(\tau)$ is a restriction of $T_1(\tau)$.

9 LEMMA. *Let f be a function whose square is integrable over every compact subinterval of I. Suppose that*

$$\int_I f(t)\overline{\tau^* g(t)}\,dt = 0$$

for all g in $H_0^n(I)$. Then (after modification on a set of measure 0) $f \in C^\infty(I)$ and $\tau f = 0$.

PROOF. In terms of the operators T_0 and T_1 defined in the preceding paragraph, the lemma states that every function which is orthogonal to the range of $T_0(\tau^*)$ is in C^∞ and belongs to the null space of $T_1(\tau)$.

The proof is divided into the following five steps:

(A) It suffices to prove the lemma under the assumption that I is a closed interval.

For, if the restriction of f to every compact subinterval of I is a C^∞ solution of the equation $\tau\sigma = 0$, then f is such a solution in the whole interval I. We shall henceforth assume that I is a compact interval $[a, b]$, and f is orthogonal to the range of $T_0(\tau^*)$.

(B) Let Σ denote the n-dimensional space of solutions of the equation $\tau\sigma = 0$. By Corollary 1.4, $\Sigma \subseteq C^\infty$. It therefore suffices to show that $f \in \Sigma$. Notice that by IV.3.2 the finite dimensional space Σ is a closed subspace of $L_2(I)$, a fact which will be used in (E) below. The proof that $f \in \Sigma$ will consist in showing that any functional vanishing on $\Sigma \subseteq L_2(I)$ necessarily vanishes on f. For this purpose we require the supplementary information contained in steps (C) and (D).

(C) Suppose that a function w in $H_0^n(I)$ is orthogonal to Σ, regarded as a subspace of $L_2(I)$:

$$[*] \qquad \int_I \sigma(t)\overline{w(t)}dt = 0, \qquad \sigma \in \Sigma.$$

Then there exists a function g in $H_0^n(I)$ such that $\tau^* g = w$.

Indeed, by Corollary 1.8 there is a unique function g in $C^n(I)$ satisfying the equation $\tau^* g = w$ and the boundary conditions:

$$[**] \qquad g(a) = g'(a) = \ldots = g^{(n-1)}(a) = 0.$$

By Green's formula and by [*] we have

$$\begin{aligned} 0 &= \int_a^b g(t)\overline{(\tau\sigma)(t)}dt \\ &= \int_a^b (\tau^* g)(t)\overline{\sigma(t)}dt + \overline{F_b(g, \sigma)} - \overline{F_a(g, \sigma)} = \overline{F_b(g, \sigma)} \end{aligned}$$

for every $\sigma \in \Sigma$. Since there exist solutions with any preassigned values $\sigma(b), \sigma'(b), \ldots, \sigma^{(n-1)}(b)$, it follows from the non-singularity of the form $F_t(g, \sigma)$ (cf. Lemma 2) that

$$[***] \qquad g(b) = g'(b) = \ldots = g^{(n-1)}(b) = 0.$$

Thus g vanishes at both end points of I, while w vanishes on some pair of intervals $[a, a+\varepsilon]$, $[b-\varepsilon, b]$. Now on each of these two

intervals g is the unique solution of $\tau^* f - w = 0$ with the boundary conditions [**] and [***] respectively. Consequently, g must vanish on $[a, a+\varepsilon]$ and on $[b-\varepsilon, b]$, that is, (Definition 8) $g \in H_0^n(I)$.

(D) It follows that any function w in $H_0^n(I)$ which is orthogonal to Σ is also orthogonal to f, i.e.,

$$\int_I f(t)\overline{w(t)}dt = 0.$$

For, by (C), there exists a function g in $H_0^n(I)$ such that $\tau^* g = w$, and the assertion reduces to

$$\int_a^b f(t)\overline{\tau^* g(t)}dt = 0$$

which is the hypothesis of the theorem.

(E) Suppose now that some linear functional φ on $L_2(I)$, represented by a function $\overline{h(\cdot)}$ in $L_2(I)$ (cf. Theorem IV.8.1), vanishes on Σ, that is,

$$\int_I \sigma(t)\overline{h(t)}dt = 0, \qquad \sigma \in \Sigma.$$

We show that φ also vanishes on f. This will complete the proof of the lemma, because if f did not belong to Σ, then, using the Hahn-Banach theorem, we could find a linear functional φ on $L_2(I)$ vanishing on Σ but not on f.

To show $\int_I f(t)\overline{h(t)}dt = 0$, let $\sigma_1, \ldots, \sigma_n$ be an orthonormal basis for Σ, and approximate each σ_i in the topology of $L_2(I)$ by a function $\varphi_i \in H_0^n(I)$ so closely that the matrix $\{a_{ij}\} = \{\int_I \varphi_i(t)\overline{\sigma_j(t)}dt\}$ (which approximates $\{\delta_{ij}\}$) is non-singular. Let $\{h_m\}$ be a sequence of functions in $H_0^n(I)$, uniformly bounded in $L_2(I)$, which converge almost everywhere to h. If $\{b_{ij}\}$ is the matrix inverse to $\{a_{ij}\}$, the sequence

$$[*] \qquad g_m = h_m - \sum_{i,j=1}^{n} b_{ij}\varphi_j \int_I h_m(t)\overline{\sigma_i(t)}dt$$

satisfies

$$\int_I g_m(t)\overline{\sigma_k(t)}dt = \int_I h_m(t)\overline{\sigma_k(t)}dt - \sum_{i,j=1}^{n} b_{ij}a_{jk}\int_I h_m(t)\overline{\sigma_i(t)}dt$$
$$= \int_I h_m(t)\overline{\sigma_k(t)}dt - \int_I h_m(t)\overline{\sigma_k(t)}dt = 0, \qquad k = 1, \ldots, n.$$

Thus $\int_I g_m(t)\overline{\sigma(t)}dt = 0$ for $\sigma \in \Sigma$. Since $g_m \in H_0^n(I)$, it follows from (D) that $\int_I g_m(t)\overline{f(t)}dt = 0$. As $m \to \infty$, $\int_I h_m(t)\overline{\sigma_k(t)}dt \to \int_I h(t)\overline{\sigma_k(t)}dt = 0$. Thus it follows from [*] that $\{g_m\}$ is a uniformly bounded sequence of functions converging almost everywhere to h as $m \to \infty$, so that

$$\int_I h(t)\overline{f(t)}dt = \lim_{n\to\infty}\int_I g_n(t)\overline{f(t)}dt = 0. \qquad \text{Q.E.D.}$$

10 THEOREM. *Let τ be a formal differential operator of order n defined on an interval I. Then $T_1(\tau) = T_0(\tau^*)^*$.*

REMARK. Using XII.1.5 it follows immediately from this theorem that $T_1(\tau)$ is a closed operator. Thus $T_0(\tau) \subseteq T_1(\tau)$ has at least one closed extension, and thus has a minimal closed extension $\overline{T_0(\tau)}$.

PROOF OF THEOREM. If $f \in H_\tau^n$ and $g \in H_0^n$, then by Green's formula (Corollary 5),

$$\int_I f(t)\overline{(\tau^* g)(t)}dt = \int_I \tau f(t)\overline{g(t)}dt$$

which shows that $f \in \mathfrak{D}(T_0(\tau^*)^*)$ and that $T_1(\tau)f = T_0(\tau^*)^*f$, that is,

$$T_1(\tau) \subseteq T_0(\tau^*)^*.$$

To complete the proof of the theorem it suffices to show that

$$\mathfrak{D}(T_0(\tau^*)^*) \subseteq \mathfrak{D}(T_1(\tau)).$$

Suppose that f is in $\mathfrak{D}(T_0(\tau^*)^*)$. This means that there exists a function g in $L_2(I)$ such that for all h in $H_0^n(I)$,

$$\int_I f(t)\overline{(\tau^* h)(t)}dt = \int_I g(t)\overline{h(t)}dt.$$

We wish to prove that f is in the domain of $T_1(\tau)$. By Theorem 1.8 there exists an $f_0 \in A^n(I)$ such that $\tau f_0 = g$. From Green's formula (cf. Corollary 5) it is seen that

$$\int_I f_0(t)\overline{(\tau^* h)(t)}dt = \int_I g(t)\overline{h(t)}dt, \qquad h \in H_0^n(I)$$

and consequently that

$$\int_I (f(t)-f_0(t))\,\overline{(\tau^* h(t))}dt = 0.$$

It follows from Lemma 9 that $f - f_0 \in C^\infty(I)$, and hence that $f = (f - f_0) + f_0$ is in A^n, so that τf is defined almost everywhere. By Lemma 9, $\tau(f - f_0) = 0$ so that $\tau f = \tau f_0$ is in $L_2(I)$ proving that f is in $H^n_\tau(I) = \mathfrak{D}(T_1(\tau))$. Q.E.D.

11 LEMMA. *If the (regular or irregular) formal differential operator τ is formally self adjoint then the operator $T_0(\tau)$ is symmetric.*

PROOF. Clearly $T_0(\tau) \subseteq T_1(\tau)$. Corollary 5 shows that $T_1(\tau) \subseteq T_0(\tau)^*$. Q.E.D.

We recall (cf. Definition XII.4.9) that if τ is formally self adjoint, the positive and negative deficiency spaces of $T_0(\tau)$ are the manifolds

$$\mathfrak{D}_+ = \{f \in \mathfrak{D}(T_1(\tau)) \mid (T_1(\tau) - iI)f = 0\}$$

and

$$\mathfrak{D}_- = \{f \in \mathfrak{D}(T_1(\tau)) \mid (T_1(\tau) + iI)f = 0\}$$

respectively.

12 COROLLARY. *If τ is formally self adjoint, the positive and negative deficiency spaces $\mathfrak{D}_+$, $\mathfrak{D}_-$ of $T_0(\tau)$ consist precisely of those solutions of the differential equations $(\tau - i)f = 0$, $(\tau + i)f = 0$, respectively, which belong to $L_2(I)$.*

13 COROLLARY. *If τ is a formally self adjoint formal differential operator of order n, both deficiency indices of $T_0(\tau)$ are less than or equal to n.*

We have seen in Corollary XII.4.13 that $T_0(\tau)$ has self adjoint extensions if and only if $\mathfrak{D}_+$ and $\mathfrak{D}_-$ have the same dimension. The numbers $n_+ = \dim \mathfrak{D}_+$ and $n_- = \dim \mathfrak{D}_-$ were called the positive and negative deficiency indices of $T_0(\tau)$. The information on n_+ and n_- which can be obtained depends in general on whether the interval I is closed, half open or open. There is, however, an important observation which can be made irrespective of the nature of the interval.

14 COROLLARY. *If, in the formally self adjoint differential operator*

$$\tau = \sum_{j=0}^{n} a_j(t) \left(\frac{d}{dt}\right)^j,$$

the coefficient functions a_j are real, then the corresponding symmetric

operator $T_0(\tau)$ in $L_2(I)$ has equal deficiency indices, and every maximal symmetric extension of $T_0(\tau)$ is self adjoint.

PROOF. Under these hypotheses the solutions of $(\tau - i)\sigma = 0$ are precisely the complex conjugates of the solutions of $(\tau + i)\sigma = 0$. Q.E.D.

If I is a finite closed interval, every solution of $(\tau \pm i)\sigma = 0$ is in $C^\infty(I)$, and hence in $L_2(I)$. Thus, from Corollary 12, we have:

15 COROLLARY. *If I is a finite closed interval and τ is a formally self adjoint differential operator of order n, then $n_+ = n_- = n$.*

16 LEMMA. *Let τ be a formal differential operator of order n defined on the interval I, and let J be a compact subinterval of I.*

(a) *The space $H^n(J)$ is complete in the norm*

$$|f| = \sum_{i=0}^{n-1} \max_{t \in J} |f^{(i)}(t)| + \left\{\int_J |f^{(n)}(t)|^2 dt\right\}^{\frac{1}{2}}.$$

(b) *If $\{f_n\}$ is a sequence in $\mathfrak{D}(T_1(\tau))$ such that $\{f_n\}$ and $\{\tau f_n\}$ converge (converge weakly) in $L_2(I)$, then the sequence $\{f_n\}$ converges (converges weakly) in the topology of $H^n(J)$ defined by the above norm.*

PROOF. (a) If $\{f_m\}$ is a Cauchy sequence in $H^n(J)$, there are functions f_0 and g_0 such that

$$\lim f_m^{(i)}(t) = f_0^{(i)}(t), \qquad i = 0, \ldots, n-1$$

uniformly on J and $f^{(n)} \to g_0$ in $L_2(J)$. Thus for c in J,

$$f_0^{(n-1)}(t) - f_0^{(n-1)}(c) = \lim_{m\to\infty} \int_c^t f_m^{(n)}(s)ds = \int_c^t g_0(s)ds,$$

which shows that $f_0^{(n)} = g_0$ and that f_0 is in $H^n(J)$. Thus $H^n(J)$ is complete.

(b) Consider the two norms

$$\begin{aligned} |f|_1 &= \left[\int_I |f(t)|^2 dt\right]^{\frac{1}{2}} + \left[\int_I |\tau f(t)|^2 dt\right]^{\frac{1}{2}} \\ &= |f| + |T_1(\tau)f| \end{aligned}$$

and

$$|f|_2 = |f|_1 + \sum_{i=0}^{n-1} \max_{t\in J} |f^{(i)}(t)| + \left\{\int_J |f^{(n)}(t)|^2 dt\right\}^{\frac{1}{2}}$$

for $\mathfrak{D}(T_1(\tau))$. From Definition 8(b) it is seen that both norms are

defined and finite on $\mathfrak{D}(T_1(\tau))$. The first norm is the norm of the pair $[f, T_1 f]$ as an element of the graph of $T_1(\tau)$. Now $T_1(\tau)$ is an adjoint (Theorem 10); therefore (cf. XII.1.6) $\mathfrak{D}(T_1(\tau))$ is complete in the norm $|f|_1$. Since the two additional terms in $|f|_2$ are the norm of f as an element of $H^n(J)$ it follows easily that $\mathfrak{D}(T_1(\tau))$ is also complete under the norm $|f|_2$. As $|f|_1 \leq |f|_2$ it follows from Theorem II.2.5 that the two norms are equivalent. The lemma follows immediately from this observation. Q.E.D.

We now turn to a discussion of the specific form assumed in the present case by the abstract "boundary values" introduced in the last chapter. We shall see that the discussion leads us to a number of results about deficiency indices. Our discussion will be less restrictive than that of Section XII.4 since we shall be able, for the most part, to handle differential operators which are not formally self adjoint. We shall also be able to locate boundary values at one or another of the end points of the interval, and to represent boundary values in a concrete analytical form. The fundamental definitions are as follows:

17 Definition. Let τ be a formal differential operator on an interval I with end points a, b. Because T_1 is closed, $\mathfrak{D}(T_1(\tau))$ becomes a Hilbert space upon the introduction of the following inner product:

$$(f, g)^* = (f, g) + (T_1(\tau)f, T_1(\tau)g).$$

A *boundary value* for τ is a continuous linear functional A on $\mathfrak{D}(T_1(\tau))$ which vanishes on $\mathfrak{D}(T_0(\tau))$. If $A(f) = 0$ for each function in the domain of $T_1(\tau)$ which vanishes in a neighborhood of a, A will be called a *boundary value at a*. The concept of a *boundary value at b* is defined similarly. By analogy with Definition XII.4.25 an equation $B(f) = 0$, where B is a boundary value for τ, is called a *boundary condition for* τ. A set of boundary conditions $B_i(f) = 0$, $i = 1, \ldots, k$, is called *stronger than* a set $C_j(f) = 0$, $j = 1, \ldots, m$, if each C_j is a linear combination of the B_i. Two sets of boundary conditions are called *equivalent* if each is stronger than the other. A *complete set of boundary values* is a maximal linearly independent set of boundary values. Similarly, a *complete set of boundary values at a* is a maximal linearly independent set of boundary values at a.

18 Lemma. *If τ is formally self adjoint, then Definition 17 of a boundary value for τ coincides with Definition XII.4.20 of a boundary value for $T_0(\tau)$.*

Proof. Recall from Theorem 10 that $T_1(\tau)$ is the adjoint of $T_0(\tau)$. Q.E.D.

The next theorem gives a basic property of boundary values of differential operators. For the sake of simplicity we shall sometimes write T_1 and T_0 in place of $T_1(\tau)$ and $T_0(\tau)$ respectively.

19 Theorem. *The space of boundary values for τ is the direct sum of the space of boundary values for τ at a and the space of boundary values for τ at b.*

Proof. First of all, it is clear from the preceding definition that the set $\mathfrak{M}_a$ (the set $\mathfrak{M}_b$) of boundary values at a (at b) is a subspace of the space $\mathfrak{M}$ of all boundary values. Let f_1 and f_2 be two functions in $C^\infty(I)$ such that $f_1(t)+f_2(t) = 1$, $t \in I$, while f_1 vanishes in a neighborhood of b and f_2 vanishes in a neighborhood of a. It is clear that if g is in $\mathfrak{D}(T_1)$, then $f_1 g$ and $f_2 g$ are also in $\mathfrak{D}(T_1)$. Since the map $g \to f_1 g$ of $\mathfrak{D}(T_1)$ into itself is clearly closed, it is, by the closed graph theorem (II.2.4), continuous. Let B be a boundary value for τ and define

$$B_1(g) = B(f_1 g),\ B_2(g) = B(f_2 g), \qquad g \in \mathfrak{D}(T_1).$$

If g is in $\mathfrak{D}(T_0)$, then $f_1 g$ and $f_2 g$ are in $\mathfrak{D}(T_0)$, so that $B_1(g) = B_2(g) = 0$. Moreover, by the continuity of the map $g \to f_1 g$, B_1 is a continuous linear functional on $\mathfrak{D}(T_1)$. A similar argument holds for B_2. Thus B_1 and B_2 are boundary values for τ. If $g(t) = 0$ in a neighborhood of a, then $f_1 g \in \mathfrak{D}(T_0)$ by Definition 8, and so $B(f_1 g) = B_1(g) = 0$, showing that B_1 is a boundary value at a. Moreover, $B = B_1 + B_2$. Thus $\mathfrak{M} = \mathfrak{M}_a + \mathfrak{M}_b$, and to prove the theorem it will suffice to show that $\mathfrak{M}_a \cap \mathfrak{M}_b = (0)$. If B is a boundary value both at a and at b, and if g is in $\mathfrak{D}(T_1)$, then $f_1 g$ vanishes in a neighborhood of b so $B(f_1 g) = 0$, and similarly, $B(f_2 g) = 0$. Thus, $B(g) = B(f_1 g) + B(f_2 g) = 0$ for each $g \in \mathfrak{D}(T_1)$. Q.E.D.

If $\tau = \sum_{i=0}^{n} a_i(t)(d/dt)^i$ is a formal differential operator of order n defined on an interval I, and J is a subinterval of I, we may consider the restriction τ' of τ to J. This is simply the formal differential

operator $\sum_{i=0}^{n} b_i(t)(d/dt)^i$, where b_i is the restriction of a_i from I to J. The next few results deal with this concept.

20 THEOREM. *Let τ be a formal differential operator on the interval I with end points a, b. Let $a < c < b$ and let τ' be the restriction of τ to $I' = I \cap [a, c]$. Then there exists a one-to-one linear mapping of the space of boundary values for τ' at a onto the space of boundary values for τ at a.*

PROOF. Choose a function h in $C^\infty(I)$ which is identically equal to one in a neighborhood of the point a and vanishes in a neighborhood of the interval $[c, b]$. Let S_1 be the linear operator defined by the equation

$$(S_1 f)(t) = h(t)f(t).$$

In the formula

$$\big(\tau(hf)\big)(t) = \sum_{i=0}^{n}\left[\sum_{k=i}^{n}\binom{k}{i} a_k(t)h^{(k-i)}(t)\right] f^{(i)}(t)$$

the term $a_n(t)h(t)f^{(n)}(t)$ is in $L_2(I')$, and the remaining terms are continuous; thus S_1 maps $\mathfrak{D}(T_1(\tau))$ into $\mathfrak{D}(T_1(\tau'))$. It is obvious that S_1 is closed and everywhere defined on the Hilbert space $\mathfrak{D}(T_1(\tau))$ (cf. Definition 17) and therefore (cf. II.2.4) S_1 is continuous. Let $\mathfrak{M}$ and $\mathfrak{M}'$ denote the spaces of boundary values at a for τ and τ' respectively. For A' in $\mathfrak{M}'$ let $\Phi_1(A') = A'S_1$. Then Φ_1 is a linear map from $\mathfrak{M}'$ to $\mathfrak{M}$. It will be shown that Φ_1 is one-to-one and that $\Phi_1(\mathfrak{M}') = \mathfrak{M}$.

Let S_2 be the map

$$(S_2 g)(t) = h(t)g(t)$$

defined on $\mathfrak{D}(T_1(\tau'))$, and clearly taking values in $\mathfrak{D}(T_1(\tau))$. By an argument similar to that which has been used above, it can be shown that S_2 is a bounded linear operator. For A in $\mathfrak{M}$ let $\Phi_2(A) = AS_2$ so that Φ_2 is a linear map of $\mathfrak{M}$ into $\mathfrak{M}'$.

If f is in $\mathfrak{D}(T_1(\tau))$, then f and $(S_2 S_1)f$ agree in a neighborhood of a. Therefore for every A in $\mathfrak{M}$, and every f in $\mathfrak{D}(T_1(\tau))$,

$$(\Phi_1\Phi_2 A)f = A\big((S_2 S_1)f\big) = Af.$$

Thus, $\Phi_1\Phi_2$ is the identity mapping, which shows that $\Phi_1(\mathfrak{M}') = \mathfrak{M}$. In the same way $\Phi_2\Phi_1$ may be seen to be the identity mapping of $\mathfrak{M}'$ into itself, which shows that Φ_1 is one-to-one. Q.E.D.

21 Corollary. *Under the hypotheses of the preceding lemma, τ and τ' have the same number of linearly independent boundary conditions at a.*

22 Corollary. *Let τ be a formal differential operator of order n defined on an interval I with end points a, b. Then τ has at most n linearly independent boundary values at a.*

Proof. Suppose that the assertion were false. Using the notations and results of the preceding theorem and corollary, it would follow that τ' had at least $n+1$ linearly independent boundary values at a. Thus, without loss of generality, we may assume that b is a fixed end point of I.

Let $\mathfrak{D}(T_2)$ be the set of all f in $\mathfrak{D}(T_1)$ which vanish on a neighborhood of a, and let T_2 be the restriction of T_1 to $\mathfrak{D}(T_2)$. By hypothesis there are at least $n+1$ linearly independent continuous linear functionals on $\mathfrak{D}(T_1)$ which vanish on $\mathfrak{D}(T_2)$, that is, the orthocomplement $\mathfrak{B}$ of $\mathfrak{D}(T_2)$ in $\mathfrak{D}(T_1)$ is at least $(n+1)$-dimensional.

If $v \in \mathfrak{B}$ and $w \in \mathfrak{D}(T_2)$, then

$$[*] \qquad 0 = (v, w)^* = (v, w) + (T_1 v, T_2 w).$$

Consequently $(T_2 w, T_1 v) = (w, -v)$, which implies that $T_1 v$ is in the domain of T_2^*. From $T_0 \subseteq T_2$ it follows that $T_2^* \subseteq T_0^* = T_1(\tau^*)$ (cf. Theorem 10). Consequently $[*]$ is equivalent to the equation

$$\tau^* \tau v + v = 0.$$

Thus v is a solution of a $2n$th order differential equation, and by Corollary 1.4, v is in $C^\infty(I)$.

From $[*]$ we have $(w, T_2^* T_1 v) = -(w, v)$, and by Green's formula,

$$\begin{aligned} F_b(w, \tau v) - F_a(w, \tau v) &= \int_a^b (\tau w)(t)\overline{\tau v(t)}\,dt \\ &\quad - \int_a^b w(t)\overline{\tau^*(\tau v)(t)}\,dt = (T_2 w, T_1 v) + (w, v) = 0. \end{aligned}$$

Since w vanishes in a neighborhood of a, $F_a(w, \tau v) = 0$, and thus $F_b(w, \tau v) = 0$. As $\mathfrak{D}(T_2)$ contains every function in $C^\infty(I)$ vanishing in a neighborhood of a, it follows from the non-singularity of the matrix $\{F_b^{jk}\}$ (cf. Lemma 2) that τv and its first $n-1$ derivatives

vanish at b. Since $\mathfrak{B}$ is at least $n+1$ dimensional, there is a non-zero $v_0 \in \mathfrak{B}$ satisfying the n linear equations $v_0(b) = v_0'(b) = \ldots = v_0^{(n-1)}(b) = 0$. Since by the above $(\tau v_0)(b) = 0$, we have $(\tau v_0)(b) = \sum_{k=0}^{n} a_k(b) v_0^{(k)}(b) = 0$, $a_n(b) \neq 0$, and it follows that $v_0^{(n)}(b) = 0$. Since $(\tau v_0)'(b) = 0$ it follows similarly that $v_0^{(n+1)}(b) = 0$. Proceeding inductively we see that $v_0^{(k)}(b) = 0$, $0 \leqq k \leqq 2n-1$. However, as v_0 satisfies an equation of order $2n$, v_0 must be identically zero. This contradiction completes the proof. Q.E.D.

23 Corollary. *Let τ be a formal differential operator of order n on an interval I with end points a, b, and suppose that the end point a is fixed. Then the functionals $A_i(f) = f^{(i)}(a)$, $i = 0, \ldots, n-1$, form a complete set of boundary values for τ at a.*

Proof. It follows from Lemma 16 that these functionals are boundary values for τ at a. They are clearly linearly independent. If the assertion of the corollary were false, it would follow that τ has a boundary value at a which is independent of the set $A_0, \ldots, A_{n-1}$, and hence has at least $n+1$ independent boundary values at a. But this is impossible by Corollary 22. Q.E.D.

24 Corollary. (*Weyl-Kodaira*) *Let τ be a formally self adjoint formal differential operator of order n defined on an interval I. Assume that at least one end point of I is fixed. Then the sum of the positive and the negative deficiency indices of τ is at least n.*

Proof. By Lemma XII.4.21, this sum is the number of linearly independent boundary conditions, and thus the result follows immediately from the preceding corollary. Q.E.D.

25 Corollary. *Let τ be a formally self adjoint formal differential operator of order n defined on an interval I with end points a and b. Let $a < c < b$, and let τ' and τ'' be respectively the restrictions of τ to the intervals $I' = I \cap [a, c]$, $I'' = I \cap [c, b]$. If d, d', and d'' are the sums of the positive and negative deficiency indices of τ, τ', and τ'' respectively, then $d = d' + d'' - 2n$.*

Proof. By Theorem 19, d equals the sum of the number of independent boundary values at a and the number of independent boundary values at b. Since c is a fixed end point, it follows from

Corollary 23 and from Theorems 19 and 20 that d' and d'' exceed by n the number of independent boundary values at a and at b respectively. The statement of the present corollary is then evident. Q.E.D.

26 Corollary. (*Kodaira*) *Under the hypotheses of the preceding corollary, let d_+, d'_+, and d''_+ be the positive, and d_-, d'_-, and d''_- the negative deficiency indices of τ, τ', and τ'' respectively. Then*

$$d_+ = d'_+ + d''_+ - n; \qquad d_- = d'_- + d''_- - n.$$

Proof. Let $\mathfrak{D}_+$ be the space of solutions of $\tau f = if$ which lie in $L_2(I)$, and let $\mathfrak{D}'_+$ and $\mathfrak{D}''_+$ be the spaces of solutions of $\tau' f = if$ and $\tau'' f = if$ which lie in $L_2(I')$ and in $L_2(I'')$ respectively. Thus $d_+ = \dim \mathfrak{D}_+$, $d'_+ = \dim \mathfrak{D}'_+$, $d''_+ = \dim \mathfrak{D}''_+$, and $\mathfrak{D}_+ = \mathfrak{D}'_+ \cap \mathfrak{D}''_+$.

If C is an arbitrary finite dimensional space, A and B are subspaces of C, and $A + B = C$, then it is well known and readily seen that

$$\dim C + \dim (A \cap B) = \dim A + \dim B.$$

Applying this rule in the present case we find that

$$d_+ + \dim (\mathfrak{D}'_+ + \mathfrak{D}''_+) = d'_+ + d''_+,$$

and thus $d_+ + n \geq d'_+ + d''_+$. Similarly, $d_- + n \geq d'_- + d''_-$. Since by the previous corollary

$$d_+ + d_- + 2n = d'_+ + d'_- + d''_+ + d''_-,$$

we must have equality in both of the above inequalities. Q.E.D.

We now give a result which reveals the concrete form of abstract boundary values for the most general formal differential operator.

27 Theorem. *Let τ be a formal differential operator defined on an interval I with end points a and b. Let w_i, $i = 0, 1, \ldots, n-1$, be a set of functions such that*

$$B(f) = \lim_{t \to a} \sum_{i=0}^{n-1} w_i(t) f^{(i)}(t)$$

exists for all f in $\mathfrak{D}(T_1(\tau))$. Then B is a boundary value for τ at a. Conversely, every boundary value for τ at a is of this form.

Proof. It is clear from Lemma 16 that for each t interior to I,

$B_t(g) = \sum_{i=0}^{n-1} w_i(t) f^{(i)}(t)$ is a continuous linear functional on the Hilbert space $\mathfrak{D}(T_1(\tau))$. If $\lim_{t\to a} B_t(f) = B(f)$ exists for each f in $\mathfrak{D}(T_1(\tau))$, then, by Theorem II.1.17, B is a continuous linear functional on $\mathfrak{D}(T_1(\tau))$. Clearly $B(f) = 0$ for those f which vanish in a neighborhood of a. Thus B is a boundary value for τ at a.

To prove the converse, let B be a boundary value at a. Choose a function h in $C^\infty(I)$ which is identically equal to one in a neighborhood of a and vanishes identically in a neighborhood of b. Clearly fh lies in $\mathfrak{D}(T_1(\tau))$ for every f in $\mathfrak{D}(T_1(\tau))$. Furthermore, fh and f take on the same values in a neighborhood of a, and consequently $B(fh) = B(f)$ for all f in $\mathfrak{D}(T_1(\tau))$. Now B is a linear functional in the Hilbert space $\mathfrak{D}(T_1(\tau))$ (cf. Definition 17). Thus (cf. Theorem IV.4.5) there exists an element g in the orthocomplement of $\mathfrak{D}(T_0(\tau))$ such that for all f in $\mathfrak{D}(T_1(\tau))$,

$$B(f) = (f, g)^*.$$

In particular, for f in $\mathfrak{D}(T_0(\tau))$ we have

$$[*] \qquad 0 = (f, g)^* = (f, g) + (T_0(\tau)f, T_1(\tau)g).$$

It follows, as in Corollary 22, that g is a solution of the equation $\tau^*\tau g + g = 0$, and is therefore infinitely differentiable. Let $v = -\tau g$; then

$$\begin{aligned} B(f) &= (f, \tau^* v) - (\tau f, v) \\ &= (fh, \tau^* v) - (\tau(fh), v) = B(fh). \end{aligned}$$

Using Green's formula and the fact that h vanishes in a neighborhood of b, it follows that

$$\begin{aligned} B(f) &= \lim_{\substack{t\to a\\ s\to b}} -\int_t^s \{[\tau(fh)](t)\overline{v(t)} - (fh)(t)\overline{(\tau^* v)(t)}\}dt \\ &= \lim_{t\to a} F_t(fh, v) = \lim_{t\to a} F_t(f, v). \end{aligned}$$

Q.E.D.

28 COROLLARY. *If B is a boundary value for τ at a, there exists an infinitely differentiable function g in the orthogonal complement of $\mathfrak{D}(T_0(\tau))$ in the Hilbert space $\mathfrak{D}(T_1(\tau))$ such that $v = -\tau g$ is in the orthocomplement of $\mathfrak{D}(T_0(\tau^*))$ in the Hilbert space $\mathfrak{D}(T_1(\tau^*))$ and*

$$B(f) = \lim_{t\to a} F_t(f, v), \quad f \in \mathfrak{D}(T_1(\tau)).$$

Proof. All the assertions of this corollary were proved in the course of the preceding proof, except the statement that

$$[\dagger] \qquad (\tau g, f) + (T_1(\tau^*)\tau g, T_1(\tau^*)f) = 0$$

for all f in $\mathfrak{D}(T_0(\tau^*))$. Since g is in $C^\infty(I)$, it follows from Green's formula that [†] is equivalent to the equation

$$(\tau g + \tau\tau^*\tau g, f) = 0,$$

and since $\tau^*\tau g + g = 0$, this is evident. Q.E.D.

Theorem 27 concludes our discussion of boundary values for a differential operator. We have shown in Theorem 19 that every boundary value is the sum of a boundary value at a and a boundary value at b. We have given a concrete representation for boundary values and obtained basic information on deficiency indices. Of principal interest is the case in which τ is formally self adjoint, and hence the operator $T_0(\tau) = T_0(\tau^*)$ is symmetric. Lemma XII.4.26 and Theorems XII.4.28, 30 and 31 of the preceding chapter then give us in explicit form all symmetric extensions of $T_0(\tau)$ and their adjoints and all self adjoint extensions of $T_0(\tau)$.

Remark. It will be convenient for later sections to extend the domain of a boundary value A to a larger class of functions than $\mathfrak{D}(T_1(\tau))$. We have seen in Theorem 27 that if A is a boundary value at a, the number $A(g)$, $g \in \mathfrak{D}(T_1(\tau))$ is uniquely determined by the values of g in any arbitrarily small neighborhood of a. Thus if f is in $L_2(I)$ and there exists a g in $\mathfrak{D}(T_1(\tau))$ such that $f(t) = g(t)$ in a neighborhood of a, we may define $A(f) = A(g)$. It is clear that this definition is unambiguous. Similar remarks apply when A is a boundary value at b. If A is a mixed boundary value, and if f is in $L_2(I)$ and there exists a function g in $\mathfrak{D}(T_1(\tau))$ which coincides with f in neighborhoods of both a and b, we define $A(f) = A(g)$.

If τ is an arbitrary formal differential operator, the restriction of the operator $T_1(\tau)$ to a domain determined by a set of boundary conditions will be called the operator *derived from τ by the imposition of the given set of boundary conditions.*

29 **Definition.** *A boundary condition for τ* of the form $B(f) = 0$ is said to be a *boundary condition at a* (at b) if B is a boundary value

for τ at a (at b). If $B(f) = 0$ is not a boundary condition either at a or at b (so that, by Theorem 19, the equation $B(f) = 0$ may be written as $B_1(f) = B_2(f)$, where B_1 and B_2 are non-zero boundary values at a and b respectively), then $B(f) = 0$ is said to be a *mixed boundary condition*. A set of boundary conditions is said to be *separated* if it (or, more generally, some set of boundary conditions equivalent to it) contains no mixed boundary conditions. In all other cases the set is said to be a *mixed set of boundary conditions*. If τ is a real formal differential operator then $\mathfrak{D}(T_1(\tau))$ is closed under complex conjugation. A boundary value A for a real operator τ is said to be *real* if $A(\bar{f}) = \overline{A(f)}$ for every f in $\mathfrak{D}(T_1(\tau))$.

We conclude this section by considering some simple examples of differential operators. The simplest example of a formally self adjoint differential operator is the operator $\tau = i(d/dt)$. We shall consider three choices for the interval I.

Case 1: $I = [0, 1]$. Here clearly $d_+ = d_- = 1$, and a complete set of boundary values is $f(0)$ and $f(1)$. It is clear that the most general complete symmetric set of boundary conditions (consisting in this case of a single condition) is $f(0) = e^{i\theta}f(1)$, where $0 < \theta \leq 2\pi$.

Case 2: $I = [0, \infty)$. Then the solution of $\tau f = if$ (of $\tau f = -if$) is $e^t(e^{-t})$. Since e^t is not square-integrable in $[0, \infty)$, $d_- = 1$ and no self adjoint operators can be derived from τ.

Case 3: $I = (-\infty, \infty)$. It follows from Corollary 27 by comparison with Case 2 that there are no boundary values at $\pm\infty$ for τ. Thus τ leads to the unique self adjoint operator $T_1(\tau)$, no boundary conditions being imposed.

Now let us consider a formally self adjoint operator τ of the form $(d/dt)[p(t)(d/dt)] + q(t)$ on an interval I with end points a, b, where the functions p and q are real. Let $a < c < b$. Then by Corollary 14, the positive and the negative deficiency indices d'_+ and d'_- of the restriction τ' of τ to $I \cap [c, b]$ are equal, and by Corollary 24, their sum is at least 2. Thus $d'_+ = d'_- \geq 1$. We recall (cf. Lemma XII.4.21) that $d'_+ + d'_-$ is the number of linearly independent boundary values for τ'. Since $d'_+ + d'_- \leqq 4$, it follows that either $d'_+ = 1$, or $d'_+ = 2$. If $d'_+ = 1$, then τ', and hence τ, has no boundary values at b, since τ' has two boundary values at c. If $d'_+ = 2$, then τ', and hence τ, has two boundary values at b. The end point a can be discussed similarly.

The following table gives the number of linearly independent solutions of $(\tau-\lambda)\sigma=0$ square integrable at a or b when $\mathscr{I}(\lambda)\neq 0$. There are four possibilities as shown by the discussion above.

Number of linearly independent solutions square-integrable:

	At a	At b
(i)	2	2
(ii)	1	2
(iii)	2	1
(iv)	1	1

The next table gives the the number of boundary values for τ in each of the cases (i)–(iv) above.

Number of linearly independent boundary values for τ:

	At a	At b
(i)	2	2
(ii)	0	2
(iii)	2	0
(iv)	0	0

In H. Weyl's terminology we say that an end point a is of *limit point* type with respect to the real second order operator τ if τ has no boundary values at a, and of *limit circle* type if τ has two boundary values at a.

The next theorem gives an important normal form for the boundary values of a second order real formally self adjoint differential operator.

30 THEOREM. *Let the second order formal differential operator τ be defined on an interval I with end points a, b and have form $\tau f=(pf')'+qf$, where p and q are real. Then in case* (iv) *above we have $(\tau f, g)=(f, \tau g)$ for $f, g\in\mathfrak{D}(T_1(\tau))$. In cases* (iii) *and* (ii) *above there exists a complete set of boundary values for τ consisting of two linearly independent real boundary values A_1 and A_2 such that*

$$(\tau f, g)-(f, \tau g)=A_1(f)\overline{A_2(g)}-A_2(f)\overline{A_1(g)}, \qquad f, g\in\mathfrak{D}(T_1(\tau)).$$

In case (i) *above there exists a complete linearly independent set of boundary values for τ consisting of four linearly independent real*

boundary values C_1, C_2, D_1, D_2 where C_1, C_2 are boundary values at a and D_1, D_2 are boundary values at b, such that

$$\begin{aligned}(\tau f, g)-(f, \tau g) = C_1(f)\overline{C_2(g)}-C_2(f)\overline{C_1(g)} \\ + D_1(f)\overline{D_2(g)}-D_2(f)\overline{D_1(g)}, \qquad f, g \in \mathfrak{D}(T_1(\tau)).\end{aligned}$$

PROOF. Let A be any boundary value for τ. Since τ is real, $\mathfrak{D}(T_1(\tau))$ is closed under the formation of complex conjugates, so the functional $\bar{A}$ defined by the formula $\bar{A}(f) = \overline{A(\bar{f})}$ is also a boundary value for τ. We may call it the *complex conjugate of the boundary value* A. The boundary value A may be written as a linear combination of real boundary values, as follows:

$$A = \left(\frac{A+\bar{A}}{2}\right) + i\left(\frac{A-\bar{A}}{2i}\right).$$

Thus τ has a complete set $A_1, \ldots, A_p$ of independent real boundary values. By Lemma XII.4.23, the bilinear form $(\tau f, g)-(f, \tau g)$ may be written uniquely in the form

$$[*] \qquad (\tau f, g)-(f, \tau g) = \sum_{i,j=1}^{p} c_{ij} A_i(f)\overline{A_j(g)}, \qquad f, g \in \mathfrak{D}(T_1(\tau)),$$

where $c_{ij} = -\bar{c}_{ji}$. Since the boundary values A_i are real and the constants c_{ij} are unique, it follows readily that the c_{ij} are real. Thus $c_{ij} = -c_{ji}$, so that $c_{ii} = 0$.

In the case (iv) discussed above there are no boundary values for τ, so that it is evident from [*] that $(\tau f, g) = (f, \tau g)$, $f, g \in \mathfrak{D}(T_1(\tau))$.

In case (ii) discussed above there are two boundary values at b, and it follows (after suitable rearrangement and normalization of the real boundary values A_1, A_2) that we may write

$$(\tau f, g)-(f, \tau g) = A_1(f)\overline{A_2(g)}-A_2(f)\overline{A_1(g)}.$$

In the case (i), it follows similarly that we may choose a complete set $\{A_1, A_2, A_3, A_4\} = \{C_1, C_2, D_1, D_2\}$ of real independent boundary values, C_1 and C_2 being boundary values at a and D_1 and D_2 being boundary values at b. If we rewrite formula [*] in terms of C_i and D_j, there will be terms on the right side of the form $d_{ij}\{C_i(f)D_j(\bar{g})-C_i(g)\bar{D}_j(\bar{f})\}$. We will show that the coefficient d_{ij} of such a term must

vanish. Suppose for example that $d_{11} \neq 0$. We can find a function f in $\mathfrak{D}(T_1(\tau))$ such that $C_1(f) = 1$, $C_2(f) = 0$, and such that f vanishes in a neighborhood of b. Similarly there exists a function g in $\mathfrak{D}(T_1(\tau))$ such that $D_1(g) = 1$, $D_2(g) = 0$, and g vanishes in a neighborhood of a. Then by Green's formula we have $(\tau f, g) - (f, \tau g) = 0$. On the other hand by [*], $(\tau f, g) - (f, \tau g) = d_{11} \neq 0$. This contradiction proves our assertion. It is clear that a similar argument will work for any values of i and j, $1 \leqq i, j \leqq 2$.

It then follows readily that (after suitable normalization of C_1, C_2, D_1, and D_2) we may write

$$(\tau f, g) - (f, \tau g) = C_1(f)C_2(\bar{g}) - C_2(f)C_1(\bar{g}) + D_1(f)D_2(\bar{g}) - D_2(f)D_1(\bar{g}). \qquad \text{Q.E.D.}$$

31 COROLLARY. *Let the hypotheses of the preceding theorem be satisfied and let T be a self adjoint extension of $T_0(\tau)$.*

In cases (ii) *and* (iii) *above, the set may be written in the form*

$$\alpha A_1(f) + \beta A_2(f) = 0, \qquad \alpha^2 + \beta^2 \neq 0,$$

where A_1 and A_2 are real boundary values which are both at b in case (ii) *and both at a in case* (iii).

In case (i), *if the set of boundary conditions is separated, it may be written in the form*

$$\alpha_1 C_1(f) + \alpha_2 C_2(f) = 0, \qquad \alpha_1^2 + \alpha_2^2 \neq 0,$$
$$\beta_1 D_1(f) + \beta_2 D_2(f) = 0, \qquad \beta_1^2 + \beta_2^2 \neq 0,$$

where C_i and D_i, $i = 1, 2$, are real boundary values at a and b respectively.

PROOF. In cases (ii) and (iii) above, it follows from Theorem XII.4.30 that the operator T is determined by one boundary condition, and it is easily seen from Theorem 30 and Definition XII.4.25 that it must have the form

$$\alpha A_1(f) + \beta A_2(f) = 0, \qquad \alpha, \beta \text{ real}, \qquad \alpha^2 + \beta^2 \neq 0.$$

In case (ii), A_1 and A_2 are boundary values at b while in case (iii) they are boundary values at a. In either case it is clear that the single

boundary condition determining T is real.

In the same way, in case (i) above the operator T, if it is determined by a separated set of boundary conditions, is determined by two boundary conditions, one at a and one at b. Using the preceding theorem and Definition XII.4.25 it is easily seen that they must have the form

$$\alpha_1 C_1(f)+\alpha_2 C_2(f) = 0,$$
$$\beta_1 D_1(f)+\beta_2 D_2(f) = 0,$$

with α_i and β_i real and $\alpha_1^2+\alpha_2^2 \neq 0$, $\beta_1^2+\beta_2^2 \neq 0$. These formulas show that in case (i) the symmetric separated set of boundary conditions is also necessarily real. Q.E.D.

If a is a fixed end point of I and τ has no boundary values at b, a complete set of boundary values for τ is $B_1(f) = f(a)$, $B_2(f) = f'(a)$. Now for $f, g \in \mathfrak{D}(T_1(\tau))$ we have (cf. Definition XII.4.2 and Green's formula),

$$\begin{aligned}(f, g) &= i\int_a^b [(pf')'(t)\overline{g(t)} - f(t)\overline{(pg')'(t)}]dt \\ &= i\big(F_a(f, g) - \lim_{s\to b} F_s(f, g)\big).\end{aligned}$$

Since τ has no boundary values at b, it follows from Theorem 27 that $\lim_{s\to b} F_s(f, g) = 0$. Thus

$$\{f, g\} = ip(a)[f'(a)\overline{g(a)} - f(a)\overline{g'(a)}].$$

By Theorem XII.4.30, the most general self adjoint extension of $T_0(\tau)$ is the restriction of $T_1(\tau)$ to the subdomain of $\mathfrak{D}(T_1(\tau))$ determined by a single symmetric boundary condition, which is necessarily a boundary condition at a. It is easily seen from the preceding equation and Definition XII.4.25 that the most general symmetric boundary condition is $\alpha f'(a)+\beta f(a) = 0$ with α and β real. Thus, in case one end point is free and there are no boundary values at the other, we have a simple explicit form for the most general self adjoint extension of $T_0(\tau)$.

The next theorem gives an important property of the second order operator which will be of use in the next section.

32 THEOREM. *Let τ have the form $\tau f = (pf')'+qf$, where p and q are real, and let T be a self adjoint operator obtained from τ by the*

imposition of a separated symmetric set of boundary conditions. Let $\mathscr{I}\lambda \neq 0$. Then the boundary conditions are real, and there is exactly one solution $\varphi(t, \lambda)$ of $(\tau - \lambda)\varphi = 0$ square-integrable at a and satisfying the boundary conditions at a, and exactly one solution $\psi(t, \lambda)$ of $(\tau - \lambda)\psi = 0$ square-integrable at b and satisfying the boundary conditions at b.

PROOF. We shall show the theorem is true in each of the four cases discussed above. In case (iv), since there are no boundary values, no boundary conditions are imposed; thus the theorem is true in this case. In case (iii), one boundary condition is to be imposed, since, according to Theorem XII.4.30 and Lemma XII.4.21, the number of boundary conditions in a symmetric set determining a self adjoint operator is half the total number of boundary values. As there are no boundary values at b, this condition must be a boundary condition at a. Hence it is clear from the above table that there is exactly one solution ψ of $(\tau - \lambda)\psi = 0$ square-integrable at b and satisfying all boundary conditions at b, and at least one solution φ of $(\tau - \lambda)\varphi = 0$ square-integrable at a and satisfying all the boundary conditions at a. Suppose there were a second solution which was linearly independent of φ and which was square-integrable at a and satisfied all the boundary conditions at a. Since the space of all solutions of $(\tau - \lambda)\varphi = 0$ is two-dimensional, all of these solutions would be square-integrable at a and satisfy all the boundary conditions at a. Thus ψ would be square-integrable over the whole interval I and satisfy all the boundary conditions defining T. Then λ would be an eigenvalue of T. But since T is self adjoint and $\mathscr{I}\lambda \neq 0$, this is impossible. Case (ii) is obviously equivalent to case (iii) by symmetry.

In case (i), Theorem XII.4.30 shows that a symmetric separated set of boundary conditions determining T contains two elements. It is clear that the argument given for case (iii) will cover case (i) also, once it is observed that it cannot happen that both conditions of this pair are conditions at the same end point. This follows immediately from Corollary 31. Q.E.D.

3. Resolvents of Differential Operators

Throughout this section the symbol τ will be used for a formal differential operator of order n which is defined on the interval I

with end points a, b. The operator $T = T(\tau)$ will be an operator obtained from τ by the imposition of a set, which may be vacuous, of k linearly independent boundary conditions $B_i(f) = 0$, $i = 1, \ldots, k$; i.e., T is the restriction of $T_1(\tau)$ (cf. Definition 2.8) to the submanifold of $\mathfrak{D}(T_1(\tau))$ determined by the conditions $B_i(f) = 0$, $i = 1, \ldots k$. Our main purpose in this section will be to obtain a concrete representation of the resolvent $R(\lambda; T)$ for λ in $\rho(T)$ as an integral operator

$$R(\lambda; T)(f, t) = \int_I K(t, s; \lambda)f(s)ds.$$

We are seeking not only information of theoretical interest, but algebraic algorithms for application to specific cases. It will be convenient in stating many of the results to suppose that the number $\lambda = 0$ is in $\rho(T)$, that is, that T has a bounded everywhere defined inverse. This convenient assumption is equivalent to the supposition that the operator τ has been replaced by $\tau - \lambda$.

Our first result is concerned with the number k of linearly independent boundary conditions which define T. Notice that k may actually be zero.

1 Lemma. *Let T have a bounded inverse. Then the number of linearly independent boundary conditions defining T is equal to the number of linearly independent solutions of the equation $\tau f = 0$ which belong to $L_2(I)$.*

Proof. Let ν be the number of linearly independent solutions of $\tau f = 0$ belonging to $L_2(I)$. If $\nu > k \geqq 0$, then $Tf = 0$ for some non-null function f, and hence T has no inverse. Thus $k \geq \nu$. Assume $k > \nu$. Since there are at least $\nu + 1$ linearly independent linear functionals on $\mathfrak{D}(T_1(\tau))$ which vanish on $\mathfrak{D}(T)$, there are at least $\nu + 1$ linearly independent linear functionals on the factor space $\mathfrak{D}(T_1(\tau))/\mathfrak{D}(T)$. Hence, this factor space is at least $\nu + 1$ dimensional. Thus there are at least $\nu + 1$ linearly independent functions f_0, $f_1, \ldots, f_\nu$ in $\mathfrak{D}(T_1(\tau))$ no non-zero linear combination of which lies in $\mathfrak{D}(T)$. Since T has a bounded everywhere defined inverse, it maps $\mathfrak{D}(T)$ onto $L_2(I)$. Therefore, there are functions $g_0, g_1, \ldots, g_\nu$ in $\mathfrak{D}(T)$ such that

$$Tg_i = T_1(\tau)f_i, \qquad i = 0, 1, \ldots, \nu.$$

Then $\tau(g_i-f_i) = Tg_i - T_1(\tau)f_i = 0$, and the functions g_i-f_i, $i = 0, 1, \ldots, \nu$ are $\nu+1$ linearly independent solutions of the equation $\tau\sigma = 0$. This contradiction completes the proof. Q.E.D.

The next result is a generalization of Corollary 2.26.

2 LEMMA. *Let T have a bounded inverse and let τ' and τ'' be the restrictions of τ to the intervals $I' = I \cap [a, c]$ and $I'' = I \cap [c, b]$, where $a < c < b$. If $\nu(\nu', \nu'')$ are respectively the number of linearly independent solutions of $\tau f = 0$ which belong to $L_2(I)$ $\big(L_2(I'), L_2(I'')\big)$, then $\nu' + \nu'' = \nu + n$.*

PROOF. Let $\mathfrak{B}(\mathfrak{B}', \mathfrak{B}'')$ be the linear set of functions consisting of all solutions of the equation $\tau f = 0$ which belong to $L_2(I)$ $\big(L_2(I'), L_2(I'')\big)$. Since $\mathfrak{B} = \mathfrak{B}' \cap \mathfrak{B}''$,

$$\dim \mathfrak{B} + \dim (\mathfrak{B}' + \mathfrak{B}'') = \dim \mathfrak{B}' + \dim \mathfrak{B}'',$$

by an elementary property of finite dimensional spaces. By Theorem 1.8, $\dim (\mathfrak{B}' + \mathfrak{B}'') \leqq n$, from which it follows that $\nu + n \geqq \nu' + \nu''$.

Let $\mathfrak{W}$ be the subspace of $L_2(I)$ consisting of all functions whose restrictions to each of the intervals I' and I'' are in $\mathfrak{B}'$ and $\mathfrak{B}''$ respectively. It is clear that $\dim \mathfrak{W} = \nu' + \nu''$. We shall prove $\nu' + \nu'' \geqq \nu + n$ by showing that $\dim \mathfrak{W} \geqq \nu + n$. Since $\mathfrak{B}$ is a subspace of $\mathfrak{W}$, this can be done by constructing n linearly independent functions in $\mathfrak{W}$ no linear combination of which lies in $\mathfrak{B}$. By Theorem 2.10 and Lemma 2.6, T^* is a restriction of the operator $T_1(\tau^*)$ (cf. Definition 2.8). Thus, the linear functionals

$$[*] \qquad A_i(f) = f^{(i)}(c), \qquad i = 0, 1, \ldots, n-1$$

are continuous on $\mathfrak{D}(T^*)$. Since there are functions f in $\mathfrak{D}\big(T_0(\tau^*)\big)$ for which the values $f^{(i)}(c)$ are arbitrarily assigned, it follows that the functionals in $[*]$ are linearly independent on $\mathfrak{D}\big(T_0(\tau^*)\big)$. By Theorem 2.10 $T^* \supseteq T_1(\tau)^* = T_0(\tau^*)^{**} \supseteq T_0(\tau^*)$, and therefore $\mathfrak{D}\big(T_0(\tau^*)\big)$ is a subspace of $\mathfrak{D}(T^*)$. Thus the functionals A_i are linearly independent on $\mathfrak{D}(T^*)$.

In $\mathfrak{D}(T^*)$ we introduce the inner product

$$(f, g)^+ = (f, g) + (T^*f, T^*g).$$

By Theorem XII.1.6 (a), T^* is closed, and by Theorem XII.1.6 (b),

it has a bounded everywhere defined inverse $(T^*)^{-1}$.

It is evident that $(T^*)^{-1}$ is a bounded linear operator from $L_2(I)$ to $\mathfrak{D}(T^*)$. Therefore the functionals $A_i((T^*)^{-1}f)$ on $L_2(I)$ are continuous, and by the representation theorem for the adjoint of Hilbert space (cf. Theorem IV.4.5) there are n functions $h_0, \ldots, h_{n-1}$ in $L_2(I)$ such that

$$A_i\big((T^*)^{-1}g\big) = (g, h_i), \qquad 0 \leqq i \leqq n-1, \qquad g \in L_2(I).$$

The linear independence of $A_0, \ldots, A_{n-1}$ implies that the functions $h_0, \ldots, h_{n-1}$ are linearly independent. For all f lying in $\mathfrak{D}\big(T_0(\tau')\big)$ or in $\mathfrak{D}\big(T_0(\tau'')\big)$ we have

$$0 = A_i f = (T^* f, h_i) = \int_{I' \text{ or } I''} \tau^* f(t)\overline{h_i(t)}dt.$$

By Lemma 2.9 we conclude that h_i lies in $C^\infty(I')$ and in $C^\infty(I'')$ and that $(\tau h_i)(t) = 0$, $t \neq c$. Thus the n linearly independent functions $h_0, \ldots, h_n$ belong to $\mathfrak{B}$. To complete the proof it suffices to verify that no non-zero linear combination $h = \sum_{j=0}^{n-1} \alpha_j h_j$ of the functions h_i lies in $\mathfrak{B}$. Suppose that h is in $\mathfrak{B}$. Then, by Green's formula, we have

$$\begin{aligned} 0 = (f, \tau h) = (T^* f, h) - \sum_{j=0}^{n-1} \bar{\alpha}_j A_j(f) \\ = \sum_{j=0}^{n-1} \bar{\alpha}_j f^{(j)}(c) \end{aligned}$$

for each f in $\mathfrak{D}\big(T_0(\tau^*)\big)$. Then clearly $\alpha_j = 0$ for $j = 0, \ldots, n-1$, so that $h = 0$. This contradiction completes the proof of the present lemma. Q.E.D.

8 LEMMA. *The adjoint T^* of T is the restriction of $T_1(\tau^*)$ determined by a set of boundary conditions $B_i^*(f) = 0$, $i = 1, \ldots, k^*$, imposed on τ^*.*

PROOF. In the proof of the preceding lemma it was observed that $T_0(\tau^*) \subsetneqq T^* \subsetneqq T_1(\tau^*)$. Now $\mathfrak{D}\big(T_1(\tau^*)\big)$ is a Hilbert space under the inner product $(f, g)^* = (f, g) + (\tau^* f, \tau^* g)$, and an elementary calculation shows that the orthocomplement $\mathfrak{B}$ of $\mathfrak{D}\big(T_0(\tau^*)\big)$ in $\mathfrak{D}\big(T_1(\tau^*)\big)$ is the set of f satisfying the equation $f + T_0(\tau^*)^* T_1(\tau^*) f =$

$f+T_1(\tau)T_1(\tau^*)f = 0$ (cf. Theorem 2.10); thus every f in $\mathfrak{B}$ satisfies the differential equation $f+\tau\tau^*f = 0$, which shows that $\mathfrak{B}$ is at most $2n$-dimensional. Since $\mathfrak{D}(T^*)$ is a closed subspace of $\mathfrak{D}(T_1(\tau^*))$ containing $\mathfrak{D}(T_0(\tau^*))$, the orthocomplement of $\mathfrak{D}(T^*)$ in $\mathfrak{D}(T_1(\tau^*))$ has a finite basis $h_1, \ldots, h_p$. Putting $B_i^*(f) = (f, h_i)^*$, it is clear that B_i^* is a boundary value for τ^*, and it follows that the equations $B_i^*(f) = 0$, $i = 1, \ldots, p$, determine the subspace $\mathfrak{D}(T^*)$ of $\mathfrak{D}(T_1(\tau^*))$. Q.E.D.

Remark. If T is a self adjoint operator derived from a formally symmetric operator or, more generally, an operator of the form $T_1 - \lambda I$, where T_1 is such a self adjoint operator, then the boundary conditions $B_i^*(f) = 0$, $i = 1, \ldots, k^*$, are equivalent to the boundary conditions $B_j(f) = 0$, $j = 1, \ldots, k$.

4 Lemma. *If R denotes the bounded inverse of T, then there is a function K defined on $I \times I$ such that $K(t, \cdot)$ is in $L_2(I)$ for each t in I, and*

$$(Rf)(t) = \int_I K(t, s)f(s)ds, \qquad f \in L_2(I).$$

Moreover

(a) *For each c in I, the function $K(c, \cdot)$ belongs to $C^\infty(I \cap [a, c])$ and $C^\infty(I \cap [c, b])$, and $\tau^*\overline{(K(c, s))} = 0$ for $s \neq c$.*

(b) *If we let $K_+(c, s) = K(c, s)$ for $s > c$, and $K_-(c, s) = K(c, s)$ for $s < c$, then*

$$\lim_{s \to c+0} K_+^{(i)}(c, s) = \lim_{s \to c-0} K_-^{(i)}(c, s), \qquad i = 0, \ldots, n-2,$$

$$\lim_{s \to c+0} K_+^{(n-1)}(c, s) - \lim_{s \to c-0} K_-^{(n-1)}(c, s) = (-1)^n \overline{[a_n(c)]}^{-1}.$$

(c) *The equations* (b) *are equivalent to the relation*

$$f(c) = F_c(f, \overline{K}_-) - F_c(f, \overline{K}_+), \qquad c \in I, \qquad f \in \mathfrak{D}(T_1(\tau)).$$

Proof. It is clear that R is a continuous one-to-one map of $L_2(I)$ onto the Hilbert space $\mathfrak{D}(T)$ (whose inner product is $(f, g)^* = (f, g) + (Tf, Tg)$). Since T is a restriction of $T_1(\tau)$, it follows from Lemma 2.16 that the functional $(Rf)(t)$ is continuous on $L_2(I)$ for each t in I. Consequently, Theorem IV.4.5 shows that, for each t in I, there is a function $K(t, \cdot)$ in $L_2(I)$ such that

$$(Rf)(t) = \int_I K(t, s)f(s)ds, \qquad f \in L_2(I).$$

Let $a < c < b$. Suppose that g lies in $\mathfrak{D}(T_0(\tau_c'))$, where τ_c' is the restriction of τ to $I \cap [a, c]$. Then

$$0 = g(c) = \int_I K(c, s)(\tau g)(s)ds.$$

It follows from Lemmas 2.6 and 2.9 that $K(c, \cdot)$ is in $C^\infty(I \cap [a, c])$ and $\tau^*\overline{K(c, s)} = 0$, $a < s < c$. By a similar argument we find that $K(c, \cdot)$ is in $C^\infty(I \cap [c, b])$ and that $\tau^*\overline{K(c, s)} = 0$, $c < s < b$.

Now let $K(t, s) = K_-(t, s)$, $s < t$, $K(t, s) = K_+(t, s)$, $s > t$. If $a < c < b$, then, by Green's formula (2.5), we have

$$
\begin{aligned}
f(c) &= \int_I K(c, s)(\tau f)(s)ds \\
&= \int_a^c K_-(c, s)(\tau f)(s)ds + \int_c^b K_+(c, s)(\tau f)(s)ds && [*] \\
&= F_c(f, \overline{K}_-) - F_c(f, \overline{K}_+) = F_c(f, \overline{K}_- - \overline{K}_+)
\end{aligned}
$$

for each f in $\mathfrak{D}(T_0(\tau))$, where the derivatives at c of $\overline{K}_+$ and $\overline{K}_-$, appearing in $F_c(f, \overline{K}_+)$ and $F_c(f, \overline{K}_-)$ are taken on the left and right respectively.

Now, by Lemma 2.2, $F_c(f, g)$ is a non-singular form in the vectors $[f(c), \ldots, f^{(n-1)}(c)]$ and $[g(c), \ldots, g^{(n-1)}(c)]$. Thus the equation

$$f(c) = F_c(f, \eta),$$

if assumed to be valid for all f in $\mathfrak{D}(T_0(\tau))$, determines η and its first $n-1$ derivatives uniquely. It is equivalent to the set of equations

$$[**] \qquad \sum_{j=0}^{n-1} F_c^{ij}(\tau)\overline{\eta^{(j)}(c)} = \delta_0^i, \qquad i = 0, \ldots, n-1,$$

where $\delta_j^i = 1$ if $i = j$ and $\delta_j^i = 0$ if $i \neq j$. From the form given for the matrix F_c^{ij} in the second displayed formula preceding Definition 2.1, we see that

$$
\begin{aligned}
F_c^{0(n-1)}(\tau) &= (-1)^{n-1}a_n(c), \\
F_c^{i(n-1)}(\tau) &= 0 \text{ if } i \neq 0.
\end{aligned}
$$

Thus the solution of the system [**] of equations is

$$\eta(c) = \ldots = \eta^{(n-2)}(c) = 0, \quad \eta^{(n-1)}(c) = (-1)^{n-1}a_n(c)^{-1}.$$

Therefore the jump of $K(c, s)$ at $s = c$ is described by the n equations

$$K_-^{(i)}(c, c-) - K_+^{(i)}(c, c+) = 0, \qquad i = 0, \ldots, n-2,$$
$$K_+^{(n-1)}(c, c+) - K_-^{(n-1)}(c, c-) = (-1)^n [\overline{a_n(c)}]^{-1}.$$

Q.E.D.

Further information on the kernel K is given in the next lemma.

5 LEMMA. *The function $\overline{K(c, \cdot)}$ obtained from the kernel defined in the preceding lemma by fixing c in I, satisfies the boundary conditions $B_i^*(f) = 0$, $i = 1, \ldots, k^*$, defining T^*.*

PROOF. The notation of the proof of the preceding lemma will be used. Let $a < c < b$ and let g be a function in $C^\infty(I)$ which coincides with $K(c, \cdot)$ in neighborhoods of both a and b and such that $\bar{g}$ is in $\mathfrak{D}(T_1(\tau^*))$ (we recall from Lemma 4 that $K(c, s)$ is infinitely differentiable except at $s = c$). An application of Green's formula yields

$$\int_I (\tau f)(s)\big(g(s) - K(c, s)\big) ds = \int_a^c (\tau f)(s)\big(g(s) - K_-(c, s)\big) ds$$
$$+ \int_c^b (\tau f)(s)\big(g(s) - K_+(c, s)\big) ds$$
$$= \int_I f(s)\overline{(\tau^* \bar{g})(s)} ds - F_c(f, \bar{K}_-) + F_c(f, \bar{K}_+), \qquad f \in \mathfrak{D}(T_1(\tau)).$$

Recalling that

$$f(c) = F_c(f, \bar{K}_-) - F_c(f, \bar{K}_+) = \int_I (\tau f)(s) K(c, s) ds$$

for f in $\mathfrak{D}(T)$, we obtain the equation

$$\int_I (\tau f)(s) g(s) ds = \int_I f(s)\overline{(\tau^* \bar{g})(s)} ds, \qquad f \in \mathfrak{D}(T).$$

Thus g lies in $\mathfrak{D}(T^*)$, so that $\bar{g}$ satisfies the boundary conditions $B_i^*(f) = 0$, $i = 1, \ldots, k^*$ of Lemma 3. Now g coincides with $K(c, \cdot)$ in neighborhoods of both a and b, and in view of the remark following Corollary 2.28 we see that $\overline{K(c, \cdot)}$ satisfies the same boundary conditions. Q.E.D.

6 LEMMA. (a) *The kernel $K(s, t)$ is infinitely differentiable in both variables if $s \neq t$.*

(b) *The functions $K_+(\cdot, \cdot)$ and $K_-(\cdot, \cdot)$ occurring in the statement of Lemma 4 are continuous in their domains of definition.*

PROOF. (a) Let $\varphi_1^*, \ldots, \varphi_{p^*}^*$ and $\psi_1^*, \ldots, \psi_{q^*}^*$ be bases for the solutions of $\tau^*\sigma = 0$ which are square-integrable in neighborhoods of a and b respectively. For c in I it follows from Lemma 4 that K can be represented in the form

$$[\dagger] \qquad \begin{aligned} K(c, s) &= \sum_{i=1}^{p^*} \alpha_i(c)\overline{\varphi_i^*(s)}, \qquad s < c, \\ &= \sum_{i=1}^{q^*} \beta_i(c)\overline{\psi_i^*(s)}, \qquad s > c. \end{aligned}$$

The p^*+q^* constants $\alpha_i(c)$ and $\beta_i(c)$ will now be computed. Consider the n linear equations

$$[1] \qquad \begin{aligned} &K^{(i)}(c, c+0) - K^{(i)}(c, c-0) = 0, \qquad 0 \leq i \leq n-2, \\ &K^{(n-1)}(c, c+0) - K^{(n-1)}(c, c-0) = (-1)^n[\overline{a_n(c)}]^{-1}. \end{aligned}$$

In view of [†], the equations [1] may be rewritten in the form

$$[1'] \qquad \begin{aligned} &\sum_{j=1}^{p^*} \alpha_j(c)\overline{\varphi_j^{*(i)}(c)} - \sum_{l=1}^{q^*} \beta_l(c)\overline{\psi_l^{*(i)}(c)} = 0, \qquad i = 0, \ldots, n-2, \\ &\sum_{j=1}^{p^*} \alpha_j(c)\overline{\varphi_j^{*(n-1)}(c)} - \sum_{l=1}^{q^*} \beta_l(c)\overline{\psi_l^{*(n-1)}(c)} = (-1)^n[\overline{a_n(c)}]^{-1}. \end{aligned}$$

By Lemma 5, the function $\overline{K(c, \cdot)}$ satisfies the equations

$$[2] \qquad B_i^*(\bar{K}) = 0, \qquad i = 1, \ldots, k^*.$$

By Theorem 2.19 we may write $B_i^* = C_i^* + D_i^*$ where C_i^* and D_i^* are boundary values at a and b respectively, and using [†], equations [2] may be rewritten in the form

$$[2'] \qquad \sum_{j=1}^{p^*} \overline{\alpha_j(c)} C_i^*(\varphi_j^*) + \sum_{l=1}^{q^*} \overline{\beta_l(c)} D_i^*(\psi_l^*) = 0, \qquad i = 1, \ldots, k^*.$$

By Lemma 1 and Lemma 2, $p^*+q^* = k^*+n$. Hence the system composed of the equations in [1′] and [2′] has a unique solution $\alpha_i(c)$, $\beta_i(c)$ if the corresponding homogeneous system of equations has only the trivial solution.

Suppose that the homogeneous system obtained from equations [1'] and [2'] has a non-trivial solution $\alpha_i^0(c)$, $\beta_i^0(c)$, and let $K_0(\cdot)$ be the function (of the variable s) obtained by substituting α_i^0 and β_i^0 for α_i and β_i in [†]. The function K_0 is differentiable n times for $s \neq c$. The equations [1'] satisfied by α_i^0 and β_i^0 imply that K_0 is also differentiable $n-1$ times at c. Thus, $\overline{K}_0 \in \mathfrak{D}(T_1(\tau))$. This being the case, equations [2'] show that $\overline{K}_0$ lies in the domain of T^*, and equations [†] show that $T^*\overline{K}_0 = 0$. Since T has a bounded inverse, T^* has a bounded inverse. Hence K_0 must vanish identically. Thus equations [1'] and [2'] have unique solutions $\alpha_i(c)$ and $\beta_i(c)$.

Observe that all coefficients in [1'] and [2'] are infinitely differentiable functions of c. The homogeneous system of p^*+q^* equations for p^*+q^* unknowns obtained from [1'] and [2'] has just been seen to have no non-zero solutions. Hence, the determinant of the system [1']+[2'] is non-zero, and the solutions of the system may be expressed (e.g., by Cramer's rule) in terms of certain determinants of its coefficients. It follows that α_i and β_i are infinitely differentiable functions of c. The remaining statement of the lemma is now an immediate consequence of formula [†]. Q.E.D.

Remark. Equations [1'] will often be referred to as the "jump equations."

To obtain a more explicit formula for K we next investigate the form of the adjoint of R.

7 Lemma. *Let K be the kernel related to R as in Lemma* 4. *Then*

(a) *for fixed c in I the function $K(\cdot, c)$ lies in $L_2(I)$. The adjoint R^* of R can be represented in the form*

$$(R^*g)(s) = \int_I \overline{K(t, s)}g(t)dt;$$

(b) *for fixed c in I and for $t \neq c$, we have $\tau K(t, c) = 0$;*

(c) *$K(\cdot, c)$ satisfies the boundary conditions defining T.*

Proof. Let J be a compact subinterval of I. Let K_1 be the kernel related to R^* as in Lemma 4. The kernels K and K_1 are bounded on J, by Lemma 6. If $f, g \in \mathfrak{D}(T_0(\tau)) = \mathfrak{D}(T_0(\tau^*))$ and if f and g vanish outside J it follows from Fubini's theorem that

$$\int_I \int_I K(s, t) f(t) \overline{g(s)} dt ds = (Rf, g)$$
$$= \overline{(R^* g, f)} = \int_I \int_I \overline{K_1(t, s) g(s)} f(t) ds dt.$$

The set of functions of the form $f(t)g(s)$, where f and g are functions in $\mathfrak{D}(T_0(\tau))$ which vanish outside J is fundamental in $L_2(J \times J)$. Hence it follows that $K(s, t) = \overline{K_1(t, s)}$ for almost all $(s, t) \in J \times J$. Since both these kernels are continuous for $t \neq s$ by Lemma 6, it follows that $K(s, t) = \overline{K_1(t, s)}$ for all points (s, t) in $J \times J$ such that $s \neq t$. The present lemma follows from this and Lemmas 4 and 5. Q.E.D

8 THEOREM. *Let τ be a formal differential operator defined on the interval I and let T be an operator obtained from τ by the imposition of k linearly independent boundary conditions $B_i(f) = 0$, $i = 1, \ldots, k$.*

Let $\{\varphi_i\}$, $i = 1, \ldots, p$, ($\{\varphi_i^*\}$, $i = 1, \ldots, p^*$) be a basis for those solutions of $\tau\sigma = 0$ ($\tau^*\sigma = 0$) which are square-integrable in a neighborhood of a, and let $\{\psi_i\}$, $i = 1, \ldots, q$, ($\{\psi_i^*\}$, $i = 1, \ldots, q^*$) be a basis for those solutions of $\tau\sigma = 0$($\tau^*\sigma = 0$) which are square-integrable in a neighborhood of b. Then there exist unique scalar matrices $\Gamma = (\gamma_{ij})$, $i = 1, \ldots, q$, $j = 1, \ldots, p^*$, and $\Gamma' = (\gamma'_{ij})$, $i = 1, \ldots, p$, $j = 1, \ldots q^*$, such that Green's function K defined in Lemma 4 is representable in the form

[*]
$$K(t, s) = \sum_{i=1}^{q} \sum_{j=1}^{p^*} \gamma_{ij} \psi_i(t) \overline{\varphi_j^*(s)}, \qquad s < t$$
$$= \sum_{i=1}^{p} \sum_{j=1}^{q^*} \gamma'_{ij} \varphi_i(t) \overline{\psi_j^*(s)}, \qquad s > t.$$

PROOF. In the proof of Lemma 6 the kernel K has been represented in the form

[†]
$$K(t, s) = \sum_{j=1}^{p^*} \alpha_j(t) \overline{\varphi_j^*(s)}, \qquad s < t$$
$$\sum_{j=1}^{q^*} \beta_j(t) \overline{\psi_j^*(s)}, \qquad s > t.$$

where the functions $\alpha_j(t)$, $\beta_j(t)$ are infinitely differentiable. Using Lemma 7, and applying τ to both sides of these equations, we obtain

$$\sum_{j=1}^{\sigma^*} (\tau\alpha_j)(t)\overline{\varphi_j^*(s)} = 0, \qquad s < t,$$

$$\sum_{j=1}^{\sigma^*} (\tau\beta_j)(t)\overline{\psi_j^*(s)} = 0, \qquad s > t.$$

From the linear independence of the functions φ_j^* and ψ_j^* it follows that $\tau\alpha_j = \tau\beta_j = 0$.

It remains to show that the functions α_j and β_j are square-integrable in neighborhoods of b and a respectively. If Φ_j^* is any other basis for the set of solutions of the equations $\tau^*\sigma = 0$ which are square-integrable in the neighborhood of a, then the coefficients $\hat{\alpha}_i$ defined by the relation

$$K(t, s) = \sum_{j=1}^{\sigma^*} \hat{\alpha}_j(t)\overline{\Phi_j^*(s)} = \sum_{i=1}^{\sigma^*} \alpha_j(t)\overline{\varphi_j^*(s)}, \qquad s < t,$$

will be related to the coefficients α_i by a non-singular linear transformation, each α_i being a linear combination with constant coefficients of the $\hat{\alpha}_i$, and conversely. Thus the α_i will be square-integrable in a neighborhood of b if and only if this is true of the $\hat{\alpha}_i$. Thus, if $J = [c, d]$ is a compact interval interior to I, we may assume without loss of generality that the φ_i^* are orthonormal over J. It follows then from [†] that

$$\alpha_i(t) = \int_c^d K(t, s)\varphi_i^*(s)ds, \qquad t > s.$$

Now for each s in I, $(R^*f)(s)$ is a continuous linear functional on L_2, and for each f, $(R^*f)(s)$ is continuous in s. Thus $\sup_{s\in J} |(R^*f)(s)| < \infty$ for f in $L_2(I)$, and from the uniform boundedness theorem it follows that the functionals $f \to (R^*f)(s)$ are uniformly bounded in norm for $s \in J$. Since

$$(R^*f)(s) = \int_I \overline{K(t, s)}f(t)dt, \qquad f \in L_2(I),$$

it follows from Theorem IV.8.1 that $\sup_{s\in J} \int_I |K(t, s)|^2 dt < \infty$. Hence

$$\int_J \int_I |K(t, s)|^2 \, dt ds < \infty,$$

and by Schwarz' inequality and Fubini's theorem

$$\begin{aligned}\int_a^b |\alpha_i(t)|^2 dt &\le \int_I \left| \int_J K(t, s)\varphi_i^*(s) ds \right|^2 dt \\ &\le \int_I \int_J |K(t, s)|^2 ds\, dt \\ &= \int_J \int_I |K(t, s)|^2 dt\, ds < \infty.\end{aligned}$$

Thus the functions α_i are square-integrable in a neighborhood of b. In the same way it may be shown that the functions β_i are square-integrable in a neighborhood of a. Q.E.D.

The following corollary enables us to use Theorem 8 as a computational algorithm in connection with specific examples.

9 Corollary. *The matrices $\Gamma = (\gamma_{ij})$ and $\Gamma' = (\gamma'_{ij})$ in the preceding theorem are uniquely determined by the jump equations and by the boundary conditions defining T.*

Proof. We have seen in the derivation of Theorem 8 that the functions $\alpha_i(t)$ and $\beta_i(t)$ are uniquely determined by the jump equations and by the boundary conditions $B_i^*(\bar{K}) = 0$, $i = 1, \ldots, k^*$, satisfied by $\overline{K(c, \cdot)}$. It has also been observed in Lemma 7 that $\overline{K(s, t)}$ is the kernel for R^*. By symmetry K is also uniquely determined by the jump equations and the boundary conditions defining T. Hence, since the functions ψ_i form a linearly independent set, and the functions φ_i, ψ_i^*, φ_i^* do also, the matrices Γ and Γ' are uniquely determined by the jump equations and the boundary conditions defining T. Q.E.D.

The calculation of K by means of Corollary 9 can be simplified if the set of boundary conditions for T or T^* is separated, or, more generally, if this set contains a certain number of boundary conditions at a or at b. In describing this simplification, it is convenient to introduce some notations, which will be used in the next few theorems. As before, let $\varphi_1, \ldots, \varphi_p$ be a basis for the space $\mathfrak{B}_a$ of those solutions of $\tau\sigma = 0$ which are square-integrable in a neighborhood of a; similarly, let $\psi_1, \ldots, \psi_q$ be a basis for $\mathfrak{B}_b$, the space of solutions which are square-integrable near b. Let the set of boundary conditions for T (or some equivalent set) be written in the form $[B_1, \ldots, B_k] = [C_1, \ldots, C_\mu] \cup [D_1, \ldots, D_\nu] \cup [E_1, \ldots, E_\omega]$ where C_i, D_i and E_i are respectively boundary conditions at a, at b, and mixed boundary

conditions. Suppose that φ_i and ψ_i are so chosen that $\varphi_1, \ldots, \varphi_u$, $u \leqq p$, forms a basis for the subspace $\mathfrak{W}_a$ of $\mathfrak{B}_a$ consisting of those members of $\mathfrak{B}_a$ satisfying $C_i(f) = 0$, $i = 1, \ldots, \mu$, and $\psi_1, \ldots, \psi_v$, $v \leqq q$, is a basis for the subspace $\mathfrak{W}_b$ of $\mathfrak{B}_b$ consisting of those members of $\mathfrak{B}_b$ satisfying $D_i(f) = 0$, $i = 1, \ldots, \nu$. The symbols C_i^*, $\mathfrak{B}_b^*$, φ_i^*, μ^*, u^*, etc., will denote corresponding integers, spaces of solutions, etc., associated with the equation $\tau^*\sigma = 0$ and with the boundary values $B_1^*, \ldots, B_k^*$ for T^*.

10 THEOREM. *Let T have the bounded inverse R. Then in the notations of the preceding paragraph,*

$$u = p-\mu, \quad v = q-\nu, \quad u^* = p^*-\mu^*, \text{ and } v^* = q^*-\nu^*.$$

Moreover, the kernel K for R has the representation

$$\begin{aligned} K(t, s) &= \sum_{i=1}^{v} \sum_{j=1}^{u^*} \gamma_{ij}\psi_i(t)\overline{\varphi_j^*(s)}, & s < t, \\ &= \sum_{i=1}^{u} \sum_{j=1}^{v^*} \gamma'_{ij}\varphi_i(t)\overline{\psi_j^*(s)}, & t < s, \end{aligned}$$

where the constants γ_{ij} and γ'_{ij} are uniquely determined by the jump equations and the remaining set of mixed boundary equations

$$E_i(K) = 0, \qquad i = 1, \ldots, \omega, \qquad \omega = k-(\mu+\nu).$$

PROOF. Since $\mathfrak{W}_a$ is a subspace of the p-dimensional linear manifold $\mathfrak{B}_a$ determined by μ linear conditions, it is clear that $u = \dim \mathfrak{W}_a \geqq p-\mu$. Similarly $v = \dim \mathfrak{W}_b \geqq q-\nu$. On the other hand, if $\dim \mathfrak{W}_a + \dim \mathfrak{W}_b > p+q-(\mu+\nu) = n+k-(\mu+\nu)$ (cf. Lemma 2 as applied to τ^*), then, since $\dim (\mathfrak{W}_a + \mathfrak{W}_b) \leqq n$, the equation

$$\dim (\mathfrak{W}_a + \mathfrak{W}_b) + \dim (\mathfrak{W}_a \cap \mathfrak{W}_b) = \dim \mathfrak{W}_a + \dim \mathfrak{W}_b$$

shows that $\dim (\mathfrak{W}_a \cap \mathfrak{W}_b) > k-(\mu+\nu) = \omega$. It follows that $\mathfrak{W}_a \cap \mathfrak{W}_b$ contains a non-zero vector f satisfying the remaining mixed boundary conditions $E_i(f) = 0$, $i = 1, \ldots, \omega$; thus f is in $\mathfrak{D}(T)$ and $Tf = 0$, contradicting the fact that T has an inverse. Therefore $u = p-\mu$ and $v = q-\nu$. The equations $u^* = p^*-\mu^*$ and $v^* = q^*-\nu^*$ are proved similarly.

Because $\overline{K(t, \cdot)}$ satisfies the boundary conditions $C_i^*(f) = 0$ and $D_i^*(f) = 0$, it has the representation

$$K(t, s) = \sum_{i=1}^{u^*} \alpha_i(t)\overline{\varphi_i^*(s)}, \qquad s < t,$$
$$= \sum_{i=1}^{v^*} \beta_i(t)\overline{\psi_i^*(s)}, \qquad s > t,$$

similar to the representation given by equation [†] in the proof of Lemma 6. To determine the $u^*+v^* = (p^*+q^*)-(\mu^*+\nu^*) = (n+k^*)-(\mu^*+\nu^*)$ numbers $\alpha_i(t)$ and $\beta_i(t)$ we have the n jump conditions and the remaining boundary conditions $E_i^*(\bar{K}) = 0$, $i = 1, \ldots, \omega^*$, $\omega^* = k^*-(\mu^*+\nu^*)$. An argument analogous to that used in the proof of Lemma 6 establishes that these two sets of conditions determine $\alpha_i(t)$ and $\beta_i(t)$ uniquely. Then, just as in the proof of Lemma 7, we show that $\alpha_i \in \mathfrak{B}_b$, $\beta_i \in \mathfrak{B}_a$. Thus (γ_{ij}) and (γ'_{ij}) are uniquely determined by the jump conditions and by the boundary conditions $E_i^*(\bar{K}) = 0$, $i = 1, \ldots, \omega^*$. By symmetry (γ_{ij}) and (γ'_{ij}) are also determined uniquely by the jump conditions and the boundary conditions $E_i(K) = 0$, $i = 1, \ldots, \omega$, (cf. Corollary 9). Q.E.D.

A particularly important case arises if both T and T^* are determined by separated sets of boundary conditions. Then $\mu+\nu = k$, $\mu^*+\nu^* = k^*$, and the coefficients γ_{ij} and γ'_{ij} are uniquely determined by the jump equations. By Lemmas 1 and 2, $p^*+q^* = n+k^*$, so $u^*+v^* = p^*+q^*-(\mu^*+\nu^*) = n$. Similarly $u+v = n$. If a function σ were at the same time a linear combination of $\varphi_1, \ldots, \varphi_u$ and of $\psi_1, \ldots, \psi_v$, then σ would be in $L_2(I)$, and we would have $\tau\sigma = 0$, and $B_i(\sigma) = 0$, $i = 1, \ldots, k$. Thus σ would be in the domain of T and $T\sigma = 0$, which shows that $\sigma = 0$. The sets $\{\varphi_1, \ldots, \varphi_u\}$ and $\{\psi_1, \ldots, \psi_v\}$ must therefore be linearly independent. Since $u+v = n$, they constitute a basis for all solutions of $\tau\sigma = 0$ on I. Similarly $\{\varphi_1^*, \ldots, \varphi_{u^*}^*\} \cup \{\psi_1^*, \ldots, \psi_{v^*}^*\}$ is a basis for all solutions of $\tau^*\sigma = 0$.

Writing

$$K(t, s) = \sum_{i=1}^{u^*} \alpha_i(t)\overline{\varphi_i^*(s)}, \qquad s < t,$$
$$= \sum_{i=1}^{v^*} \beta_i(t)\overline{\psi_i^*(s)}, \qquad s > t,$$

it is seen from Lemma 4(c) that the jump equations are equivalent to the relation

$$f(t) = \sum_{i=1}^{u^*} \alpha_i(t) F_t(f, \varphi_i^*) - \sum_{j=1}^{v^*} \beta_j(t) F_t(f, \psi_j^*), \quad f \in C^{n-1}(I).$$

Define $a_i = \alpha_i$, $i = 1, \ldots, u^*$, $a_i = \beta_{i-u^*}$, $i = u^*+1, \ldots, n$ and $\eta_i^* = \varphi_i^*$, $i = 1, \ldots, u^*$, $\eta_i^* = -\psi^*_{i-u^*}$, $i = u^*+1, \ldots, n$. Then this relation takes the simpler form

$$[3] \qquad f(t) = \sum_{i=1}^{n} a_i(t) F_t(f, \eta_i^*), \qquad f \in C^{n-1}(I).$$

Since F_t is a form involving only f and its first $n-1$ derivatives, equation [3] holds for all f in $C^{n-1}(I)$ if it holds for any subspace of functions for which the values of the first $n-1$ derivatives may be arbitrarily prescribed at any point; in particular for the subspace of solutions of $\tau\sigma = 0$. We have seen (cf. Theorem 10) that $\tau a_i = 0$, $i = 1, \ldots, n$. Thus choosing a basis $\{\xi_i\}$, $i = 1, \ldots, n$, for the solutions of $\tau\sigma = 0$, and defining the matrix $\{\Gamma_{ij}\}$ by the equations

$$a_i = \sum_{j=1}^{n} \Gamma_{ij} \xi_j, \qquad i = 1, \ldots, n,$$

the jump equations are seen to be equivalent to the following set of equations:

$$[4] \qquad \sum_{i,j=1}^{n} \Gamma_{ij} F_t(\xi_l, \eta_i^*) \xi_j(t) = \xi_l(t), \qquad l = 1, \ldots, n.$$

Now by Green's formula,

$$\begin{aligned} F_{t_1}(\xi_l, \eta_i^*) - F_{t_2}(\xi_l, \eta_i^*) &= \int_{t_2}^{t_1} \{(\tau\xi_l)(t)\overline{\eta_i^*(t)} - \xi_l(t)\overline{(\tau^*\eta_i^*)(t)}\} dt \\ &= 0. \end{aligned}$$

Thus $F_t(\xi_l, \eta_i^*)$ is independent of t, and [4] is equivalent to the set of equations

$$\sum_{i=1}^{n} \Gamma_{ij} F_t(\xi_l, \eta_i^*) = \delta_{jl}, \qquad 1 \leq j, l \leq n.$$

In other words, the matrix $\{\Gamma_{ij}\}$ is the inverse of the matrix $\{F_t(\xi_l, \eta_i^*)\}$. In terms of the matrix $\{\Gamma_{ij}\}$ we have

$$K(t, s) = \sum_{i=1}^{u^*} \sum_{j=1}^{n} \Gamma_{ij}\xi_j(t)\overline{\varphi_i^*(s)}, \qquad s < t,$$
$$\sum_{i=u^*+1}^{n} \sum_{j=1}^{n} \Gamma_{ij}\xi_j(t)\overline{\psi_{i-u^*}^*(s)}, \qquad s > t.$$

If, moreover, we take $\xi_j = \varphi_j$, $j = 1, \ldots, u$, and $\xi_j = \psi_{j-u}$, $j = u+1, \ldots, n$, then, since the constants Γ_{ij} which express the kernel K are evidently unique, it follows from the form for K given in Theorem 10, that we have $\Gamma_{ij} = 0$ if both $i < u^*$ and $j < u$, or if both $i > u^*$ and $j > u$.

These remarks are summarized in the following theorem, in which the notations introduced above will be employed.

11 Theorem. *Assume that T and T^* are defined by sets of separated boundary conditions; let $\xi_i = \varphi_i$, $i = 1, \ldots, u$, and $\xi_i = \psi_{i-u}$, $i = u+1, \ldots, n$; $\eta_i^* = \varphi_i^*$, $i = 1, \ldots, u^*$ and $\eta_i^* = -\psi_{i-u^*}^*$ $i = u^*+1, \ldots, n$. Then:*

(a) *the sets $\{\xi_i\}$ and $\{\eta_i^*\}$ are bases for the spaces of all solutions of $\tau\sigma = 0$ and $\tau^*\sigma = 0$ respectively;*

(b) *the matrix $(F_t(\xi_i, \eta_j^*))$ is independent of t and non-singular;*

(c) *if (Γ_{ij}) denotes the inverse of the matrix $(F_t(\xi_i, \eta_j^*))$, then $\Gamma_{ij} = 0$ if both $i \leq u^*$ and $j \leq u$ or if both $i > u^*$ and $j > u$;*

(d) *the kernel K for the inverse R of T is given by the formula*

$$K(t, s) = \sum_{i=1}^{u^*} \sum_{j=u+1}^{n} \Gamma_{ij}\overline{\varphi_i^*(s)}\psi_{j-u}(t), \qquad s < t,$$
$$-\sum_{i=u^*+1}^{n} \sum_{j=1}^{u} \Gamma_{ij}\overline{\psi_{i-u^*}^*(s)}\varphi_j(t), \qquad s > t.$$

For future reference, we will now write out the form which Theorems 10 and 11 take when stated for the resolvent of a differential operator.

12 Corollary. *Let τ be a formal differential operator, and let T be a densely defined operator obtained from τ by the imposition of a set $\{B_i\}$ of boundary conditions. Let the sets of boundary conditions defining T and T^* each be divided into three subsets as described in the paragraph preceding Theorem 10. For each λ in $\rho(T)$, let $\varphi_i(\cdot, \lambda)$, $1 \leq i \leq u$, $\{\varphi_i^*(\cdot, \bar{\lambda})$, $1 \leq i \leq u^*\}$ and let $\psi_i(\cdot, \lambda)$, $1 \leq i \leq v$, $\{\psi_i^*(\cdot, \bar{\lambda})$, $1 < i < v^*\}$ be bases*

for all solutions of $(\tau-\lambda)\sigma = 0$ $\left((\tau^*-\bar{\lambda})\sigma = 0\right)$ *which are square-integrable in a neighborhood of a and b respectively, and which satisfy the boundary conditions at a and at b respectively. Then the resolvent* $R(\lambda; T) = (\lambda I - T)^{-1}$ *is given by the expression*

$$\left(R(\lambda; T)f\right)(t) = \int_I f(s)K(t, s; \lambda)ds, \qquad f \in L_2(I),$$

where the kernel K *has the representation*

$$\begin{aligned} K(t, s; \lambda) &= \sum_{i=1}^{v}\sum_{j=1}^{u^*}\gamma_{ij}(\lambda)\psi_i(t;\lambda)\overline{\varphi_j^*(s;\bar{\lambda})}, & s < t, \\ &= \sum_{i=1}^{u}\sum_{j=1}^{v^*}\gamma'_{ij}(\lambda)\varphi_i(t;\lambda)\overline{\psi_j^*(s;\bar{\lambda})}, & s > t. \end{aligned}$$

The functions $\gamma_{ij}(\cdot)$ *and* $\gamma'_{ij}(\cdot)$ *are uniquely determined by the mixed boundary conditions defining* T, *and by the following set of jump equations:*

$$\begin{aligned} &K^{(i)}(c, c+0) - K^{(i)}(c, c-0) = 0, \quad 0 \leq i \leq n-2, \\ &K^{(n-1)}(c, c+0) - K^{(n-1)}(c, c-0) = (-1)^n\left[\overline{a_n(c)}\right]^{-1}. \end{aligned}$$

13 COROLLARY. *Let* τ *be a formal differential operator, and let* T *be the operator in Hilbert space obtained from* τ *by the imposition of a separated set of boundary conditions. Suppose that* T^* *is also defined by a separated set of boundary conditions. In the notation of the preceding corollary, let* $\xi_i = \varphi_i$, $1 \leq i \leq u$, $\xi_i = \psi_{i-u}$, $u+1 \leq i \leq n$; $\eta_i^* = \varphi_i^*$, $1 \leq i \leq u^*$, *and* $\eta_i^* = \psi_{i-u^*}$, $u^*+1 \leq i \leq n$, *for* $\lambda \in \rho(T)$. *Then under the hypotheses of the preceding corollary we have:*

(a) *for each* λ *in* $\rho(T)$ *the sets* $\{\xi_i\}$ *and* $\{\eta_i^*\}$ *are bases for the spaces of solutions of* $\tau\sigma = \lambda\sigma$ *and* $\tau^*\sigma = \bar{\lambda}\sigma$ *respectively;*

(b) *the matrix* $\{F_t(\xi_i, \eta_j^*)\}$ *is independent of* t *and nonsingular for each* λ *in* $\rho(T)$;

(c) *if* $\{\Gamma_{ij}(\lambda)\}$ *denotes the inverse of the matrix* $\{F_t(\xi_i, \eta_j^*)\}$, *then* $\Gamma_{ij} = 0$ *if both* $i \leq u^*$ *and* $j \leq u$ *or if both* $i > u^*$ *and* $j > u$;

(d) *the kernel* K *defining the resolvent* $R(\lambda; T)$ *is given by the formula*

$$\begin{aligned} K(t, s; \lambda) &= \sum_{i=1}^{u^*}\sum_{j=u+1}^{n}\Gamma_{ij}(\lambda)\xi_j(t, \lambda)\overline{\varphi_i^*(s, \bar{\lambda})}, & s < t, \\ &= -\sum_{i=u^*+1}^{n}\sum_{j=1}^{u}\Gamma_{ij}(\lambda)\xi_j(t, \lambda)\overline{\psi_{i-u^*}^*(s, \bar{\lambda})}, & s > t. \end{aligned}$$

Let τ be a real formally symmetric formal differential operator. A boundary value A for τ is said to be *real* if $A(\bar{f}) = \overline{A(f)}$ for each f in $\mathfrak{D}(T_1(\tau))$. A boundary condition $A(f) = 0$ is said to be real if the boundary value A is real: a set of boundary conditions is said to be real if all its members are real. If T is a self adjoint operator obtained from τ by imposing a real symmetric set of boundary conditions, a further simplification can be obtained by noticing that the solutions of the equation $(\tau^* - \bar{\lambda})\sigma = 0$ are complex conjugates of the solutions of the equations $(\tau - \lambda)\sigma = 0$, so that for each λ in $\rho(T)$ we may take $\varphi_i(\cdot, \lambda) = \overline{\varphi_i^*(\cdot, \bar{\lambda})}$ and $\psi_i(\cdot, \lambda) = \overline{\psi_i^*(\cdot, \bar{\lambda})}$. Then the operator $T - \lambda I$ has a bounded inverse for each non-real λ, since T is self adjoint (XII.2.2).

14 COROLLARY. *In the notation of Corollary 12, and under the further assumption that τ is a real formally self adjoint formal differential operator and that T is a self adjoint operator obtained from τ by the imposition of a real symmetric set of boundary conditions, the resolvent $R(\lambda; T)$ of T is given by the formula*

$$(R(\lambda; T)f)(t) = \int_I f(s)K(t, s; \lambda)ds, \qquad f \in L_2(I),$$

where the kernel K has the representation:

$$K(t, s; \lambda) = \begin{cases} \displaystyle\sum_{i=1}^{v} \sum_{j=1}^{u} \gamma_{ij}(\lambda)\psi_i(t, \lambda)\varphi_j(s, \lambda), & s < t, \\ \displaystyle\sum_{i=1}^{u} \sum_{j=1}^{v} \gamma'_{ij}(\lambda)\varphi_i(t, \lambda)\psi_j(s, \lambda), & s > t. \end{cases}$$

The functions γ_{ij} and γ'_{ij} are uniquely determined by the mixed boundary conditions for T and the following set of jump equations:

$$K^{(i)}(c, c+0) - K^{(i)}(c, c-0) = 0, \qquad 0 \leq i \leq n-2,$$
$$K^{(n-1)}(c, c+0) - K^{(n-1)}(c, c-0) = (-1)^n[\overline{a_n(c)}]^{-1} = [a_n(c)]^{-1}.$$

15 COROLLARY. *Under the hypotheses and with the notation of the preceding corollary and under the further assumption that the boundary conditions imposed on τ are separated, the kernel K has the form*

$$K(t, s; \lambda) = -\sum_{i=1}^{u} \sum_{j=u+1}^{n} \Gamma_{ij}(\lambda)\psi_i(t, \lambda)\varphi_j(s, \lambda), \qquad s < t,$$

$$= \sum_{i=u+1}^{n} \sum_{j=1}^{u} \Gamma_{ij}(\lambda)\varphi_i(t, \lambda)\psi_j(s, \lambda), \qquad s > t.$$

The matrix $\{\Gamma_{ij}(\lambda)\}$ *is the inverse of the matrix* $\{F_{ij}\}$ *where* $F_{ij} = F_t(\xi_i, \bar{\xi}_j)$ *for* $i \leq u$, $F_{ij} = F_t(\xi_i, \bar{\xi}_j)$ *for* $i > u$, *and* $\xi_i = \varphi_i$, $i = 1, \ldots u$, $\xi_i = \psi_{i-u}$, $i = u+1, \ldots, n$.

An interesting and important case of Corollary 13 is provided when $n = 2$, when the operator τ has the form $\tau f = (pf')' + qf$, and when T and T^* are determined by separated boundary conditions. Suppose also there is only one solution φ (φ^*) of $\tau\sigma = 0$ ($\tau^*\sigma = 0$) which is square-integrable at a and which satisfies the boundary conditions at a for T (T^*), and similarly that there are single solutions ψ and ψ^* corresponding to the end point b. For this operator

$$F_t(f, g) = p(t)[f'(t)\overline{g(t)} - f(t)\overline{g'(t)}] = p(t)W_t(f, \bar{g})$$

where $W_t(f, g) = f'(t)g(t) - f(t)g'(t)$ is the ***Wronskian*** of the functions f and g. The matrix $\{\Gamma_{ij}\}$ is a two-by-two matrix with vanishing diagonal elements, and it follows immediately that its inverse $\{F_t(\xi_i, \eta_j^*)\}$, ($\xi_1 = \varphi$, $\xi_2 = \psi$, $\eta_1^* = \varphi^*$, $\eta_2^* = -\psi^*$) also has vanishing diagonal elements. An elementary calculation shows that

$$\Gamma_{12} = \frac{1}{p(t)W_t(\psi, \overline{\varphi^*})}, \quad \Gamma_{21} = \frac{1}{p(t)W_t(\varphi, -\overline{\psi^*})}.$$

Thus

$$K(t, s; \lambda) = \frac{\psi(t, \lambda)\overline{\varphi^*(s, \bar{\lambda})}}{p(t)W_t(\overline{\varphi^*(\bar{\lambda})}, \psi(\lambda))}, \qquad s < t,$$

$$= \frac{\varphi(t, \lambda)\overline{\psi^*(s, \bar{\lambda})}}{p(t)W_t(\varphi(\lambda), \overline{\psi^*(\bar{\lambda})})}, \qquad s > t.$$

If we assume in addition that τ is formally symmetric (and hence real) and that the set of boundary conditions is symmetric and real, these formulas specialize as in Corollary 15 to

$$K(t, s; \lambda) = \frac{\psi(t, \lambda)\varphi(s, \lambda)}{p(t)W_t(\varphi(\lambda), \psi(\lambda))}, \qquad s < t,$$

[*]

$$= \frac{\varphi(t, \lambda)\psi(s, \lambda)}{p(t)W_t(\varphi(\lambda), \psi(\lambda))}, \qquad s > t.$$

The next theorem shows that the resolvent is given by the formula [*] for a large and important class of second order differential operators.

16 THEOREM. *Let T be a self adjoint operator derived from a real formal differential operator $\tau = -(d/dt)[p(t)(d/dt)]+q(t)$ by the imposition of a separated symmetric set of boundary conditions. Let $\mathscr{I}\lambda \neq 0$. Then the boundary conditions are real, and there is exactly one solution $\varphi(t, \lambda)$ of $(\tau-\lambda)\sigma = 0$ square-integrable at a and satisfying the boundary conditions at a, and exactly one solution $\psi(t, \lambda)$ of $(\tau-\lambda)\sigma = 0$ square-integrable at b satisfying the boundary conditions at b. The resolvent $R(\lambda; T)$ is an integral operator whose kernel $K(t, s; \lambda)$ is given by formula* [*].

PROOF. The result follows immediately from Theorem 2.32 and the preceding discussion. Q.E.D.

Finally, it should be noted that Corollary 14 does not cover all self adjoint cases, and that if complex coefficients occur in a formally symmetric τ, as they may, we must fall back on Corollary 12 or Corollary 13 in calculating the resolvent. Consider, for instance, the formal operator $\tau = i(d/dt)$ on $(-\infty, +\infty)$ introduced as an example in the last section. Here, as indicated in the last section, there are no boundary values. Thus Corollary 12 can be applied to calculate the inverse of $\lambda I - T$ as follows. If $\mathscr{I}\lambda > 0$, then $\tau\sigma = \lambda\sigma$ has the single solution $ce^{-i\lambda t}$ square-integrable in the neighborhood of $-\infty$, and no solution square-integrable in the neighborhood of $+\infty$. The equation $(\tau^*-\bar{\lambda}) = 0$ has the solution $c'e^{-i\bar{\lambda}t}$ square-integrable in the neighborhood of $+\infty$ and none square-integrable in the neighborhood of $-\infty$. Thus our desired kernel will be of the form

$$K_\lambda(t, s; \lambda) = c\varphi(t)\overline{\psi^*(s)} = ce^{-i\lambda(t-s)}, \qquad s > t,$$
$$= 0 \qquad\qquad s < t,$$

where c is to be determined by the jump conditions of Corollary 12. i.e., $c = -i$. That is: for $\mathscr{I}\lambda > 0$, we have the formula

$$((\lambda I - T)^{-1}f)(t) = -i\int_t^\infty e^{-i\lambda(t-s)}f(s)ds.$$

It is easy to see that for $\mathscr{I}(\lambda) < 0$ this must be replaced by the formula

$$((\lambda I - T)^{-1}f)(t) = i\int_{-\infty}^t e^{-i\lambda(t-s)}f(s)ds.$$

4. Spectral Theory: Compact Resolvents

We saw in Section 2 that with each formally self adjoint formal differential operator τ there may be associated a symmetric operator $T_0(\tau)$ in the Hilbert space $L_2(I)$. and saw how to obtain self adjoint extensions T of $T_0(\tau)$ by the imposition of boundary conditions on $\mathfrak{D}(T_0^*)$. The present section will be devoted to the spectral theory of such extensions T in the important special case in which the resolvent $R(\lambda; T)$ is compact for λ non-real. We shall make a detailed examination of the specific analytical form which the spectral theorem assumes in the case of such a self adjoint extension T.

Drawing on the results of the preceding section, we begin by identifying two important cases in which the resolvent of T is compact.

1 THEOREM. *Let τ be a formally symmetric formal differential operator defined on an interval I. Let T be a self adjoint extension of the symmetric operator $T_0(\tau)$. The resolvent $R(\lambda; T)$ is compact for every non-real λ if either*

(1) *the interval I is compact*

or

(2) *the deficiency indices of $T_0(\tau)$ are equal to the order of the differential operator τ.*

PROOF. Since, in case (1), every solution of an equation $\tau\sigma = \lambda\sigma$ belongs to $C(I)$, and since, in this case, $C(I) \subseteqq L_2(I)$, case (1) is included in case (2).

It follows from Corollary 3.12 that for $\mathscr{I}\lambda \neq 0$, $R(\lambda; T)$ may be expressed by a kernel

$$(R(\lambda; T)f)(t) = \int_I f(s)K(t, s; \lambda)ds.$$

Since, in case (2), we assume that for $\mathscr{I}\lambda \neq 0$ every solution of $\tau\sigma = \lambda\sigma$ belongs to $L_2(I)$, the expression for $K(t, s; \lambda)$ given by Corollary 3.12 shows immediately that

$$\int_I\int_I |K(t, s; \lambda)|^2\, ds dt < \infty.$$

To complete the proof it is therefore sufficient to show that every integral operator in $L_2(I)$ defined by a kernel K with

$$\|K\|^2 = \int_I\int_I |K(t, s)|^2\, ds dt < \infty$$

is compact. This is a special case of Exercise VI.9.52, but, for the sake of completeness, a proof will be given here.

Note first, that by Schwarz' inequality,

$$\int_I \left|\int_I K(t, s) f(s) ds\right|^2 dt \leqq \left\{\int_I\int_I |K(t, s)|^2\, ds dt\right\}\left\{\int_I |f(s)|^2\, ds\right\}.$$

Hence, the norm of the operator in $L_2(I)$ defined by the kernel K is less than or equal to $\|K\|$. Since the set of compact operators is closed (cf. VI.5.3), to prove that $R(\lambda; T)$ is compact it will suffice to observe that the integrable simple functions in $L_2(I \times I)$ define compact operators (since such operators have finite dimensional ranges) and that such functions are dense in $L_2(I \times I)$. Q.E.D.

The next theorem gives the spectral theory of an unbounded self adjoint differential operator with a compact resolvent.

2 THEOREM. (*Spectral Theorem*) *Let τ be a formally symmetric formal differential operator of order n. Let T be a self adjoint extension of $T_0(\tau)$ such that $R(\lambda; T)$ is compact for non real λ. Then*

(a) *the spectrum of T is a sequence of points on the real axis with no finite limit point;*

(b) *each λ in the spectrum of T belongs to the point spectrum of T. Moreover,* $\dim E(\{\lambda\})L_2(I) \leq n$;

(c) *there is a complete orthonormal set $\{\varphi_m\}$, $m = 0, 1, \ldots,$ of eigenfunctions for T. If φ is an eigenfunction corresponding to the eigenvalue λ, then $\varphi \in C^\infty(I)$ and φ is a solution of the equation $\tau\varphi - \lambda\varphi = 0$.*

PROOF. If $R(\lambda; T)$ is compact, then (cf. VII.4.5) its spectrum consists of a sequence of points converging to zero. By the spectral mapping theorem (XII.2.9(b)), $\sigma(T) \subseteq f(\sigma(R(\lambda; T)))$, where $f(\mu) =$

$\lambda = \mu^{-1}$ for $\mu \neq 0$. Since T is self adjoint, $\sigma(T)$ is real. Statement (a) follows from these remarks.

If λ is in $\sigma(T)$, then, since λ is an isolated point of $\sigma(T)$, it follows immediately from XII.2.9(b) and X.3.8(i) that $E(\lambda) \neq 0$.

If $\lambda \in \sigma(T)$, then $(T-\lambda I)E(\lambda)f = 0$; hence, the spectrum of T is identical with its point spectrum. Since $T = T^* \subseteqq T_0(\tau)^* = T_1(\tau)$ by Theorem 2.10, it follows from Theorem 1.8 that every function in the range of $E(\lambda)$ is a C^∞ solution of $\tau\sigma = \lambda\sigma$. Hence, by the remark following Corollary 1.5, the range of $E(\lambda)$ is at most n-dimensional. This proves (b).

To prove (c), we have only to show that the set of eigenfunctions of T is complete. Since for each element f of our Hilbert space we have

$$f = \sum_{\lambda \in \sigma(T)} E(\lambda)f,$$

and since we have observed above that $E(\lambda)f$ is an eigenfunction of T, this is obvious. Q.E.D.

Before passing to the next section, which deals with spectral representation theory of T in the cases where T does not have a compact resolvent, it is worth pausing for a moment to give an elementary but useful result on the pointwise convergence of eigenfunction expansions

3 THEOREM. *Let τ be a formally symmetric formal differential operator defined on an interval I. Let T be a self adjoint extension of $T_0(\tau)$. Suppose that T has a complete orthonormal set $\{\varphi_n\}$ of eigenfunctions. Then for f in $\mathfrak{D}(T)$, the eigenfunction expansion*

$$f = \sum_{n=0}^{\infty} (f, \varphi_n)\varphi_n$$

converges uniformly and absolutely on each finite closed subinterval of I. The series may be differentiated term by term $(n-1)$ times, each differentiated series retaining the properties of absolute and uniform convergence.

PROOF. Let $f \in \mathfrak{D}(T)$ and $\lambda \in \rho(T)$. Then

$$f - \sum_{i=0}^{n} (f, \varphi_i)\varphi_i = R(\lambda; T)g_n,$$

where $g_n = (\lambda I - T)f - \sum_{i=0}^{n}((\lambda I - T)f, \varphi_i)\varphi_i$. Since $g_n \to 0$ and $TR(\lambda; T)$ is a bounded operator, it follows that $R(\lambda; T)g_n$ and $TR(\lambda; T)g_n$ converge to zero in $L_2(I)$. Thus by Lemma 2.16 the series $\sum_{i=0}^{\infty}(f, \varphi_i)\varphi_i$ converges to f in the topology of $C^{n-1}(J)$ for each compact interval J of I. Since the series converges unconditionally in $L_2(I)$, it follows that it converges unconditionally in $C^{n-1}(J)$. Consequently each of the differentiated series converges absolutely. Q.E.D.

5. Spectral Theory: General Case

In the preceding section the spectral theory of self adjoint operators T derived from formal differential operators in the case where the resolvent of T is compact was discussed. We turn in this section to the study of the general case in which T is allowed to have a continuous spectrum. The methods to be used are largely those of the spectral representation and spectral multiplicity theory developed in Section XII.3. It will appear, in fact, that most of what is necessary to prove the fundamental eigenfunction expansion theorem formulated as Theorem 1 below has already been established in Theorems XII.3.11 and XII.3.19.

1 THEOREM. *Let τ be a formally self adjoint formal differential operator on an interval I, and let T be a self adjoint extension of $T_0(\tau)$. Let U be an ordered representation of $L_2(I)$ relative to T, with measure μ, multiplicity sets e_i, and multiplicity m. Then m is not greater than the order n of τ. There exist kernels $W_i(t, \lambda)$, $i = 1, \ldots, m$, measurable with respect to the product of Lebesgue measure v and μ, which vanish for λ in the complement of e_i, belong to $C^\infty(I)$ for each fixed λ, and satisfy the differential equation $(\tau - \lambda)W_i(\cdot, \lambda) = 0$ for each fixed λ. Moreover the kernels W_i have the property that*

$$v\text{-ess}\sup_{t \in J} \int_e |W_i(t, \lambda)|^2 \mu(d\lambda) < \infty$$

for each compact subinterval J of I and bounded Borel set e, and are such that

$$\text{(i)} \qquad (Uf)_i(\lambda) = \int_I f(t)\overline{W_i(t, \lambda)}dt, \qquad f \in L_2(I),$$

the integral existing in the mean square sense in $L_2(\mu, e_i)$;

(ii) *for each Borel function* F,

$$U\mathfrak{D}(F(T)) = \left\{ [f_i] \,\middle|\, \sum_{i=1}^{m} \int_{-\infty}^{+\infty} |F(\lambda)|^2 |f_i(\lambda)|^2 \mu(d\lambda) < \infty \right\}$$

and

$$(UF(T)g)_i(\lambda) = F(\lambda)(Ug)_i(\lambda), \qquad g \in \mathfrak{D}(F(T)), \qquad -\infty < \lambda < +\infty.$$

PROOF. By Corollaries XII.8.13, and XII.8.14 we find that there exists a μ-null set N and kernels W_i, $i = 1, \ldots, m$, satisfying formulae (i) and (ii), such that for $\lambda \in e_i - N$, $(T_0(\tau) - \lambda)^* W_i(\cdot, \lambda) = 0$. Thus, by Theorem 2.10, if we put $W_i(\cdot, \lambda) = 0$ for $\lambda \notin N$, and modify $W_i(t, \lambda)$ on a suitable Lebesgue null set $M(\lambda)$ for each $\lambda \in N$, we obtain a function $\tilde{W}_i$ such that

$$\tau \tilde{W}_i(t, \lambda) = \lambda \tilde{W}_i(t, \lambda), \qquad t \in I,$$

for all λ. If $\tilde{W}_i$ is measurable with respect to the product of μ and Lebesgue measure, it will follow from Fubini's theorem that we can take $W_i = \tilde{W}_i$. To see that $\tilde{W}_i$ is measurable note that, by the fundamental theorem of calculus,

$$\begin{aligned} \tilde{W}_i(t, \lambda) &= \lim_{n \to \infty} \frac{1}{2n} \int_{[t-1/n,\, t+1/n]} \tilde{W}_i(s, \lambda) ds \\ &= \lim_{n \to \infty} \frac{1}{2n} \int_{[t-1/n,\, t+1/n]} W_i(s, \lambda) ds \end{aligned}$$

for every t interior to I and $\lambda \notin N$. This establishes all parts of the theorem except the assertion that $m \leq n$. However, since by XIII.3.19 the functions $W_1(\cdot, \lambda), \ldots, W_m(\cdot, \lambda)$ are linearly independent for μ-almost all λ in e_m, and since $\tau\sigma = \lambda\sigma$ has at most n linearly independent solutions for any λ, this too is evident. Q.E.D.

2 COROLLARY. (*Inversion Formula*) *Let* I, W_i, *etc., be as in the preceding theorem. Then, for each* $f \in L_2(I)$, *we have*

$$f(t) = \lim_{A \to \infty} \int_{-A}^{+A} \sum_{i=1}^{m} (Uf)_i(\lambda) W_i(t, \lambda) \mu(d\lambda),$$

the limit existing in the mean square sense in $L_2(I)$.

PROOF. The corollary follows from the preceding theorem and from Corollary XII.3.12. Q.E.D.

3 COROLLARY. *Let the operator T and kernels $W_1, \ldots, W_m$ be defined as in Theorem 1 and let F be a bounded Borel measurable function vanishing outside a bounded Borel set e of the real axis. Then the bounded operator $F(T)$ has the representation*

$$(F(T)f)(t) = \int_I f(s)K(F; t, s)ds, \qquad f \in L_2(I),$$

where

$$K(F; t, s) = \sum_{i=1}^{m} \int_e F(\lambda)W_i(t, \lambda)\overline{W_i(s, \lambda)}\mu(d\lambda),$$

and

$$\sup_{t \in J} \int_I |K(F; t, s)|^2 ds < \infty,$$

where J is any compact subinterval of I.

PROOF. From the boundedness of $F(T)$ and Lemma 2.16, it follows that the map $f \to F(T)f$ is a continuous map of $L_2(I)$ into $C(J)$. Thus there is a constant $M(J)$ such that

$$\sup_{t \in J} |(F(T)f)(t)| \leq M(J)|f|, \qquad f \in L_2(I).$$

It follows from Theorem 1(ii) and Corollary 2 that

$$(F(T)f)(t) = \int_e \sum_{i=1}^{m} F(\lambda)W_i(t, \lambda)\int_I f(s)\overline{W_i(s, \lambda)}ds\mu(d\lambda),$$

where the integral $\int_e f(s)\overline{W_i(s, \lambda)}ds$ exists in the mean square sense in $L_2(\mu)$. Let $\mathfrak{H}_0$ denote the dense set of those f in $L_2(I)$ each of which vanishes outside a compact subinterval of I. If $f \in \mathfrak{H}_0$ we may interchange the order of the integrations in the formula above, obtaining the equation

$$[*] \qquad (F(T)f)(t) = \int_I f(s)K(F; t, s)ds, \qquad f \in \mathfrak{H}_0,$$

and the inequality

$$\sup_{t \in J} \left| \int_I f(s) K(F; t, s) ds \right| \leq M(J) |f|, \qquad f \in \mathfrak{H}_0,$$

where

$$K(F; t, s) = \sum_{i=1}^{m} \int_e F(\lambda) W_i(t, \lambda) \overline{W_i(s, \lambda)} \mu(d\lambda).$$

It follows from Theorem IV.8.1 that

$$\left[\int_I |K(F; t, s)|^2 ds \right]^{\frac{1}{2}} \leq M(J), \qquad t \in J,$$

and that equation [*] holds for all f in $L_2(I)$. Q.E.D

As in Definition XII.3.15, the integer m of Theorem 1 is called the spectral multiplicity of the operator T. The next theorem is useful for determining the spectral multiplicity in certain cases.

4 Theorem. *Let the operator T, the kernels $W_1, \ldots, W_m$, and the measure μ be defined as in Theorem 1. Let a be a fixed end point of I, and let $B(f) = 0$ be a boundary condition at a for τ satisfied by all f in $\mathfrak{D}(T)$. Then $B(W_i(\cdot, \lambda)) = 0$, μ-almost everywhere for each of the kernels W_i.*

Proof. By Corollary 2.28 there is an f in $\mathfrak{D}(T_1(\tau))$ such that $B(g) = \lim_{t \to a} F_t(g, f) = F_a(g, f)$ for all $g \in \mathfrak{D}(T_1(\tau))$. It is clear we can assume without loss of generality that f vanishes in a neighborhood of b. By Green's formula (2.5), we have

$$B(\bar{g}) = (\tau f, g) - (f, \tau g) \tag{[*]}$$

for g in $\mathfrak{D}(T_1(\tau))$. Hence $(\tau f, g) = (f, Tg)$ for all g in $\mathfrak{D}(T)$. Since T is self adjoint, it follows that f is in $\mathfrak{D}(T)$. By Theorem 1,

$$\int_a^b \{(\tau f)(t) - \lambda f(t)\} \overline{W_i(t, \lambda)} dt = 0$$

for μ-almost all λ, the integral existing in the mean square sense in $L_2(\mu, e_i)$. However, since f vanishes near b and $W_i(t, \lambda)$ is square-integrable near a, the integral exists in the ordinary sense. Consequently, since $(\tau - \lambda) W_i(\cdot, \lambda) = 0$, formula [*] gives

$$B(W_i(\cdot, \lambda)) = \int_a^b \{(\tau f)(t) - \lambda f(t)\} \overline{W_i(t, \lambda)} dt = 0$$

for μ-almost all λ. Q.E.D.

5 Corollary. *Let τ be a real second order operator defined on an interval I and let a be a fixed end point of I. If T is a self adjoint operator obtained by restricting τ by a set of boundary conditions including at least one boundary condition at a, then the spectral multiplicity of T is one.*

Proof. It is clear that the multiplicity m must be at least one. On the other hand, if $B(f) = 0$ is any non-trivial boundary condition at a, it is clear that the common set of solutions of $(\tau-\lambda)\sigma = 0$ and $B(\sigma) = 0$ is one-dimensional. Since by the preceding theorem the functions W_i of Theorem 1 satisfy both these equations, it follows from their linear independence (cf. Theorem XII.8.19) that $m \leqq 1$. Q.E.D.

We have seen in Theorem 1 and Corollary 2 that an arbitrary vector f in $L_2(I)$ has an expansion of "Fourier integral" type in terms of eigenfunctions $W_i(t, \lambda)$ of the differential operator τ. Unfortunately, the interest of Theorem 1 is more theoretical than practical, since it is difficult to construct the functions $W_i(t, \lambda)$ explicitly. In practice it is more convenient to choose some suitable basis $\sigma_i(t, \lambda)$, $i = 1, \ldots, n$, for the set of solutions of $(\tau-\lambda)\sigma = 0$, and to express the expansion of an arbitrary function f in terms of the functions $\sigma_i(t, \lambda)$. While the behavior of the W_i as functions of the variable λ may be quite complicated, the basis $\sigma_i(t, \lambda)$ can be chosen to be continuous in the pair (t, λ) and even analytic in λ (cf. Corollary 1.5). However, when the expansion theorem is reformulated in terms of an arbitrary basis $\sigma_1, \ldots, \sigma_n$ for the set of solutions of $(\tau-\lambda)\sigma = 0$, the details of the discussion of the convergence of the resulting series or integral expansions become more complicated, the Hilbert space $\sum L_2(\mu_i)$ being replaced by an appropriate L_2 space with respect to a positive semi-definite matrix of set functions. We shall now examine these measure-theoretic questions.

6 Definition. Let $\{\mu_{ij}\}$, $1 \leqq i, j \leqq n$, be a family of complex valued set functions defined on the bounded Borel subsets of the real line. The family $\{\mu_{ij}\}$ will be called an *n by n positive matrix measure* if

(i) the matrix $\{\mu_{ij}(e)\}$ is Hermitian and positive semi-definite for each bounded Borel set e,

(ii) we have

$$\mu_{ij}\Big(\bigcup_{m=1}^{\infty} e_m\Big) = \sum_{m=1}^{\infty} \mu_{ij}(e_m)$$

for each sequence of disjoint Borel sets with bounded union.

7 LEMMA. *Let $\{\mu_{ij}\}$ be a positive matrix measure whose elements μ_{ij} are continuous with respect to a positive σ-finite measure μ. If the matrix of densities $\{m_{ij}\}$ is defined by the equations*

$$\mu_{ij}(e) = \int_e m_{ij}(\lambda)\mu(d\lambda),$$

where e is any bounded Borel set, then the matrix $\{m_{ij}(\lambda)\}$ is positive semi-definite for μ-almost all λ.

PROOF. We observe first that the set e_0 of all λ for which the matrix $\{m_{ij}(\lambda)\}$ is positive semi-definite is measurable, for it is the set of λ such that

$$\sum_{i,j=1}^{n} m_{ij}(\lambda)\xi_i\bar{\xi}_j \geqq 0$$

for every vector $[\xi_1, \ldots, \xi_n]$ in E^n whose elements are rational. If the lemma is false, it follows easily that there exists a vector $[\xi_1, \ldots, \xi_n]$ with rational elements and a set $e \subseteqq e_0'$ of positive μ-measure such that

$$\sum_{i,j=1}^{n} m_{ij}(\lambda)\xi_i\bar{\xi}_j < 0$$

for λ in e. But then

$$0 \leqq \sum_{i,j=1}^{n} \mu(e)\xi_i\bar{\xi}_j = \int_e \Big\{\sum_{i,j=1}^{n} m_{ij}(\lambda)\xi_i\bar{\xi}_j\Big\}\mu(d\lambda) < 0,$$

contradicting the hypothesis that $\{\mu_{ij}(e)\}$ is positive semi-definite. Q.E.D.

8 DEFINITION. Let $\{\mu_{ij}\}$ be a positive n by n matrix measure on the real line and let μ be a σ-finite positive regular measure with respect to which all the set functions μ_{ij} are absolutely continuous. If $\{m_{ij}\}$ denotes the matrix of densities of $\{\mu_{ij}\}$ with respect to μ, the

family of n-tuples $F = [f_1, \ldots, f_n]$ of Borel measurable functions defined on the real axis for which

$$|F|^2 = \int_{-\infty}^{+\infty} \left\{ \sum_{i,j=1}^{n} m_{ij}(\lambda) f_i(\lambda) \overline{f_j(\lambda)} \right\} \mu(d\lambda) < \infty$$

will be denoted by $L_2^0(\{\mu_{ij}\})$.

An element F of $L_2^0(\{\mu_{ij}\})$ will be said to be a $\{\mu_{ij}\}$-*null function* if $|F| = 0$. The set of all equivalence classes of elements of $L_2^0(\{\mu_{ij}\})$ modulo $\{\mu_{ij}\}$-null functions will be denoted by $L_2(\{\mu_{ij}\})$.

We observe that by Lemma 7, the integrand in the integral above is non-negative for μ-almost all λ. Also, if F, G are in $L_2^0(\{\mu_{ij}\})$, we may regard their values $[f_1(\lambda), \ldots, f_n(\lambda)]$ and $[g_1(\lambda), \ldots, g_n(\lambda)]$ for a particular value of λ as elements of n-dimensional unitary space E^n. Applying the Schwarz inequality for the positive semi-definite inner product $\sum_{i,j=1}^{n} m_{ij}(\lambda)\xi_i\bar{\xi}_j$ in E^n (cf. the remark following Theorem IV.4.1), we obtain the inequality

$$\left| \sum_{i,j=1}^{n} m_{ij}(\lambda) f_i(\lambda) \overline{g_j(\lambda)} \right|$$
$$\leqq \left\{ \sum_{i,j=1}^{n} m_{ij}(\lambda) f_i(\lambda) \overline{f_j(\lambda)} \right\}^{\frac{1}{2}} \left\{ \sum_{i,j=1}^{n} m_{ij}(\lambda) g_i(\lambda) \overline{g_j(\lambda)} \right\}^{\frac{1}{2}}$$

for μ-almost all λ. From these remarks follows the existence of the integral

$$[*] \qquad (F, G) = \int_{-\infty}^{+\infty} \left\{ \sum_{i,j=1}^{n} m_{ij}(\lambda) f_i(\lambda) \overline{g_j(\lambda)} \right\} \mu(d\lambda),$$

and the inequality

$$|(F, G)| \leqq |F|\,|G|.$$

Since clearly $|F+G|^2 = |F|^2 + (F, G) + (G, F) + |G|^2$, it follows from this inequality that the sum of two $\{\mu_{ij}\}$-null elements is a $\{\mu_{ij}\}$-null element. Since a scalar multiple of a $\{\mu_{ij}\}$-null element is evidently a $\{\mu_{ij}\}$-null element, the family $N(\{\mu_{ij}\})$ of $\{\mu_{ij}\}$-null elements is a linear subspace of $L_2^0(\{\mu_{ij}\})$. We shall follow the usual practice of denoting elements of $L_2^0(\{\mu_{ij}\})$ and $L_2(\{\mu_{ij}\})$ by the same symbol $[f_1, \ldots, f_n]$.

If $F = [f_1, \ldots, f_n]$ and $G = [g_1, \ldots, g_n]$ belong to $L_2(\{\mu_{ij}\})$, and e is a bounded Borel set, we shall often write

$$\int_e \left\{ \sum_{i,j=1}^{n} f_i(\lambda)\overline{g_j(\lambda)} \right\} \mu_{ij}(d\lambda)$$

instead of

$$\int_e \left\{ \sum_{i,j=1}^{n} m_{ij}(\lambda) f_i(\lambda)\overline{g_j(\lambda)} \right\} \mu(d\lambda).$$

To prove that the integral [*] is independent of the measure μ we may argue as follows. Let $\tilde{\mu}$ be another σ-finite positive regular measure with respect to which the set functions μ_{ij} are continuous. Let $\{\tilde{m}_{ij}\}$ be the corresponding matrix of densities, and let $\{n_{ij}\}$ be the matrix of densities of the μ_{ij} with respect to the measure $(\mu+\tilde{\mu})$. If m is the density of μ with respect to $(\mu+\tilde{\mu})$, then

$$\mu_{ij}(e) = \int_e m_{ij}(\lambda)\mu(d\lambda) = \int_e m_{ij}(\lambda) m(\lambda)(\mu+\tilde{\mu})(d\lambda)$$

for every bounded Borel set e. Hence, $m_{ij}m = n_{ij}$ for $(\mu+\tilde{\mu})$-almost all λ. Given measurable functions f_i and g_i, $i = 1, \ldots, n$, it follows from Corollary III.10.6 that

$$\int_{-\infty}^{\infty} \left\{ \sum_{i,j=1}^{n} m_{ij}(\lambda) f_i(\lambda)\overline{g_j(\lambda)} \right\} \mu(d\lambda) = \int_{-\infty}^{\infty} \left\{ \sum_{i,j=1}^{n} n_{ij}(\lambda) f_i(\lambda)\overline{g_j(\lambda)} \right\} (\mu+\tilde{\mu})(d\lambda).$$

By an exactly similar argument we obtain an analogous formula in which μ and m_{ij} are replaced by $\tilde{\mu}$ and $\tilde{m}_{ij}$ on the left hand side. Thus

$$\int_{-\infty}^{\infty} \left\{ \sum_{i,j=1}^{n} m_{ij}(\lambda) f_i(\lambda)\overline{g_j(\lambda)} \right\} \mu(d\lambda) = \int_{-\infty}^{\infty} \left\{ \sum_{i,j=1}^{n} \tilde{m}_{ij}(\lambda) f_i(\lambda)\overline{g_j(\lambda)} \right\} \tilde{\mu}(d\lambda).$$

9 LEMMA. *The space $L_2(\{\mu_{ij}\})$ is a normed linear space with a positive definite Hermitian inner product defined by the formula*

$$(F, G) = \int_{-\infty}^{\infty} \left\{ \sum_{i,j=1}^{n} m_{ij}(\lambda) f_i(\lambda)\overline{g_j(\lambda)} \right\} \mu(d\lambda)$$

for every pair $F = [f_1, \ldots, f_n]$ and $G = [g_1, \ldots, g_n]$ in $L_2(\{\mu_{ij}\})$.

PROOF. It was shown in the discussion above that $L_2(\{\mu_{ij}\})$ is a linear space and that

$$|(F, G)| \leq |F|\,|G|, \qquad F, G \in L_2(\{\mu_{ij}\}).$$

From this it follows that $|F+G| \leq |F|+|G|$ (cf. the remark following Theorem IV.4.1). Q.E.D.

In order to show that $L_2(\{\mu_{ij}\})$ is a Hilbert space, it remains to prove that it is complete.

10 THEOREM. *If $\{\mu_{ij}\}$ is an $n \times n$ positive matrix measure defined on the real line, then $L_2(\{\mu_{ij}\})$ is a Hilbert space.*

The proof of this theorem will be based on the following lemma.

11 LEMMA. *Let $\{\mu_{ij}\}$ be a positive matrix measure whose elements are continuous with respect to a positive σ-finite measure μ. If $\{m_{ij}\}$ is the matrix of densities of μ_{ij} with respect to μ, then there exist non-negative μ-measurable functions φ_i, $i = 1, \ldots, n$, μ-integrable over each bounded interval, and μ-measurable functions a_{ij}, $1 \leq i, j \leq n$, such that for μ-almost all λ,*

$$\text{(a)} \qquad \sum_{j=1}^{n} a_{ij}(\lambda)\overline{a_{jl}(\lambda)} = \delta_{il},$$

and

$$\text{(b)} \qquad \sum_{j=1}^{n} \varphi_j(\lambda)a_{ji}(\lambda)\overline{a_{jl}(\lambda)} = m_{il}(\lambda).$$

Before giving the proof of Lemma 11 we shall use it to complete the proof of Theorem 10.

PROOF OF THEOREM 10. Let the functions φ_i and a_{ij} have the properties of Lemma 11. For each i let ν_i be the positive measure defined by the formula

$$\nu_i(e) = \int_e \varphi_i(\lambda)\mu(d\lambda).$$

Let $\mathfrak{H}$ be the direct sum of the Hilbert spaces $L_2(\nu_i)$, $i = 1, \ldots, n$. Then $\mathfrak{H}$ is the Hilbert space of all n-tuples $F = [f_1, \ldots, f_n]$ of μ-measurable functions for which

$$|F| = \sum_{i=1}^{n} \left\{\int_{-\infty}^{\infty} |f_i(\lambda)|^2 \varphi_i(\lambda)\mu(d\lambda)\right\}^{\frac{1}{2}} < \infty.$$

Now let A be the mapping that assigns to each $G = [g_1, \ldots, g_n]$ in $L_2(\{\mu_{ij}\})$ the n-tuple of functions

$$f_i = \sum_{j=1}^{n} a_{ij} g_j.$$

Since

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left\{ \sum_{i,j=1}^{n} m_{ij}(\lambda) g_i(\lambda) \overline{g_j(\lambda)} \right\} \mu(d\lambda) \\
&= \int_{-\infty}^{\infty} \left\{ \sum_{i,j,k=1}^{n} [a_{ki}(\lambda) g_i(\lambda)] [\overline{a_{kj}(\lambda) g_j(\lambda)}] \varphi_k(\lambda) \right\} \mu(d\lambda) \\
&= \int_{-\infty}^{\infty} \sum_{k=1}^{n} |f_k(\lambda)|^2 \varphi_k(\lambda) \mu(d\lambda),
\end{aligned}$$

it is evident that A defines an isometric isomorphism of $L_2(\{\mu_{ij}\})$ into $\mathfrak{H}$. On the other hand, let B be the mapping which assigns to each $[f_1, \ldots, f_n]$ in $\mathfrak{H}$ the element $[g_1, \ldots, g_n]$ defined by

$$g_i(\lambda) = \sum_{j=1}^{n} \overline{a_{ij}(\lambda)} f_j.$$

Since by Lemma 11(a) the matrix $\{\overline{a_{ij}(\lambda)}\}$ is the inverse of the matrix $\{a_{ij}(\lambda)\}$ for μ-almost all λ, we see that $B = A^{-1}$. Consequently, A is an isometric mapping of $L_2(\{\mu_{ij}\})$ onto all of $\mathfrak{H}$, showing that $L_2(\{\mu_{ij}\})$ is complete. Q.E.D.

It remains to prove Lemma 11.

PROOF OF LEMMA 11. We shall first make an observation which will be used repeatedly in the proof. Suppose that for each set e_n in a sequence of sets we can find μ-measurable matrices $\{a_{ij}^{(n)}\}$ and functions $\{\varphi_i^{(n)}\}$ such that

$$\sum_{j=1}^{n} a_{ij}^{(n)}(\lambda) \overline{a_{jl}^{(n)}(\lambda)} = \delta_{il}$$

and

$$\sum_{j=1}^{k} \varphi_j^{(n)}(\lambda) a_{ji}^{(n)}(\lambda) \overline{a_{jl}^{(n)}(\lambda)} = m_{il}(\lambda)$$

for μ-almost all $\lambda \in e_n$. Then, if we define

$$a_{ij}(\lambda) = a_{ij}^{(n)}(\lambda)$$

and

$$\varphi_i(\lambda) = \varphi_i^{(n)}(\lambda)$$

for $\lambda \in e_n - \bigcup_{p=1}^{n-1} e_p$, it is clear that functions a_{ij} and φ_i satisfy the requirements (a) and (b) of the lemma for all λ in the set $\bigcup_{n=1}^{\infty} e_n$.

Since μ is σ finite the real axis is the union of a sequence of sets of finite μ-measure. By Lusin's lemma (XII.3.17) each set of finite measure differs by a null set from the union of a sequence of measurable sets on each of which the functions m_{ij} are continuous. Thus it suffices to prove that one can construct the functions a_{ij} and φ_i on any measurable set σ_0 of finite measure on which the functions m_{ij} are continuous.

A further reduction of the problem may be obtained as follows. Let the functions m_{ij} be continuous on σ_0, $\mu(\sigma_0) < \infty$, and let $M(\lambda)$, $\lambda \in \sigma_0$, be the Hermitian operator in n-dimensional unitary space whose matrix is $\{m_{ij}(\lambda)\}$. Let λ_0 be a point of σ_0, let ζ_0 be an eigenvalue of $M(\lambda_0)$, and U be a neighborhood of ζ_0 whose closure contains no other eigenvalues of $M(\lambda_0)$. By Corollary X.7.3,

$$\lim_{\substack{\lambda \to \lambda_0 \\ \lambda \in \sigma_0}} E(M(\lambda); U) = E(M(\lambda_0); \zeta_0),$$

where $E(M(\lambda); \cdot)$ is the spectral resolution of $M(\lambda)$. Thus $E(M(\lambda); U)$ is non-zero for λ near λ_0, $\lambda \in \sigma_0$, and it follows that for λ sufficiently close to λ_0, $\sigma(M(\lambda)) \cap U$ is non-void. Thus if $n(\lambda)$ denotes the number of distinct points in the spectrum of $M(\lambda)$, the sets $\{\lambda \in \sigma_0 | n(\lambda) \geqq s\}$ are relatively open in σ_0, and hence the sets $b_s = \{\lambda \in \sigma_0 | n(\lambda) = s\}$, $s = 1, \ldots, n$, are Borel sets. Thus, to prove the lemma, it suffices to show that we may construct the functions a_{ij} and φ_i on each of the sets b_s. On the other hand, by the regularity of μ, such a set differs by a null set from the union of a sequence of compact sets. Thus it suffices to show that if e_0 is any compact subset of b_s, and λ_0 is any point of e_0, then there is a neighborhood of λ_0 in e_0 in which there exist functions a_{ij} and φ_i having the properties of the lemma.

Let ε be so small that no two distinct eigenvalues of $M(\lambda_0)$ differ by a quantity of modulus less than ε. We saw above that for $\lambda \in e_0$ sufficiently close to λ_0, the $\varepsilon/2$-neighborhood of each point in $\sigma(M(\lambda_0))$ contains at least one point of $\sigma(M(\lambda))$. Since $\sigma(M(\lambda_0))$ and $\sigma(M(\lambda))$ have the same number of points (since $\lambda, \lambda_0 \in e_0$), it follows that if $\hat{\varphi}_1(\lambda_0), \ldots, \hat{\varphi}_k(\lambda_0)$ are the distinct eigenvalues of $M(\lambda_0)$, then, for each $\lambda \in e_0$ sufficiently close to λ_0, there exists a unique point

$\hat{\varphi}_i(\lambda) \in \sigma(M(\lambda))$ such that

$$|\hat{\varphi}_i(\lambda)-\hat{\varphi}_i(\lambda_0)| < \varepsilon/2, \qquad i = 1, \ldots, k.$$

Moreover, $\{\hat{\varphi}_1(\lambda), \ldots, \hat{\varphi}_k(\lambda)\} = \sigma(M(\lambda))$. Thus the functions $\varphi_i(\lambda) = \hat{\varphi}_i(\lambda)$ are defined on a neighborhood N_1 of λ_0 in e_0. It is readily seen, by a similar argument, that $\varphi_i(\lambda)$ depends continuously on λ. It then follows easily from Corollary X.7.8 that $E_i(\lambda) = E(M(\lambda); \varphi_i(\lambda))$ depends continuously on λ, $i = 1, \ldots, k$. Let $v_1, \ldots, v_n$ be an orthonormal basis for E^n such that $E_i(\lambda_0)v_j = v_j$, $n_{i-1} < j \leq n_i$, $0 = n_0 < n_1 < n_2 < \ldots < n_k = n$. Put $\hat{v}_j(\lambda) = E_i(\lambda)v_j$, $n_{i-1} < j \leq n_i$. Then $\hat{v}_j(\lambda)$ depends continuously on λ for $\lambda \in N_1$, and $M(\lambda)\hat{v}_j(\lambda) = \varphi_i(\lambda)\hat{v}_j(\lambda)$ for $n_{i-1} < j \leq n_i$. It follows readily (by an argument depending on the non-vanishing of the determinant det $(\hat{v}_i(\lambda), v_j)$ for λ near λ_0) that $\hat{v}_j(\lambda)$ form a basis for E^n for each λ in a neighborhood $N_2 \subseteq N_1$ of λ_0 in e_0. Since $E_i(\lambda)\hat{v}_j(\lambda) = \hat{v}_j(\lambda)$ for $n_{i-1} < j \leq n_i$, it follows that $\hat{v}_j(\lambda)$ and $\hat{v}_{j'}(\lambda)$ are orthogonal if there exists an $i \leq k$ such that $j \leq n_i < j'$. Let $v_1(\lambda), \ldots, v_n(\lambda)$ be the orthonormal basis for E^n obtained by applying the Gram-Schmidt orthonormalization process to $\hat{v}_1(\lambda), \ldots, \hat{v}_n(\lambda)$; that is, we define v_i inductively by the formulae

$$v_1(\lambda) = \frac{\hat{v}_1(\lambda)}{|\hat{v}_1(\lambda)|},$$

$$v_{i+1}(\lambda) = \frac{\hat{v}_{i+1}(\lambda) - \sum_{j=1}^{i} (\hat{v}_{i+1}(\lambda), v_j(\lambda))v_j(\lambda)}{|\hat{v}_{i+1}(\lambda) - \sum_{j=1}^{i} (\hat{v}_{i+1}(\lambda), v_j(\lambda))v_j(\lambda)|}.$$

It is readily seen by induction that $E_i(\lambda)v_j(\lambda) = v_j(\lambda)$ for $n_{i-1} < j \leq n_i$, so that $M(\lambda)v_j(\lambda) = \varphi_i(\lambda)v_j(\lambda)$ for $n_{i-1} < j \leq n_i$. In this way, we have constructed a continuously varying orthonormal basis for E^n, defined for $\lambda \in N_2$, and a set $\varphi_1, \ldots, \varphi_n$ of continuous functions defined for $\lambda \in N_2$, such that

$$M(\lambda)v_i(\lambda) = \varphi_i(\lambda)v_i(\lambda), \qquad i = 1, \ldots, n.$$

If $\{u_i\}$ is the basis in E^n whose element u_i is the n-tuple $[\delta_{ij}]$, $j = 1, \ldots, n$, define the functions a_{ij} by the formulae

$$u_j = \sum_{i=1}^{n} a_{ij}(\lambda) v_i(\lambda).$$

They are clearly continuous in $N(\lambda_0)$. Since $(v_i(\lambda), v_j(\lambda)) = \delta_{ij}$, we have

$$\sum_{i=1}^{n} a_{ij}(\lambda)\overline{a_{ik}(\lambda)} = (u_j, u_k) = \delta_{jk}, \qquad \lambda \in N(\lambda_0).$$

Similarly,

$$\begin{aligned} m_{jk}(\lambda) &= (M(\lambda)u_j, u_k) \\ &= \sum_{i=1}^{n} a_{ij}(\lambda)\overline{a_{ik}(\lambda)}(M(\lambda)v_i(\lambda), v_i(\lambda)) \\ &= \sum_{i=1}^{n} \varphi_i(\lambda) a_{ij}(\lambda)\overline{a_{ik}(\lambda)}, \qquad \lambda \in N(\lambda_0). \end{aligned}$$

Q.E.D.

With the proof of Theorem 10, the theory of the spaces $L_2(\{\mu_{ij}\})$ is placed on an equal footing with the theory of Hilbert spaces of the form $L_2(\mu)$. In what follows, we shall make free use of the Hilbert space properties of $L_2(\{\mu_{ij}\})$ established in Lemma 9 and Theorem 10, and also of a number of other elementary measure theoretic properties which these spaces have in common with the spaces $L_2(\mu)$. However, a few words of caution are in order. If $[f_1, \ldots, f_n]$ is in $L_2(\{\mu_{ij}\})$, it does not follow that any of the n-tuples $[f_1, 0, \ldots, 0]$, $\ldots, [0, \ldots, 0, f_n]$ belong to $L_2(\{\mu_{ij}\})$. For example, let μ be a finite positive regular measure, let $n = 2$, and let $\{\mu_{ij}(e)\}$ be the matrix

$$\begin{Bmatrix} \mu(e) & -\mu(e) \\ -\mu(e) & \mu(e) \end{Bmatrix}.$$

Then the density matrix of $\{\mu_{ij}\}$ with respect to μ is

$$\begin{Bmatrix} +1 & -1 \\ -1 & +1 \end{Bmatrix},$$

so that

$$\int_{-\infty}^{\infty} \left\{ \sum_{i,j=1}^{2} f_i(\lambda)\overline{f_j(\lambda)} m_{ij}(\lambda) \right\} \mu(d\lambda) = \int_{-\infty}^{\infty} |f_1(\lambda) - f_2(\lambda)|^2 \mu(d\lambda).$$

Thus, if $[f_1, f_2]$ is any pair of functions whose difference belongs to $L_2(\mu)$, $[f_1, f_2] \in L_2(\{\mu_{ij}\})$, even though neither $[f_1, 0]$ nor $[0, f_2]$ need belong to $L_2(\{\mu_{ij}\})$. This consideration should also make it plain that none of the individual terms in the symbolic integral

$$\int_e \left\{ \sum_{i,j=1}^n f_i(\lambda)\overline{f_j(\lambda)}\mu_{ij}(d\lambda) \right\}$$

need exist individually if the functions f_i are not bounded, so that the integral cannot in general be regarded as a sum of simple integrals. Of course, if the individual terms $\int_e f_i(\lambda)\overline{f_j(\lambda)}\mu_{ij}(d\lambda)$ do exist, then

$$\int_e \left\{ \sum_{i,j=1}^n f_i(\lambda)\overline{f_j(\lambda)}\mu_{ij}(d\lambda) \right\} = \sum_{i,j=1}^n \int_e f_i(\lambda)\overline{f_j(\lambda)}\mu_{ij}(d\lambda).$$

Having built up a sufficient basis in the measure theory of positive matrix measures, we now return to the eigenfunction expansion theory of differential equations.

What we wish to do is to express the expansion formula given by Theorem 1 in a form more suitable for specific calculations. This process involves a number of unavoidable technical complications. In order to explain the nature of these complications, we shall make a preliminary heuristic survey of the body of theory which comprises the remainder of the present section.

The kernels $W_i(t, \lambda)$ of Theorem 1 are C^∞ solutions of $(\tau-\lambda)W_i(\cdot, \lambda)$ for each fixed λ, but, as functions of λ, they are merely known to be measurable, so that they may be quite "wild". We can overcome some of this "wildness" as follows. Rather than dealing directly with the vector-valued functions $W_i(\cdot, \lambda)$ of λ, we choose a basis $\sigma_1(\cdot, \lambda), \ldots, \sigma_n(\cdot, \lambda)$ for the space of solutions of $\tau\sigma = \lambda\sigma$, taking our basis to be continuous in $t \times \lambda$, or even, if we like, C^∞ in t and analytic in λ. (Such a basis may, for instance, be determined by a specification of initial values such as $\sigma_{i+1}^{(j)}(c, \lambda) = \delta_i^j$, $0 \leq i, j \leq n-1$ (cf. Corollary 1.5)). Then $W_i(\cdot, \lambda)$ can be written uniquely in the form

$$W_i(\cdot, \lambda) = \sum_{j=1}^n a_{ij}(\lambda)\sigma_j(\cdot, \lambda).$$

The variation of W_i with λ can now be comprehended more readily

than before, since it is now expressed in terms of the variation of the finite set of coefficients a_{ij} with λ.

However, this simplification introduces new complications. For instance, the formula of Corollary 3, if written in terms of the functions $\sigma_i(\cdot, \lambda)$, becomes

$$K(F; t, s) = \sum_{i=1}^{m} \int_{e} \sum_{j,k=1}^{n} F(\lambda) a_{ij}(\lambda) \overline{a_{ik}(\lambda)} \sigma_j(t, \lambda) \overline{\sigma_k(s, \lambda)} \mu(d\lambda),$$

which may be written as

$$K(F: t, s) = \int_{e} \left\{ \sum_{j,\, k=1}^{n} F(\lambda) \sigma_j(t, \lambda) \overline{\sigma_k(s, \lambda)} \rho_{jk}(d\lambda) \right\}$$

in terms of the matrix measure ρ_{jk} defined by

$$\rho_{jk}(e) = \sum_{i=1}^{m} \int_{e} a_{ij}(\lambda) \overline{a_{ik}(\lambda)} \mu(d\lambda).$$

In this way matrix measures are unavoidably introduced into our expansion theory.

The very process of choosing a basis $\sigma_1(\cdot, \lambda), \ldots, \sigma_n(\cdot, \lambda)$ for the space of solutions of $\tau\sigma = \lambda\sigma$ introduces difficulties We can, of course, always follow the course indicated above; i.e., choose some point c in the interval I on which τ is defined, and define the basis $\sigma_1(\cdot, \lambda), \ldots, \sigma_n(\cdot, \lambda)$ by the requirement that

$$\sigma_{i+1}^{(j)}(c, \lambda) = \delta_i^j, \qquad 0 \leqq i, j \leqq n-1.$$

This procedure has the evident advantage that it makes $\sigma_i(\cdot, \lambda)$ an entire function of the complex variable λ; but it has drawbacks which, though less evident, are nevertheless decisive.

Suppose, for example, that we study the self adjoint operator T obtained from the formal differential operator $-(d/dt)^2$ on the interval $[0, \infty)$ by imposition of the boundary condition $f(0)+f'(0) = 0$. How shall the point c in $[0, \infty)$ be chosen to best effect? It is clear that any choice of c other than $c = 0$ will introduce an unnecessary asymmetry into all computations. Thus $c = 0$ appears as a natural choice. In this case an elementary calculation shows that the solutions σ_1 and σ_2 of $\tau\sigma = \lambda\sigma$ defined by the boundary conditions $\sigma^{(j)}{}_{i+1}(0) = \delta_i^j$, $i, j = 0, 1$, are $\cos \lambda^{\frac{1}{2}} t$ and $\lambda^{-\frac{1}{2}} \sin \lambda^{\frac{1}{2}} t$, respectively. These functions

are in fact entire in λ. In the range $\lambda > 0$ they form a perfectly suitable basis for the solutions of $\tau\sigma = \lambda\sigma$. However, in the range $\lambda < 0$, $\lambda^{\frac{1}{2}}$ is imaginary, and an analytic expression like $\cos \lambda^{\frac{1}{2}}t$ is hard to work with because of the apparent ambiguity in the definition of $\lambda^{\frac{1}{2}}$. An analysis of the spectrum of T shows that the only eigenvalue of T is $\lambda = -1$, and that the corresponding orthonormal eigenfunction φ (written in terms of σ_1 and σ_2) is

$$\varphi(t) = \sqrt{2}(i \sin it + \cos it).$$

Since

$$\sin t = \frac{e^{it}-e^{-it}}{2i}, \qquad \cos t = \frac{e^{it}+e^{-it}}{2},$$

this is simply $\sqrt{2}\, e^{-t}$. This fact makes evident what could readily be suspected before: that from the point of view of the negative real axis, a more favorable choice of basis for the solutions of $\tau\sigma = \lambda\sigma$ is

$$\hat{\sigma}_1(t, \lambda) = e^{-t\sqrt{-\lambda}}, \qquad \hat{\sigma}_2(t, \lambda) = e^{t\sqrt{-\lambda}}.$$

Of course, as long as we are dealing with the formal operator $\tau_0 = -(d/dt)^2$, so that the solutions of $\tau_0\sigma = \lambda\sigma$ are the familiar trigonometric (or exponential) functions, the problems arising out of any particular choice of basis are not particularly significant. But if, for instance, we consider the formal differential operator $\tau_1 = -(d/dx)^2 + (\alpha/x^2)$ on the interval $(0, \infty)$, so that the solutions of $\tau\sigma = \lambda\sigma$ are Bessel (Hankel, Neumann, cylinder) functions, these problems may become quite annoying. In the first place, if we intend to hold to the above method for choosing a basis σ_1, σ_2 for the solutions of $\tau\sigma = \lambda\sigma$, we must first choose some arbitrary point c in $(0, \infty)$; and this choice introduces an unnecessary asymmetry into all our subsequent calculations. The effect of such a choice may be inferred from the fact that a corresponding choice made in connection with the formal differential operator $\tau_0 = -(d/dx)^2$ would lead to the basis

$$\hat{\sigma}_1(t) = \lambda^{-\frac{1}{2}}(\sin \lambda^{\frac{1}{2}}t \cos \lambda^{\frac{1}{2}}c - \cos \lambda^{\frac{1}{2}}t \sin \lambda^{\frac{1}{2}}c),$$

$$\hat{\sigma}_2(t) = \cos \lambda^{\frac{1}{2}}t \cos \lambda^{\frac{1}{2}}c + \sin \lambda^{\frac{1}{2}}t \sin \lambda^{\frac{1}{2}}c.$$

Moreover, the analogue for Bessel functions of the relation

$$\cos it + i \sin it = e^{-t}$$

is a relation which expresses the Hankel function in terms of Bessel and Neumann functions. These remarks illustrate the following general principles.

(a) In order to avoid introducing inessential and confusing complications into the computation of resolvents, spectral resolutions, etc., of differential operators, care must be exercised in choosing a suitable basis for the solutions of the equation $\tau\sigma = \lambda\sigma$.

(b) The analytical continuation of a basis convenient for the study of a certain range of λ is not necessarily the basis convenient for the study of another range of λ.

For the above reasons, we wish to reserve the right to investigate different ranges of λ separately. The detailed definitions, results, etc., to which this desire leads us will be given below. At this point, we shall only remark that it is necessary to make a number of slight generalizations of the theory of positive matrix measures presented above. Definition 6 must be generalized as follows.

12 DEFINITION. Let Λ be an open interval of the real line, and let $\{\mu_{ij}\}$, $1 \leqq i, j \leqq n$, be a family of complex valued set functions defined on the family of Borel subsets of Λ whose closures are compact and contained in Λ. The family $\{\mu_{ij}\}$ will be called an $n \times n$ ***positive matrix measure on*** Λ if

(i) the matrix $\mu_{ij}(e)$ is Hermitian and positive semi-definite for each Borel subset e of Λ whose closure is compact and contained in Λ;

(ii) we have

$$\mu_{ij}\Big(\bigcup_{m=1}^{\infty} e_m\Big) = \sum_{m=1}^{\infty} \mu_{ij}(e_m)$$

for each sequence of disjoint Borel subsets of Λ whose union has compact closure contained in Λ.

Lemma 7, Definition 8, Lemma 9, Theorem 10, Lemma 11, then all have corresponding generalizations. For example, Definition 8 is to be generalized as follows: Let Λ be a subinterval of the real line. Let $\{\mu_{ij}\}$ be a positive $n\times n$ matrix measure on Λ, and let μ be a

positive σ-finite regular Borel measure on Λ with respect to which all the set functions $\{\mu_{ij}\}$ are continuous. Let $\{m_{ij}\}$ denote the matrix of densities of $\{\mu_{ij}\}$ with respect to μ. The family of n-tuples $F = [f_1, \ldots, f_n]$ of Borel-measurable functions defined on Λ for which

$$|F|^2 = \int_\Lambda \left\{ \sum_{i,j=1}^{n} m_{ij}(\lambda) f_i(\lambda) \overline{f_j(\lambda)} \right\} \mu(d\lambda) < \infty$$

will be denoted by $L_2^0(\Lambda, \{\mu_{ij}\})$. An element F of $L_2^0(\Lambda, \{\mu_{ij}\})$ will be said to be a $\{\mu_{ij}\}$*-null function* if $|F| = 0$. The set of all equivalence classes of elements of $L_2^0(\Lambda, \{\mu_{ij}\})$ modulo $\{\mu_{ij}\}$-null functions will be denoted by $L_2(\Lambda, \{\mu_{ij}\})$.

Proceeding as in Lemma 9 and Theorem 10 it may be shown that $L_2(\Lambda, \{\mu_{ij}\})$ is a complete Hilbert space.

It should be noted that if $\{\rho_{ij}\}$ is an $n \times n$ positive matrix measure on the real line, and if $\{\mu_{ij}\}$ is the $n \times n$ positive matrix measure on Λ defined by $\mu_{ij}(e) = \rho_{ij}(e)$ for $e \subseteq \Lambda$, then $L_2(\Lambda, \{\mu_{ij}\})$ may be regarded as being isometrically imbedded in $L_2(\{\rho_{ij}\})$, as the set of n-tuples $[f_1(\cdot), \ldots, f_n(\cdot)]$ of $L_2(\{\rho_{ij}\})$ whose components vanish outside Λ. (Compare Section III.8, in particular the paragraphs of discussion between Lemma III.8.2, and Lemma III.8.8.) In what follows, this and similar elementary measure-theoretic ideas will be used, sometimes implicitly. In particular, a sequence of elements of $L_2(\Lambda, \{\mu_{ij}\})$ converging in the metric of $L_2(\Lambda, \{\mu_{ij}\})$ will sometimes be said to converge *in the mean square sense in* $L_2(\Lambda, \{\mu_{ij}\})$. Let I be an interval of the real axis, ν a Borel measure on I, $\sigma_1, \ldots, \sigma_n$ a set of Borel measurable functions defined on $I \times \Lambda$, and h a ν-measurable function defined on I. Suppose that for each compact subinterval J of I the integrals

$$H_i^J(\lambda) = \int_J \sigma_i(t, \lambda) h(t) \nu(dt), \qquad i = 1, \ldots, n,$$

exist for all $\lambda \in \Lambda$ and define an element H^J of $L_2(\Lambda, \{\mu_{ij}\})$, and that if J_p is any increasing sequence of compact subintervals of I with union I, $\lim_{p\to\infty} H^{J_p}$ exists in the topology of $L_2(\Lambda, \{\mu_{ij}\})$. Then we will sometimes say that *the family of integrals*

$$\int_I \sigma_i(t, \lambda) h(t) \nu(dt)$$

exists in the mean square sense in $L_2(\Lambda, \{\mu_{ij}\})$.

13 THEOREM. (*Weyl-Kodaira*). *Let τ be a formally self adjoint formal differential operator of order n defined on an interval I with end-points a, b. Let T be a self adjoint extension of $T_0(\tau)$. Let Λ be an open interval of the real axis, and suppose that there is given a set $\sigma_1, \ldots, \sigma_n$ of functions, defined and continuous on $I\times\Lambda$, such that for each fixed λ in Λ, $\sigma_1(\cdot, \lambda), \ldots, \sigma_n(\cdot, \lambda)$ forms a basis for the space of solutions of $\tau\sigma = \lambda\sigma$. Then there exists a positive $n\times n$ matrix measure $\{\rho_{ij}\}$ defined on Λ, such that*

(i) *the limit*

$$[(Vf)_i(\lambda)] = \lim_{\substack{c\to a\\ d\to b}} \left[\int_c^d f(t)\overline{\sigma_i(t, \lambda)}dt\right]$$

exists in the topology of $L_2(\Lambda, \{\rho_{ij}\})$ for each f in $L_2(I)$ and defines an isometric isomorphism V of $E(\Lambda)L_2(I)$ onto $L_2(\Lambda, \{\rho_{ij}\})$:

(ii) *for each Borel function G defined on the real line and vanishing outside Λ,*

$$V\mathfrak{D}\big(G(T)\big) = \{[f_i]\in L_2(\Lambda, \{\rho_{ij}\}) | [Gf_i] \in L_2(\Lambda, \{\rho_{ij}\})\},$$

and

$$\big(VG(T)f\big)_i(\lambda) = G(\lambda)(Vf)_i(\lambda), \quad i = 1, \ldots, n, \lambda\in\Lambda, f\in\mathfrak{D}\big(G(T)\big).$$

PROOF. Let $W_1, \ldots, W_m$, μ and e_i, $i = 1, \ldots, m$ be as in Theorem 1. Then we can find functions a_{ij} such that

$$[*] \qquad W_i(t, \lambda) = \sum_{j=1}^{n} a_{ij}(\lambda)\sigma_j(t, \lambda), \qquad t\in I, \lambda\in\Lambda, i = 1, \ldots, m.$$

We begin by showing that the functions a_{ij} are μ-measurable. Let J be a bounded closed subinterval of I. If the restrictions of the functions $\sigma_1(\cdot, \lambda), \ldots, \sigma_n(\cdot, \lambda)$ to J were linearly dependent for any λ, so that there existed a non-vanishing set of constants $c_1, \ldots, c_p$ such that $\sum_{i=1}^{p} c_i\sigma_i(t, \lambda) = 0$ for t in J, then it would follow by the uniqueness assertion of Theorem 1.3 that $\sum_{i=1}^{p} c_i\sigma_i(t, \lambda) = 0$ for all t in I. Thus, $\sigma_1(\cdot, \lambda), \ldots, \sigma_p(\cdot, \lambda)$ would be linearly dependent, contrary to assumption. This shows that the restrictions $\sigma_i(\lambda)$ of $\sigma_i(\cdot, \lambda)$ to J are linearly independent, and hence that equation [*] determines the coefficients $a_{ij}(\lambda)$ uniquely, even if we only demand that [*] hold for all t in J. Since they are continuous and hence bounded, the restric-

tions $W_i(\lambda)$ and $\sigma_i(\lambda)$ of $W_i(\cdot, \lambda)$ and $\sigma_i(\cdot, \lambda)$ to J belong to $L_2(J)$. Let λ_0 be a fixed point in Λ, and, using the Hahn-Banach Theorem II.3.13, select vectors $h_j \in L_2(J)$ such that

$$(\sigma_i(\lambda_0), h_j) = \delta_{ij}, \qquad i, j = 1, \ldots, n.$$

Then for any fixed integer $k, 1 \leqq k \leqq m$, we have

$$(W_k(\lambda), h_j) = \sum_{i=1}^{n} a_{ki}(\lambda)(\sigma_i(\lambda), h_j), \qquad j = 1, \ldots, n.$$

When $\lambda = \lambda_0$, the matrix $\{b_{ij}(\lambda)\} = \{(\sigma_i(\lambda), h_j)\}$ is the identity matrix. The matrix $\{b_{ij}(\lambda)\}$ is evidently continuous in λ. Hence, there exists a neighborhood N of λ_0 such that the determinant $\det \{b_{ij}(\lambda)\}$ is non-zero in $\Lambda \cap N$. Consequently, for λ in $\Lambda \cap N$ the matrix $\{b_{ij}(\lambda)\}$ has an inverse $\{c_{ij}(\lambda)\}$, which is continuous in λ for λ in N. Thus

$$[\dagger] \quad a_{ki}(\lambda) = \sum_{j=1}^{n} (W_k(\lambda), h_j)c_{ji}(\lambda), \qquad k = 1, \ldots, m,$$
$$i = 1, \ldots, n, \qquad \lambda \in \Lambda \cap N.$$

This shows that the restriction of a_{kj} to $\Lambda \cap N$ is μ-measurable. By Theorem 1, $(h_j, W_k(\lambda)) = (Uh_j)_k(\lambda)$ is in $L_2(\mu)$ for $k = 1, \ldots, m$, $j = 1, \ldots, n$. Taken together with [†], this shows that

$$\int_{\Lambda \cap M \cap N} |a_{ki}(\lambda)|^2 \mu(d\lambda) < \infty, \qquad i = 1, \ldots, n, \qquad k = 1, \ldots, m,$$

provided that M is a compact subset of Λ. Since Λ may be covered by a sequence of sets of the form $\Lambda \cap N$, a_{ki} is μ-measurable on Λ. Since any Borel set e whose closure is compact and contained in Λ can be covered by a finite sequence of sets of the form $\Lambda \cap N$, it follows that

$$\int_e |a_{ki}(\lambda)|^2 \mu(d\lambda) < \infty, \qquad k = 1, \ldots, m, \qquad i = 1, \ldots, n,$$

for each Borel set e with compact closure contained in Λ.

Let ρ_{jk} be the complex valued set functions defined for each Borel set e whose closure is compact and contained in Λ by the formulas

$$\rho_{jk}(e) = \sum_{i=1}^{m} \int_e a_{ik}(\lambda)\overline{a_{ij}(\lambda)}\mu(d\lambda), \qquad j, k = 1, \ldots, n.$$

Let $[\xi_1, \ldots, \xi_n]$ be an n-tuple of complex numbers. Then

$$\sum_{j,k=1}^{n} \rho_{jk}(e)\xi_j\bar{\xi}_k = \sum_{i=1}^{m} \int_e |\sum_{j=1}^{n} a_{ij}(\lambda)\bar{\xi}_j|^2 \mu(d\lambda) \geqq 0.$$

Thus $\{\rho_{jk}(\cdot)\}$ satisfies condition (i) of Definition 12. It is clear that $\{\rho_{jk}(\cdot)\}$ also satisfies condition (ii) of Definition 12 provided that the closure of the union $\bigcup_{i=1}^{\infty} e_i$ is compact and contained in Λ. Thus $\{\rho_{ij}\}$ is an $n \times n$ positive matrix measure defined on the interval Λ. Clearly, all the set functions ρ_{jk} are μ-continuous, and the density of ρ_{jk} with respect to μ is the function

$$m_{ij} = \sum_{k=1}^{m} a_{kj}\overline{a_{ki}}.$$

For any n-tuple $F = [f_1, \ldots, f_n]$ of Borel-measurable functions defined on Λ let AF be the m-tuple $[g_1, \ldots, g_m]$ defined by the equations

$$g_i(\lambda) = \sum_{k=1}^{n} \overline{a_{ik}(\lambda)} f_k(\lambda), \qquad i = 1, \ldots, m, \qquad \lambda \in \Lambda.$$

Then, since $a_{ik}(\lambda)$ vanishes for $\lambda \notin e_i$, and since

$$\int_\Lambda \left\{ \sum_{i,j=1}^{n} m_{ij}(\lambda) f_i(\lambda) \overline{f_j(\lambda)} \right\} \mu(d\lambda) = \int_\Lambda \{ \sum_{k=1}^{m} |g_k(\lambda)|^2 \} \mu(d\lambda),$$

the mapping A is evidently an isometric isomorphism of $L_2(\Lambda, \{\rho_{ij}\})$ into the subspace $\sum_{i=1}^{m} L_2(\mu, \Lambda e_i)$ of $\sum_{i=1}^{m} L_2(\mu, e_i)$. This same equation establishes the following remark,

[††] *If F is an n-tuple of Borel measurable functions, then $AF \in \sum_{i=1}^{m} L_2(\Lambda e_i, \mu)$ if and only if $F \in L_2(\Lambda, \{\rho_{ij}\})$.*

Let U be the isometric isomorphism of $L_2(I)$ onto $\sum_{i=1}^{m} L_2(\mu_i)$ given in Theorem 1, and let $\{J_q\}$ be an increasing sequence of compact intervals whose union is I. It follows from Theorem 1 that for each integer q and each f in $L_2(I)$ the functions $\int_{J_q} f(t)\overline{W_k(t, \cdot)}dt$ belong to $L_2(\mu, e_k)$, $k = 1, \ldots, m$, and

$$(Uf)_k(\lambda) = \lim_{q\to\infty} \int_{J_q} f(t)\overline{W_k(t, \lambda)}dt, \qquad k = 1, \ldots, m,$$

the limit existing in the topology of $L_2(\mu, e_k)$. Moreover, the mapping

$$f \to \left[\chi_\Lambda(\cdot)\int_I f(t)W_i(t, \cdot)dt\right], \qquad f \in E(\Lambda)L_2(I),$$

is an isometric isomorphism of $E(\Lambda)L_2(I)$ onto all of $\sum_{i=1}^m L_2(\mu, \Lambda e_i)$.

For each f in $L_2(I)$ define

$$(V_q f)_i(\lambda) = \int_{J_q} f(t)\overline{\sigma_i(t, \lambda)}dt, \qquad \lambda \in \Lambda, \ q \geqq 1, \ i = 1, \ldots, n.$$

The equations

$$\begin{aligned}
\int_\Lambda \left\{\sum_{i,j=1}^n m_{ij}(\lambda)(V_q f)_i(\lambda)\overline{(V_q f)_j(\lambda)}\right\}\mu(d\lambda) \\
&= \int_\Lambda \left\{\sum_{i,j=1}^m \sum_{k=1}^m a_{kj}(\lambda)\overline{a_{ki}(\lambda)}(V_q f)_i(\lambda)\overline{(V_q f)_j(\lambda)}\right\}\mu(d\lambda) \\
&= \int_\Lambda \sum_{k=1}^m \left|\sum_{i=1}^n \overline{a_{ki}(\lambda)}(V_q f)_i(\lambda)\right|^2 \mu(d\lambda) \\
&= \sum_{k=1}^m \int_\Lambda |(AV_q f)_k(\lambda)|^2 \mu(d\lambda);
\end{aligned}$$

$$\begin{aligned}
(AV_q f)_k(\lambda) &= \sum_{i=1}^m \overline{a_{ki}(\lambda)} \int_{J_q} \overline{\sigma_i(t, \lambda)} f(t)dt \\
&= \int_{J_q} \overline{W_k(t, \lambda)} f(t)dt, && \lambda \in \Lambda. \\
(AV_q f)_k(\lambda) &= 0, && \lambda \notin \Lambda;
\end{aligned}$$

and

$$(AV_q f)_k(\lambda) = \chi_\Lambda(\lambda)\int_{J_q} \overline{W_k(t, \lambda)} f(t)dt,$$

show that the n-tuple $[(V_q f)_i]$ belongs to $L_2(\Lambda, \{\rho_{ij}\})$, that the n-tuple of integrals

$$[(Vf)_i] = \left[\int_I f(t)\sigma_i(t, \cdot)dt\right], \qquad i = 1, \ldots, n,$$

exists in the mean square sense in the Hilbert space $L_2(\Lambda, \{\rho_{ij}\})$, and (using Theorem 1(ii)), that

$$UE(\Lambda)f = AVf, \qquad f \in L_2(I).$$

From this equation it follows immediately (since $UE(\Lambda) = AV$ is a mapping of $E(\Lambda)L_2(I)$ onto all of the subspace $\sum_{i=1}^m L_2(\mu, \Lambda e_i)$ of

$\sum_{i=1}^{m} L_2(\mu, e_i)$) that V is an isometric isomorphism of $E(\Lambda)L_2(I)$ onto $L_2(\Lambda, \{\rho_{ij}\})$ and that A is an isometric isomorphism of $L_2(\Lambda, \{\rho_{ij}\})$ onto the subspace $\sum_{i=1}^{m} L_2(\mu, \Lambda e_i)$ of $\sum_{i=1}^{m} L_2(\mu, e_i)$.

To prove (ii), note that since G vanishes outside of Λ, it follows from XII.2.6(a) that f is in $\mathfrak{D}(G(T))$ if and only if $E(\Lambda)f$ is in $\mathfrak{D}(G(T))$. By Theorem 1(ii), this is the case if and only if $[G(\cdot)(UE(\Lambda)f)_i(\cdot)]$ is in $\sum_{i=1}^{m} L_2(e_i, \mu)$. From the definition of A it follows immediately that $[G(\cdot)(AF)_i(\cdot)] = [(A\tilde{F})_i(\cdot)]$ for each n-tuple $F = [f_1(\cdot), \ldots, f_n(\cdot)]$ of Borel-measurable functions defined on Λ, where $\tilde{F} = [G(\cdot)f_1(\cdot), \ldots, G(\cdot)f_n(\cdot)]$. Since $UE(\Lambda)f = AVf$, we have $[G(\cdot)(UE(\Lambda)f)_i(\cdot)] = [(AH)_i(\cdot)]$, where H is the n-tuple $[G(\cdot)(Vf)_j(\cdot)]$. Thus, by remark [††], $f \in \mathfrak{D}(G(T))$ if and only if $[G(\cdot)(Vf)_j(\cdot)]$ is in $L_2(\Lambda, \{\rho_{ij}\})$, which proves the first part of (ii).

If f is in $\mathfrak{D}(G(T))$, then, by XII.2.7(d), $E(\Lambda)G(T)f = G(T)f = G(T)E(\Lambda)f$. Since $AV = UE(\Lambda)$, and since, by Theorem 1(ii), $UG(T)f = [G(\cdot)(UE(\Lambda)f)_i(\cdot)]$, it follows by the above remarks that $UG(T)f$ is the n-tuple AH, where H is the n-tuple $[G(\cdot)(Vf)_j(\cdot)]$. Since $UG(T)f = UE(\Lambda)G(T)f = AVG(T)f$, it follows that $AVG(T)f = AH$. Since A is an isomorphic mapping of $L_2(\Lambda, \{\rho_{ij}\})$, $VG(T)f = H$. Q.E.D.

14 Theorem. (*Weyl-Kodaira*). *Let $T, \Lambda, \{\rho_{ij}\}$, etc., be as in Theorem* 13. *Let λ_0 and λ_1 be the end points of Λ. Then*

(i) *the inverse of the isometric isomorphism V of $E(\Lambda)L_2(I)$ onto $L_2(\Lambda, (\rho_{ij}))$ is given by the formula*

$$(V^{-1}F)(t) = \lim_{\substack{\mu_0 \to \lambda_0 \\ \mu_1 \to \lambda_1}} \int_{\mu_0}^{\mu_1} \left\{ \sum_{i,j=1}^{n} F_i(\lambda)\sigma_j(t, \lambda)\rho_{ij}(d\lambda) \right\},$$

where $F = [F_1, \ldots, F_n] \in L_2(\Lambda, (\rho_{ij}))$, the limit existing in the topology of $L_2(I)$;

(ii) *if G is a bounded Borel function vanishing outside a Borel set e whose closure is compact and contained in Λ, then $G(T)$ has the representation*

$$\big(G(T)f\big)(t) = \int_I f(s)K(G, t, s)ds,$$

where

$$K(G, t, s) = \sum_{i,j=1}^{n} \int_e G(\lambda)\overline{\sigma_i(s, \lambda)}\sigma_j(t, \lambda)\rho_{ij}(d\lambda).$$

Moreover, given any compact interval $J \subseteq I$,

$$\sup_{t \in J} \int_I |K(G, t, s)|^2 ds < \infty.$$

PROOF. Let f in $L_2(I)$ be such that $(Uf)_i(\cdot)$ is bounded for $i = 1, \ldots, n$, and let g in $L_2(I)$ vanish outside a compact subinterval J of I. Let $\Lambda_0 = (\mu_0, \mu_1)$ be an open subinterval of Λ with compact closure. Then, by Theorem 13(i) and (ii),

$$\begin{aligned}(E(\Lambda_0)f, g) &= (E(\Lambda_0)f, E(\Lambda_0)g) = (VE(\Lambda_0)f, VE(\Lambda_0)g) \\ &= \int_{\Lambda_0} \left\{ \sum_{i,j=1}^{n} (Vf)_i(\lambda)\overline{(Vg)_j(\lambda)}\rho_{ij}(d\lambda) \right\} \\ &= \int_{\Lambda_0} \left\{ \sum_{i,j=1}^{n} (Vf)_i(\lambda) \left\{ \int_J \sigma_j(t, \lambda)\overline{g(t)}dt \right\} \rho_{ij}(d\lambda) \right\}.\end{aligned}$$

Since all the measures ρ_{ij}, $i, j = 1, \ldots, n$, have finite restrictions to Λ_0, and since σ_j is continuous and hence bounded on the compact set $\bar{\Lambda}_0 \times J$, we may interchange the orders of integration and summation in this formula, and find

$$[*] \qquad (E(\Lambda_0)f, g) = \int_J \left\{ \int_{\Lambda_0} \sum_{i,j=1}^{n} \{(Vf)_i(\lambda)\sigma_j(t, \lambda)\rho_{ij}(d\lambda)\}\overline{g(t)} \right\} dt.$$

Let f be in $L_2(I)$, and, using the fact that the family of bounded n-tuples is dense in $L_2(\Lambda, \{\rho_{ij}\})$, let f_k be a sequence of functions in $L_2(I)$ such that $f_k \to f$ in the topology of $L_2(I)$ and such that Vf_k is a bounded n-tuple for $k \geq 1$. Then, from the boundedness of $\sigma_i(t, \lambda)$ in $J \times \Lambda_0$, it is clear that

$$\int_{\Lambda_0} \left\{ \sum_{i,j=1}^{n} (Vf_k)_i(\lambda)\sigma_j(t, \lambda)\rho_{ij}(d\lambda) \right\} \to \int_{\Lambda_0} \left\{ \sum_{i,j=1}^{n} (Vf)_i(\lambda)\sigma_j(t, \lambda)\rho_{ij}(d\lambda) \right\}$$

as $k \to \infty$, the limit relation holding for each t in J and boundedly for t in J. Applying [*] to each function f_k, and letting $k \to \infty$, it is seen that

$$[**] \qquad (E(\Lambda_0)f, g) = \int_I (S_{\Lambda_0} Vf)(t)\overline{g(t)}dt,$$

where for each n-tuple $F = [F_1(\cdot), \ldots, F_n(\cdot)]$ in $L_2(\Lambda, \{\rho_{ij}\})$ we have put

$$(S_{\Lambda_0}F)(t) = \int_{\Lambda_0} \left\{ \sum_{i,j=1}^{n} F_i(\lambda)\sigma_j(t, \lambda)\rho_{ij}(d\lambda) \right\}, \qquad t \in I.$$

It follows from Theorem IV.8.1 that $\int_I |(S_{\Lambda_0} Vf)(t)|^2 dt < \infty$, and that [$**$] holds for all g in $L_2(I)$. Thus

$$\big(E(\Lambda_0)f\big)(t) = (S_{\Lambda_0}Vf)(t).$$

If we let the end points μ_0 and μ_1 of Λ_0 approach the end points λ_0 and λ_1 of Λ, it follows that

$$\big(E(\Lambda)f\big)(t) = \lim_{\substack{\mu_0 \to \lambda_0 \\ \mu_1 \to \lambda_1}} \int_{\mu_0}^{\mu_1} \left\{ \sum_{i,j=1}^{n} (V_i f)(\lambda)\sigma_j(t, \lambda)\rho_{ij}(d\lambda) \right\},$$

proving part (i) of the theorem.

To prove (ii), we argue as follows (compare the proof of Corollary 8). From the boundedness of $G(T)$ and Lemma 2.16, it follows that the map $f \to G(T)f$ is a continuous map of $L_2(I)$ into $C(J)$. Thus there is a constant $M(J)$ such that

$$|\big(G(T)f\big)(t)| \leq M(J)|f|, \qquad f \in L_2(I).$$

It follows from Theorem 13(ii) and part (i) of the present theorem that

$$\big(G(T)f\big)(t) = \int_e \sum_{i,j=1}^{n} G(\lambda) \left\{ \int_I f(s)\overline{\sigma_i(s, \lambda)}ds \right\} \sigma_j(t, \lambda)\rho_{ij}(d\lambda),$$

where the integrals $\int_I f(s)\overline{\sigma_i(s, \lambda)}ds$ exist in the mean square sense in $L_2(\Lambda, \{\rho_{ij}\})$. Let $\mathfrak{H}_0$ denote the dense set of those $f \in L_2(I)$ each of which vanishes outside a compact subinterval of I. If f is in $\mathfrak{H}_0$ the individual integrals in the formula displayed above all exist, and we interchange the orders of integration and summation in that formula to obtain

$$[\dagger] \qquad \big(G(T)f\big)(t) = \int_I f(s)K(G; t, s)ds, \qquad f \in \mathfrak{H}_0,$$

and the inequality

$$\left| \int_I f(s)K(G; t, s)ds \right| \leq M(J)|f|, \qquad f \in \mathfrak{H}_0,$$

where

$$K(G; t, s) = \sum_{i,j=1}^{n} \int_e G(\lambda)\overline{\sigma_i(s, \lambda)}\sigma_j(t, \lambda)\rho_{ij}(d\lambda).$$

It follows from Theorem IV.8.1 that

$$\left[\int_I |K(G; t, s)|^2 ds\right]^{\frac{1}{2}} \leqq M(J), \qquad t \in J,$$

and that equation [†] holds for all f in $L_2(I)$. Q.E.D.

15 Corollary. *Let T, Λ, and $\{\rho_{ij}\}$ be defined as in Theorem* 14. *The complement of $\sigma(T)$ in Λ is the largest open subset e_0 of Λ such that $\{\rho_{ij}(e)\} = 0$ for every open subset of e_0 whose closure is compact and contained in Λ.*

Proof. We observe first that if e is a Borel set whose closure is compact and contained in Λ and $\{\rho_{ij}(e)\} = 0$, then $\{\rho_{ij}(e_1)\} = 0$ for every Borel subset $e_1 \subseteq e$. To prove this, let $\xi_1, \ldots, \xi_n$ be any n-tuple of complex numbers, and let $F = [f_1, \ldots, f_n]$ where $f_i = \xi_i\chi_e$, $i = 1, \ldots, n$. Then $VE(e_1)V^{-1}F = [\chi_{e_1}f_1, \ldots, \chi_{e_1}f_n]$. Since $\{\rho_{ij}(e)\} = 0$,

$$0 = \sum_{i,j=1}^{n} \rho_{ij}(e)\xi_i\bar{\xi}_j = |F|^2 \geqq |VE(e_1)V^{-1}F|^2$$

$$= \sum_{i,j=1}^{n} \rho_{ij}(e_1)\xi_i\bar{\xi}_j \geqq 0.$$

Consequently, $\sum_{i,j=1}^{n} \rho_{ij}(e_1)\xi_i\bar{\xi}_j = 0$ for every vector $[\xi_1, \ldots, \xi_n]$ in E^n. If $[\zeta_1, \ldots, \zeta_n] \in E^n$, the Schwarz inequality for the inner product $\sum_{i,j=1}^{n} \rho_{ij}(e_1)\xi_i\bar{\zeta}_j$ in E^n shows

$$\left|\sum_{i,j=1}^{n} \rho_{ij}(e_1)\xi_i\bar{\zeta}_j\right|^2 \leqq \left\{\sum_{i,j=1}^{n} \rho_{ij}(e_1)\xi_i\bar{\xi}_j\right\}\left\{\sum_{i,j=1}^{n} \rho_{ij}(e_1)\zeta_i\bar{\zeta}_j\right\}$$

for arbitrary vectors $[\xi_i]$, $[\zeta_i]$ in E^n (cf. the remark following Theorem IV.4.1). It follows that $\sum_{i=1}^{n} \rho_{ij}(e_1)\xi_i = 0$, $j = 1, \ldots, n$, $[\xi_i] \in E^n$. Hence $\rho_{ij}(e_1) = 0$, $1 \leqq i, j \leqq n$.

Now let e_0 be the union of all open subsets e whose closures are compact and contained in Λ and for which $\{\rho_{ij}(e)\} = 0$. Since e_0 is the union of a sequence of such sets, it follows from Theorem 15 (ii) and the preceding paragraph that $E(e) = 0$, $e \subseteq e_0$. Thus $E(e_0) = 0$.

Hence by Theorem XII.2.9(b), $\sigma(T) \cap e_0 = \phi$. On the other hand, if $e_0 \cap \sigma(T) = \phi$ so that $E(e_0) = 0$ by Theorem XII.2.9(b), then by Theorem 13(i) and (ii)

$$\int_{-\infty}^{\infty} \left\{ \sum_{i,j=1}^{n} f_i(\lambda) g_j(\lambda) \rho_{ij}(d\lambda) \right\} = 0, \qquad [f_i], [g_i] \in L_2(\Lambda, (\rho_{ij})),$$

provided f_i and g_j vanish outside e_0. If e is a Borel set whose closure is compact and contained in e_0, then taking $F = [\xi_1\chi_e, \ldots, \xi_n\chi_e]$ and $G = [\zeta_1\chi_e, \ldots, \zeta_n\chi_e]$ we see by the argument above that $(\rho_{ij}(e)) = 0$, and hence $\{\rho_{ij}(e_0)\} = 0$. Q.E.D.

The next theorem gives useful information on the convergence of the inversion integrals appearing in Theorem 14. It is the analogue for the continuous case of Theorem 4.3.

16 THEOREM. *Let the operator T, the isomorphism V and the matrix measure (ρ_{ij}) be defined as in Theorems* 13 *and* 14. *Let μ be a positive σ-finite measure with respect to which all the set functions ρ_{ij} are continuous and let m_{ij} be the Radon-Nikodým derivative of ρ_{ij} with respect to μ. If f is in $\mathfrak{D}(T) \subseteq L_2(I)$, then the integral*

$$\int_{\Lambda} \left\{ \sum_{i,j=1}^{n} m_{ij}(\lambda) \sigma_i(t, \lambda)(Vf)_j(\lambda) \right\} \mu(d\lambda)$$

converges absolutely and uniformly on each finite closed subinterval of I, and may be differentiated under the sign of integration $(n-1)$ times, each differentiated integral retaining the properties of absolute and uniform convergence.

PROOF. The proof is similar to that of Theorem 4.3. If $\{\Lambda_k\}$ is an increasing sequence of compact subintervals of Λ whose union is Λ and λ_0 belongs to $\rho(T)$, then

$$\int_{\Lambda-\Lambda_k} \left\{ \sum_{i,j=1}^{n} m_{ij}(\lambda) \sigma_i(t, \lambda)(Vf)_j(\lambda) \right\} \mu(d\lambda) = R(\lambda_0; T) g_k,$$

where

$$g_k = E(\Lambda - \Lambda_k)(\lambda_0 I - T) f.$$

Since g_k converges to zero and $TR(\lambda_0; T)$ is a bounded operator, it

follows that $R(\lambda_0; T)g_k$ and $TR(\lambda_0; T)g_k$ converge to zero in $L_2(I)$. Thus by Lemma 2.16, the integrals

$$[*] \qquad \int_{\Lambda_k}\left\{\sum_{i,j=1}^{n} m_{ij}(\lambda)\sigma_i(t, \lambda)(Vf)_j(\lambda)\right\}\mu(d\lambda)$$

converge to $E(\Lambda)g$ in the topology of $C^{n-1}(J)$ for each compact interval J of I. Since the integrals converge unconditionally in $L_2(I)$, they converge unconditionally in $C^{n-1}(J)$. The functions σ_i are continuous on $I \times \Lambda$ and belong to $C^\infty(I)$ for each λ in Λ. It is easily seen that for any $k \geqq 0$ we have the limit

$$\lim_{\Delta t \to 0+} \frac{\sigma^{(k)}(t+\Delta t, \lambda) - \sigma^{(k)}(t, \lambda)}{\Delta t} = \sigma^{(k+1)}(t, \lambda),$$

uniformly for t in any compact subinterval of I and for λ in any compact subinterval of Λ. Thus if

$$\eta_k(t, \lambda, \Delta t) = \frac{\sigma^{(k)}(t+\Delta t, \lambda) - \sigma^{(k)}(t, \lambda)}{t} - \sigma^{(k+1)}(t, \lambda),$$

it follows readily that

$$\lim_{\Delta t \to 0} \int_{\Lambda_k}\left\{\sum_{i,j=1}^{n} \eta_i(t, \lambda, \Delta t)\eta_j(t, \lambda, \Delta t)\rho_{ij}(d\lambda)\right\}$$
$$= \lim_{\Delta t \to 0} \sum_{i,j=1}^{n} \int_{\Lambda_k} \eta_i(t, \lambda, \Delta t)\eta_j(t, \lambda, \Delta t)\rho_{ij}(d\lambda) = 0, \qquad k \geqq 1.$$

Thus, by the Schwarz inequality,

$$\lim_{\Delta t \to 0}\int_{\Lambda_k}\left\{\sum_{i,j=1}^{n} m_{ij}(\lambda)\eta_i(t, \lambda, \Delta t)(Vf)(\lambda)\right\}\mu(d\lambda) = 0, \qquad k \geqq 1;$$

i.e., for each $k \geqq 1$, the integral in [*] may be differentiated arbitrarily often under the integral sign

Consequently, each of the differentiated integrals converges absolutely. Q.E.D.

Observe that a point λ_0 in Λ is an eigenvalue of T if and only if the matrix $\{\rho_{ij}(\{\lambda_0\})\}$ is different from zero. For by Theorem 14(ii), $E(\{\lambda_0\})$ is the integral operator whose kernel is

$$E(\{\lambda_0\}, t, s) = \sum_{i,j=1}^{n} \sigma_i(t, \lambda_0)\overline{\sigma_j(s, \lambda_0)}\rho_{ji}(\{\lambda_0\}).$$

Since the functions $\sigma_1(\cdot, \lambda_0), \ldots, \sigma_n(\cdot, \lambda_0)$ are linearly independent, $E(\{\lambda_0\}) = 0$ if and only if $\{\rho_{ij}(\{\lambda_0\})\} = 0$. However, as in the proof of Lemma X.3.3(ii), it is seen that $E(\{\lambda_0\}) \neq 0$ if and only if λ_0 is an eigenvalue of T.

We shall now show that when the matrix $S(\lambda_0) = \{\rho_{ij}(\{\lambda_0\})\}$ is not zero, one may construct a complete set of orthogonal eigenfunctions corresponding to λ_0 by diagonalizing this positive semi-definite Hermitian matrix. This observation will be useful later. It follows from the spectral theorem that there exist matrices U and A such that $S(\lambda_0) = UAU^{-1}$, where $U = \{u_{ij}\}$ is a unitary matrix and A is the matrix $\{a_{ij}\} = \{\lambda_i\delta_{ij}\}$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $S(\lambda_0)$ each repeated according to its multiplicity. Thus there exist quantities u_{ij}, $1 \leqq i, j \leqq n$, such that

$$\sum_{k=1}^{n} u_{ik}\bar{u}_{jk} = \delta_{ij}, \qquad \rho_{ij}(\{\lambda_0\}) = \sum_{k=1}^{n} \lambda_k u_{ik}\bar{u}_{jk}.$$

Since the eigenvalues of $S(\lambda_0)$ are non-negative, we can suppose that $\lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_n \geqq 0$. Suppose that $\lambda_p > 0$ while $\lambda_{p+1} = 0$. Define

$$\psi_k(t) = \sqrt{\lambda_k} \sum_{i=1}^{n} u_{ik}\sigma_i(t, \lambda_0), \qquad k = 1, \ldots, p.$$

Then the functions ψ_k are linearly independent solutions of $(\tau - \lambda_0)\sigma = 0$ and

$$[*] \qquad E(\{\lambda_0\}, t, s) = \sum_{k=1}^{p} \psi_k(t)\overline{\psi_k(s)}.$$

If J is any compact subinterval of I, then $\psi_1, \ldots, \psi_p$ are linearly independent in $L_2(J)$. Thus we can find functions $g_1, \ldots, g_p$ in $L_2(I)$ vanishing outside of J such that

$$\int_I g_i(s)\overline{\psi_j(s)}ds = \int_J g_i(s)\overline{\psi_j(s)}ds = \delta_{ij}, \qquad 1 \leqq i, j \leqq p.$$

By Theorem 14(ii) and formula [*]

$$(E(\{\lambda_0\})g_i)(t) = \int_I g_i(s)E(\{\lambda_0\}, t, s)ds = \psi_i(t), \qquad t \in I,$$

which shows that ψ_i is in $L_2(I)$ and, since $E(\lambda_0)g_i = \psi_i$, that ψ_i is in

$\mathfrak{D}(T)$. Hence $T\psi_i = \lambda_0\psi_i$, $i = 1, \ldots, p$, so $E(\{\lambda_0\})\psi_i = \psi_i$, $i = 1, \ldots, p$. Consequently, formula [*] implies that

$$\psi_i(t) = \sum_{j=1}^{p} \psi_j(t) \int_I \psi_i(s)\overline{\psi_j(s)}ds,$$

which, by linear independence, implies that we have $(\psi_i, \psi_j) = \delta_{ij}$, $1 \leq i, j \leq p$. Hence the vectors $\psi_1, \ldots, \psi_p$ form an orthonormal set of eigenvectors of T belonging to the eigenvalue λ_0. If ψ is any other such eigenvector, we have $E(\{\lambda_0\})\psi = \psi$, so that

$$\psi = \sum_{j=1}^{p} (\psi, \psi_j)\psi_j$$

by [*]. Thus ψ is linearly dependent on the vectors ψ_j, showing that $\{\psi_1, \ldots, \psi_p\}$ is a complete orthonormal set of eigenfunctions of T belonging to the eigenvalue λ_0.

In Theorem 13 we proved the existence of a positive matrix measure $\{\rho_{ij}\}$ associated with any self adjoint extension of $T_0(\tau)$ and basis $\sigma_1, \ldots, \sigma_n$ of solutions of $(\tau - \lambda)\sigma = 0$ continuous on $I \times \Lambda$. We now consider the problem of how to calculate this matrix explicitly, the functions $\sigma_i(\cdot, \lambda)$ being a basis depending analytically on λ for each λ in a neighborhood U of the complex plane containing Λ. It was shown in Section 3 that the resolvent $R(\lambda; T)$ of T is an integral operator whose kernel $K(t, s; \lambda)$ could be expressed in terms of products of solutions of $(\tau - \lambda)\sigma = 0$ and $(\tau - \bar{\lambda})\sigma = 0$. Thus, it is a consequence of Corollary 3.12 that there exist functions $\theta_{ij}^{\pm}$ such that for λ in $U \cap \rho(T)$, then

$$\begin{aligned} K(t, s; \lambda) &= \sum_{i,j=1}^{n} \theta_{ij}^{-}(\lambda)\sigma_i(t, \lambda)\overline{\sigma_j(s, \bar{\lambda})}, \qquad t < s, \\ &= \sum_{i,j=1}^{n} \theta_{ij}^{+}(\lambda)\sigma_i(t, \lambda)\overline{\sigma_j(s, \bar{\lambda})}, \qquad s < t. \end{aligned}$$

We shall show that the functions $\theta_{ij}^{\pm}$ are analytic and prove that

$$[**] \quad \rho_{ij}((\lambda_1, \lambda_2)) = \lim_{\delta \to 0} \lim_{\varepsilon \to 0+} \frac{1}{2\pi i} \int_{\lambda_1+\delta}^{\lambda_2-\delta} [\theta_{ij}^{\pm}(\lambda - i\varepsilon) - \theta_{ij}^{\pm}(\lambda + i\varepsilon)]d\lambda,$$

where (λ_1, λ_2) is any bounded open interval contained in Λ. The

basis for this formula is found in Theorem XII.2.10 which asserts that the projection in the resolution of the identity for T corresponding to (λ_1, λ_2) may be calculated from the resolvent by the formula

$$E((\lambda_1, \lambda_2))f = \lim_{\delta\to 0}\lim_{\varepsilon\to 0+} \frac{1}{2\pi i}\int_{\lambda_1+\delta}^{\lambda_2-\delta} [R(\lambda-i\varepsilon; T) - R(\lambda+i\varepsilon; T)]f d\lambda.$$

The problem we face is that of passing from this latter formula involving the resolvent to a formula involving the individual terms $\theta^+_{ij}(\lambda)$ and $\theta_{ij}(\lambda)$ as in formula [**]. Many of the technical difficulties in the proof to follow are due to the fact that the functions $\sigma_i(\cdot, \lambda)$ are not necessarily square-integrable on I. The next lemma will be an important tool in establishing formula [**].

17 LEMMA. *Let $\mathfrak{X}$ be a complex B-space and G be an open set in the complex plane. Let V and $Q_1, \ldots, Q_N$ be analytic functions defined in G having values in $B(\mathfrak{X})$ and let $g_1, \ldots, g_N$ be scalar functions defined in G such that $V(\lambda) = \sum_{i=1}^N g_i(\lambda)Q_i(\lambda)$ for λ in G. Suppose that $Q_1(\lambda), \ldots, Q_N(\lambda)$ are linearly independent for each λ in G. If $\mathfrak{K}$ denotes the linear space of continuous linear functionals φ on $B(\mathfrak{X})$ of the form $\varphi(T) = \sum_{i=1}^k x_i^* T x_i$, $x_i^* \in \mathfrak{X}^*$, $x_i \in \mathfrak{X}$, then given any point λ_0 in G, there exist linearly independent functionals φ_i in $\mathfrak{K}$, a neighborhood $G(\lambda_0) \subseteq G$ of λ_0, and analytic functions P_{ij} defined in $G(\lambda_0)$ such that*

$$\sum_{j=1}^N P_{kj}(\lambda)\varphi_j Q_i(\lambda) = \delta_{ki}, \qquad \lambda \in G(\lambda_0),$$

and

$$g_k(\lambda) = \sum_{j=1}^N P_{kj}(\lambda)\varphi_j V(\lambda), \qquad \lambda \in G(\lambda_0).$$

The functions g_i are analytic for λ in G.

PROOF. We observe first that if $A \in B(\mathfrak{X})$ and $\varphi(A) = 0$ for all φ in $\mathfrak{K}$, then clearly $A = 0$. It will now be shown that there exists a set $\varphi_1, \ldots, \varphi_N \in \mathfrak{K}$ such that $\varphi_i Q_j(\lambda_0) = \delta_{ij}$. There is certainly an element φ_1 such that $\varphi_1 Q_1(\lambda_0) = 1$. Suppose now that $1 < p < N$ and that functionals $\varphi_1, \ldots, \varphi_p$ such that $\varphi_i Q_j(\lambda_0) = \delta_{ij}$, $1 \leq i$, $j \leq p$ have been constructed. Then there must exist an element $\psi \in \mathfrak{K}$ such that $\psi(Q_{p+1}(\lambda_0)) - \sum_{i=1}^p \psi(Q_i(\lambda_0)\varphi_i(Q_p(\lambda_0))) \neq 0$, for if not, then,

by the remark above, we would have $Q_{p+1}(\lambda_0) = \sum_{i=1}^{p} Q_i(\lambda_0)\varphi_i(Q_p(\lambda_0))$, contradicting the linear independence of the operators $Q_1(\lambda_0), \ldots, Q_{p+1}(\lambda_0)$. Choosing such a ψ, define $\psi_1 = \psi - \sum_{i=1}^{p}\psi(Q_i(\lambda_0))\varphi_i$ and $\hat{\varphi}_{p+1} = \psi_1/\psi_1(Q_{p+1}(\lambda_0))$. Then put

$$\hat{\varphi}_j = \varphi_j - \varphi_j(Q_{p+1}(\lambda_0))\hat{\varphi}_{p+1}, \qquad 1 \leq j \leq p.$$

It is easily seen that the set $\hat{\varphi}_1, \ldots, \hat{\varphi}_{p+1}$ satisfies $\varphi_i Q_j(\lambda_0) = \delta_{ij}$, $1 \leq i, j \leq p+1$. It follows by induction that we can construct the required functionals $\varphi_1, \ldots, \varphi_N$ in $\mathfrak{K}$.

We now select a neighborhood $G(\lambda_0)$ of λ_0 such that the analytic matrix $\{\varphi_i Q_j(\lambda)\}$ has a non-vanishing determinant for $\lambda \in G(\lambda_0)$. It follows easily that $\{\varphi_i Q_j(\lambda)\}$ has an inverse $\{P_{ij}(\lambda)\}$ analytic for $\lambda \in G(\lambda_0)$. Thus

$$\sum_{j=1}^{N} P_{ij}(\lambda)\varphi_j Q_k(\lambda) = \delta_{ik}, \qquad \lambda \in G(\lambda_0),$$

and

$$\begin{aligned}\sum_{j=1}^{N} P_{ij}(\lambda)\varphi_j V(\lambda) &= \sum_{j=1}^{N}\sum_{k=1}^{N} P_{ij}(\lambda) g_k(\lambda)\varphi_j Q_k(\lambda_0) \\ &= \sum_{k=1}^{N} g_k(\lambda)\delta_{ik} = g_i(\lambda), \qquad \lambda \in G(\lambda_0).\end{aligned}$$

It is clear from the last formula that g_i is analytic in G. Q.E.D.

18 Theorem. (*Titchmarsh-Kodaira*). *Let Λ be an open interval of the real axis and U be an open set in the complex plane containing Λ. Let $\sigma_1, \ldots, \sigma_n$ be a set of functions which form a basis for the solutions of the equation $(\tau - \lambda)\sigma = 0$, $\lambda \in U$, and which are continuous on $I \times U$ and analytically dependent on λ for λ in U. Suppose that the kernel $K(t, s; \lambda)$ for the resolvent $R(\lambda; T)$ has the representation*

$$\begin{aligned}K(t, s; \lambda) &= \sum_{i,j=1}^{n} \theta_{ij}^{-}(\lambda)\sigma_i(t, \lambda)\overline{\sigma_j(s, \bar{\lambda})}, \qquad t < s \\ &= \sum_{i,j=1}^{n} \theta_{ij}^{+}(\lambda)\sigma_i(t, \lambda)\overline{\sigma_j(s, \bar{\lambda})}, \qquad t > s,\end{aligned}$$

for all λ in $\rho(T) \cap U$, and that $\{\rho_{ij}\}$ is a positive matrix measure on Λ associated with T as in Theorem 13. Then the functions $\theta_{ij}^{\pm}$ are analytic in

$U \cap \rho(T)$, and given any bounded open interval $(\lambda_1, \lambda_2) \subseteq \Lambda$, we have for $1 \leqq i, j \leqq n$,

$$\rho_{ij}\big((\lambda_1, \lambda_2)\big) = \lim_{\delta \to 0} \lim_{\varepsilon \to 0+} \frac{1}{2\pi i} \int_{\lambda_1+\delta}^{\lambda_2-\delta} [\theta_{ij}^{-}(\lambda - i\varepsilon) - \theta_{ij}^{-}(\lambda + i\varepsilon)] d\lambda$$
$$= \lim_{\delta \to 0} \lim_{\varepsilon \to 0+} \frac{1}{2\pi i} \int_{\lambda_1+\delta}^{\lambda_2-\delta} [\theta_{ij}^{+}(\lambda - i\varepsilon) - \theta_{ij}^{+}(\lambda + i\varepsilon)] d\lambda.$$

PROOF. For λ in U define

$$\begin{aligned} \psi_{ij}^{-}(t, s; \lambda) &= \sigma_j(t, \lambda)\overline{\sigma_i(s, \bar{\lambda})}, & t < s, \\ &= 0 & t > s; \\ \psi_{ij}^{+}(t, s; \lambda) &= 0, & t < s, \\ &= \sigma_j(t, \lambda)\overline{\sigma_i(s, \bar{\lambda})}, & t > s. \end{aligned}$$

Let J be any fixed compact subinterval of I. Since the restrictions $\{\sigma_i(\cdot, \lambda)|J\}$ are linearly independent in $L_2(J)$ for each $\lambda \in U$, it follows easily that the restrictions $\{\psi_{ij}^{\pm}(\cdot, \cdot, \lambda)|J \times J\}$, $1 \leqq i, j \leqq n$, form a linearly independent family of functions in $L_2(J \times J)$. Consequently, the continuous operators $Q_{ij}^{+}(\lambda)$ and $Q_{ij}^{-}(\lambda)$ on $L_2(J)$ defined by the equations

$$\big(Q_{ij}^{\pm}(\lambda)f\big)(t) = \int_J \psi_{ij}^{\pm}(t, s; \lambda) f(s) ds,$$
$$f \in L_2(J), \qquad t \in J, \qquad \lambda \in U,$$

are linearly independent for each λ in U. Since the functions $\{\sigma_i(\cdot, \lambda)|J\}$ are analytic in $L_2(J)$ for $\lambda \in U$, it follows easily that the operators $Q_{ij}^{\pm}$ are analytic functions of λ, $\lambda \in U$.

For convenience let A_J be the map of $L_2(I)$ into $L_2(J)$ which assigns to any function f in $L_2(I)$ the restriction $f|J$, and let B_J denote the map of $L_2(J)$ into $L_2(I)$ which assigns to any g in $L_2(J)$ the function $g\psi_J$ in $L_2(I)$. Then from the kernel representation for the resolvent we have the formula

$$A_J R(\lambda; T) B_J f = \sum_{i,j=1}^{n} \theta_{ij}^{-}(\lambda) Q_{ij}^{-}(\lambda) f + \sum_{i,j=1}^{n} \theta_{ij}^{+}(\lambda) Q_{ij}^{+}(\lambda) f,$$
$$f \in L_2(J), \qquad \lambda \in \rho(T) \cap U.$$

In Lemma 17 let $G = \rho(T) \cap U$, $V(\lambda) = A_J R(\lambda; T) B_J$. Moreover let $Q_1, \ldots, Q_N$ be an enumeration of the functions $Q_{ij}^{\pm}$, $1 \leq i, j \leq n$, and $g_1, \ldots, g_N$ be the corresponding enumeration of the functions $\theta_{ij}^{\pm}$. Then it follows that the functions $\theta_{ij}^{\pm}$ are analytic in $U \cap \rho(T)$.

We shall next establish certain properties of the operator $A_J R(\lambda; T) B_J$.

First observe that

$$\text{(a)} \quad |\mathscr{I}(\lambda) A_J R(\lambda; T) B_J| \leq 1, \qquad \lambda \in \rho(T).$$

This follows from the inequality $|R(\lambda; T)| \leq 1/|\mathscr{I}(\lambda)|$ (cf. Lemma XII.2.1) and the fact that $|A_J| = |B_J| = 1$.

Next we have

$$\text{(b)} \quad \lim_{\varepsilon \to 0} \varepsilon A_J R(\lambda \pm i\varepsilon; T) B_J f = 0, \qquad f \in L_2(U),$$

for almost all λ in $[\lambda_1, \lambda_2]$.

To prove (b) observe that

$$\varepsilon R(\lambda \pm i\varepsilon; T) f = \int_{\sigma(T)} \frac{\varepsilon}{(\lambda - \mu) \pm i\varepsilon} E(d\mu) f \to \mp i E(\{\lambda\}) f$$

for every f in $L_2(I)$ by Theorem XII.2.9(v). Since $\sum |E(\{\lambda\}) f|^2 \leq |f|^2$, where the summation is over all λ for which $E(\{\lambda\}) f \neq 0$, it follows that $\lim_{\varepsilon \to 0} \varepsilon R(\lambda \pm i\varepsilon; T) f = 0$ except possibly for λ in a countable set. Statement (b) follows immediately from these observations.

(c) If F is any continuous function defined on $[\lambda_1, \lambda_2]$, then

$$\lim_{\delta \to 0} \lim_{\varepsilon \to 0+} \frac{1}{2\pi i} \int_{\lambda_1 + \delta}^{\lambda_2 - \delta} F(\lambda) A_J \{R(\lambda - i\varepsilon; T) - R(\lambda + i\varepsilon; T)\} B_J f \, d\lambda$$
$$= \sum_{i, j=1}^{n} \int_{(\lambda_1, \lambda_2)} F(\lambda) [Q_{ij}^{-}(\lambda) + Q_{ij}^{+}(\lambda)] f \rho_{ij}(d\lambda), \qquad f \in L_2(J).$$

By Theorem XII.2.11 we have

$$\lim_{\delta \to 0} \lim_{\varepsilon \to 0+} \frac{1}{2\pi i} \int_{\lambda_1 + \delta}^{\lambda_2 - \delta} F(\lambda) \{R(\lambda - i\varepsilon; T) - R(\lambda + i\varepsilon; T)\} f \, d\lambda$$
$$= \int_{(\lambda_1, \lambda_2)} F(\lambda) E(d\lambda) f, \qquad f \in L_2(I).$$

By Theorem 14(ii)

$$\int_{(\lambda_1, \lambda_2)} F(\lambda)E(d\lambda)f(t) = \int_I K(F, t, s)f(s)ds,$$

where

$$K(F, t, s) = \sum_{i,j=1}^{n} \int_{(\lambda_1, \lambda_2)} F(\lambda)\sigma_j(t, \lambda)\overline{\sigma_i(s, \lambda)}\rho_{ij}(d\lambda).$$

Recalling the definition of $Q_{ij}^{\pm}$ we see that for f in $L_2(J)$

$$A_J \int_{(\lambda_1, \lambda_2)} F(\lambda)E(d\lambda)B_J f = \sum_{i,j=1}^{n} \int_{(\lambda_1, \lambda_2)} F(\lambda)[Q_{ij}^{-}(\lambda) + Q_{ij}^{+}(\lambda)]f\rho_{ij}(d\lambda).$$

This establishes statement (c).

Having proved statements (a), (b), and (c), the conclusion of the theorem will follow if we can establish the following lemma. In applying the lemma we take $\mathfrak{X} = L_2(J)$, $W = U$, $V(\lambda) = A_J R(\lambda; T)B_J$, put $\rho_{ij}^{+} = \rho_{ij}^{-} = \rho_{ij}$, and let $Q_1, \ldots, Q_N$, $g_1, \ldots, g_N$, and $\mu_1, \ldots, \mu_N$ be corresponding enumerations of $\{Q_{ij}^{\pm}\}$, $\{\theta_{ij}^{\pm}\}$ and $\{\rho_{ij}^{\pm}\}$ as in the proof of Theorem 18.

19 LEMMA. *Let $\mathfrak{X}$ be a complex B-space and W be a neighborhood in the complex plane of the bounded interval $[\lambda_1, \lambda_2]$ of the real axis. Let $g_1, \ldots, g_N$ and $Q_1, \ldots, Q_N$ be analytic functions defined on W with g_i scalar valued and Q_i having values in $B(\mathfrak{X})$. Let V be the operator valued function defined by the formula*

$$V(\lambda) = \sum_{i=1}^{N} g_i(\lambda)Q_i(\lambda), \qquad \lambda \in W.$$

Suppose that

(a) *there is a constant M such that*

$$|\mathscr{I}(\zeta)V(\zeta)| \leq M, \qquad \zeta \in W;$$

(b) $$\lim_{\varepsilon \to 0} \varepsilon V(\lambda \pm i\varepsilon)x = 0, \qquad x \in \mathfrak{X},$$

for almost all λ in $[\lambda_1, \lambda_2]$; and

(c) *for each $i = 1, \ldots, N$ there exists a bounded Borel measure μ_i such that*

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0+}\frac{1}{2\pi i}\int_{\lambda_1+\delta}^{\lambda_2-\delta}\{V(\lambda-i\varepsilon)x-V(\lambda+i\varepsilon)x\}F(\lambda)d\lambda$$
$$=\sum_{i=1}^{N}\int_{(\lambda_1,\lambda_2)}F(\lambda)Q_i(\lambda)x\mu_i(d\lambda),\qquad x\in\mathfrak{X},$$

for each continuous function F defined on $[\lambda_1, \lambda_2]$. Then

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0+}\frac{1}{2\pi i}\int_{\lambda_1+\delta}^{\lambda_2-\delta}[g_i(\lambda-i\varepsilon)-g_i(\lambda+i\varepsilon)]d\lambda=\mu_i((\lambda_1,\lambda_2)),\ i=1,\ldots,N.$$

PROOF. Let λ_0 be a point of the open interval (λ_1, λ_2) and let $\mathfrak{R}$ denote the linear space of all functionals on $B(\mathfrak{X})$ of the form

$$\varphi(T)=\sum_{j=1}^{k}x_j^*Tx_j,$$

where $x_j^*\in\mathfrak{X}^*$, $x_j\in\mathfrak{X}$. Let $\varphi_1,\ldots,\varphi_N$ be chosen in $\mathfrak{R}$ by Lemma 17 in such a way that there exists a neighborhood $U(\lambda_0)$ of λ_0 such that the scalar matrix $\{\varphi_jQ_i(\lambda)\}$ has an analytic inverse $\{P_{ij}(\lambda)\}$ for λ in $U(\lambda_0)$. It follows from hypothesis (c) that

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0+}\frac{1}{2\pi i}\int_{\lambda_1+\delta}^{\lambda_2-\delta}\varphi_j\big(V(\lambda-i\varepsilon)-V(\lambda+i\varepsilon)\big)F(\lambda)d\lambda$$
$$=\sum_{i=1}^{N}\int_{(\lambda_1,\lambda_2)}F(\lambda)\varphi_jQ_i(\lambda)\mu_i(d\lambda),$$

for every $j=1,\ldots,N$, and every continuous F defined on $[\lambda_1, \lambda_2]$. If we replace $F(\lambda)$ in this formula by $F(\lambda)P_{kj}(\lambda)$ and sum over j we obtain the formula

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0+}\sum_{j=1}^{N}\frac{1}{2\pi i}\int_{\lambda_1+\delta}^{\lambda_2-\delta}F(\lambda)\varphi_j\big(V(\lambda-i\varepsilon)-V(\lambda+i\varepsilon)\big)P_{kj}(\lambda)d\lambda$$
$$=\sum_{j=1}^{N}\sum_{i=1}^{N}\int_{(\lambda_1,\lambda_2)}P_{kj}(\lambda)\varphi_jQ_i(\lambda)F(\lambda)\mu_i(d\lambda)$$
$$=\sum_{i=1}^{N}\int_{(\lambda_1,\lambda_2)}\sum_{j=1}^{N}P_{kj}(\lambda)\varphi_jQ_i(\lambda)F(\lambda)\mu_i(d\lambda)$$
$$=\int_{(\lambda_1,\lambda_2)}F(\lambda)\mu_k(d\lambda),$$

if F vanishes outside $U(\lambda_0)$, since $\sum_{j=1}^{N} P_{kj}(\lambda)\varphi_j Q_i(\lambda) = \delta_{ki}$ for $\lambda \in U(\lambda_0)$. However, we may write

$$
\begin{aligned}
&\sum_{j=1}^{N}\int_{\lambda_1+\delta}^{\lambda_2-\delta} F(\lambda)\varphi_j\big(V(\lambda-i\varepsilon)-V(\lambda+i\varepsilon)\big)P_{kj}(\lambda)\,d\lambda \\
&\quad = \sum_{j=1}^{N}\int_{\lambda_1+\delta}^{\lambda_2-\delta} F(\lambda)[P_{kj}(\lambda-i\varepsilon)\varphi_j V(\lambda-i\varepsilon)-P_{kj}(\lambda+i\varepsilon)\varphi_j V(\lambda+i\varepsilon)]\,d\lambda \\
&\quad\quad - \sum_{j=1}^{N}\int_{\lambda_1+\delta}^{\lambda_2-\delta} F(\lambda)[P_{kj}(\lambda-i\varepsilon)-P_{kj}(\lambda)]\varphi_j V(\lambda-i\varepsilon)\,d\lambda \\
&\quad\quad + \sum_{j=1}^{N}\int_{\lambda_1+\delta}^{\lambda_2-\delta} F(\lambda)[P_{kj}(\lambda+i\varepsilon)-P_{kj}(\lambda)]\varphi_j V(\lambda+i\varepsilon)\,d\lambda.
\end{aligned}
$$

As ε approaches zero, the function $\varepsilon^{-1}[P_{kj}(\lambda\pm i\varepsilon)-P_{kj}(\lambda)]$ remains bounded by the analyticity of P_{kj}. By hypothesis (b), $\varepsilon\varphi_j V(\lambda\pm i\varepsilon)\to 0$ for almost all λ. Thus, by hypothesis (a) and the Lebesgue dominated convergence theorem, the last two terms on the right above tend to zero as $\varepsilon\to 0$. Consequently,

$$
\begin{aligned}
&\lim_{\varepsilon\to 0+}\sum_{j=1}^{N}\int_{\lambda_1+\delta}^{\lambda_2-\delta} F(\lambda)\varphi_j\big(V(\lambda-i\varepsilon)-V(\lambda+i\varepsilon)\big)P_{kj}(\lambda) \\
&= \lim_{\varepsilon\to 0+}\sum_{j=1}^{N}\int_{\lambda_1+\delta}^{\lambda_2-\delta} F(\lambda)[P_{kj}(\lambda-i\varepsilon)\varphi_j V(\lambda-i\varepsilon)-P_{kj}(\lambda+i\varepsilon)\varphi_j V(\lambda+i\varepsilon)]\,d\lambda \\
&\qquad = \lim_{\varepsilon\to 0+}\int_{\lambda_1+\delta}^{\lambda_2-\delta} F(\lambda)[g_k(\lambda-i\varepsilon)-g_k(\lambda+i\varepsilon)]\,d\lambda,
\end{aligned}
$$

since $g_k(\lambda) = \sum_{j=1}^{N} P_{kj}(\lambda)\varphi_j V(\lambda)$ for λ in $U(\lambda_0)$. Thus

$$
\lim_{\delta\to 0}\lim_{\varepsilon\to 0+}\frac{1}{2\pi i}\int_{\lambda_1+\delta}^{\lambda_2-\delta} F(\lambda)[g_k(\lambda-i\varepsilon)-g_k(\lambda+i\varepsilon)]\,d\lambda = \int_{(\lambda_1,\lambda_2)} F(\lambda)\mu_k(d\lambda).
$$

Now using the compactness of $[\lambda_1, \lambda_2]$ we may find a finite number of continuous functions $F_1, \ldots, F_r$ of the real variable λ, each vanishing outside a compact set interior to W, with the property that $\sum_{i=1}^{r} F_i(\lambda) = 1$, $\lambda\in[\lambda_1, \lambda_2]$, and

$$
\begin{aligned}
&\lim_{\delta\to 0}\lim_{\varepsilon\to 0+}\frac{1}{2\pi i}\int_{\lambda_1+\delta}^{\lambda_2-\delta} F_i(\lambda)[g_k(\lambda-i\varepsilon)-g_k(\lambda+i\varepsilon)]\,d\lambda \\
&\qquad\qquad = \int_{(\lambda_1,\lambda_2)} F_i(\lambda)\mu_k(d\lambda), \qquad i = 1, \ldots, r.
\end{aligned}
$$

Then

$$
\begin{aligned}
\mu_k((\lambda_1, \lambda_2)) &= \lim_{\delta\to 0} \sum_{i=1}^{\tau} \int_{\lambda_1+\delta}^{\lambda_2-\delta} F_i(\lambda)\mu_k(d\lambda) \\
&= \lim_{\delta\to 0} \lim_{\varepsilon\to 0} \sum_{i=1}^{\tau} \frac{1}{2\pi i} \int_{\lambda_1+\delta}^{\lambda_2-\delta} F_i(\lambda)[g_k(\lambda-i\varepsilon) - g_k(\lambda+i\varepsilon)]d\lambda \\
&= \lim_{\delta\to 0} \lim_{\varepsilon\to 0+} \frac{1}{2\pi i} \int_{\lambda_1+\delta}^{\lambda_2-\delta} [g_k(\lambda-i\varepsilon) - g_k(\lambda+i\varepsilon)]d\lambda.
\end{aligned}
$$

Q.E.D.

In case T is a real operator, Theorem 18 may be stated in a somewhat more convenient way.

20 COROLLARY. *With the hypotheses of Theorem* 18, *suppose that* τ *is a formal operator with real coefficients, that each of the functions* σ_i *is real for* t *and* λ *real, and that the operator* τ *is defined by a set of real boundary conditions. Then we may write*

$$
\begin{aligned}
\rho_{ij}((\lambda_1, \lambda_2)) &= \lim_{\delta\to 0} \lim_{\varepsilon\to 0+} \pi^{-1} \int_{\lambda_1+\delta}^{\lambda_2-\delta} \mathscr{I}\theta_{ij}^{-}(\lambda-i\varepsilon)d\lambda \\
&= \lim_{\delta\to 0} \lim_{\varepsilon\to 0+} \pi^{-1} \int_{\lambda_1+\delta}^{\lambda_2-\delta} \mathscr{I}\theta_{ij}^{+}(\lambda-i\varepsilon)d\lambda.
\end{aligned}
$$

PROOF. This will follow from Theorem 18 as soon as we establish that

$$
[*] \qquad \theta_{ij}^{-}(\bar{\lambda}) = \overline{\theta_{ij}^{-}(\lambda)} \text{ and } \theta_{ij}^{+}(\bar{\lambda}) = \overline{\theta_{ij}^{+}(\lambda)}.
$$

Since $\sigma_i(t, \lambda)$ is real for real λ and depends analytically on λ, we have $\overline{\sigma_i(t, \lambda)} = \sigma_i(t, \bar{\lambda})$. Hence [*] will follow if only we can establish that $\overline{K(t, s; \lambda)} = K(t, s; \bar{\lambda})$. Let Γ be the mapping of $L_2(I)$ into itself which sends every function into its complex conjugate. Then Γ is additive and isometric, but $\Gamma(\alpha x) = \bar{\alpha}\Gamma(x)$. Since T is defined by the real formal operator τ and by a set of real boundary conditions, we have $\Gamma T = T\Gamma$. Hence it follows that $\Gamma R(\lambda; T)\Gamma^{-1} = R(\bar{\lambda}; T)$ for each λ in the resolvent set of T. Consequently, if $K(t, s; \lambda)$ is the kernel such that

$$
\int_I K(t, s; \lambda)f(s)ds = (R(\lambda; T)f)(t),
$$

we have

$$\int_I \overline{K(t, s; \lambda)} f(s) ds = \big(\Gamma R(\lambda; T)\Gamma^{-1} f\big)(t) = \big(R(\bar{\lambda}; T)\big) f(t).$$

Since the kernel for $R(\bar{\lambda}; T)$ is unique, this shows that $\overline{K(t, s; \lambda)} = K(t, s; \bar{\lambda})$ and the present corollary is established. Q.E.D.

21 Corollary. *Let T, Λ, σ_i, etc., be as in Theorem* 18. *Then the positive matrix measure $\{\rho_{ij}\}$ on Λ is unique.*

Proof. If $\sigma_1(\cdot, \lambda), \ldots, \sigma_n(\cdot, \lambda)$ were known to be analytic in λ for λ in a neighborhood U of Λ in the complex plane, the result would follow immediately from Theorem 18. However, since this is not assumed, we are forced to take a more circuitous path.

Let c be any arbitrarily chosen point in I. For each complex λ, let $\hat{\sigma}_1(\cdot, \lambda), \ldots, \hat{\sigma}_n(\cdot, \lambda)$ be that basis for the space of solutions of $\tau\sigma = \lambda\sigma$ defined by the conditions

$$\hat{\sigma}_{i+1}^{(k)}(c, \lambda) = \delta_{ik}, \qquad i, k = 0, \ldots, n-1.$$

Then, by Corollary 1.5, $\hat{\sigma}_i(t, \lambda)$ depends analytically on λ, uniformly for t in any compact subinterval J of I. Let the coefficients a_{ij} and b_{ij} be defined for $\lambda \in \Lambda$ by the equations

$$\hat{\sigma}_i(\cdot, \lambda) = \sum_{j=1}^{n} a_{ij}(\lambda)\sigma_j(\cdot, \lambda)$$

$$\sigma_i(\cdot, \lambda) = \sum_{j=1}^{n} b_{ij}(\lambda)\hat{\sigma}_j(\cdot, \lambda).$$

Then it is clear that $\sum_{p=1}^{n} a_{ip}(\lambda)b_{pj}(\lambda) = \sum_{p=1}^{n} b_{ip}(\lambda)a_{pj}(\lambda) = \delta_{ij}$. Moreover, from the second equation displayed above, it follows on differentiating that $b_{ij}(\lambda) = \sigma_i^{(j-1)}(c, \lambda)$, $i, j = 1, \ldots, n$. Thus, by Lemma 2.16, b_{ij} depends continuously on λ for $\lambda \in \Lambda$. Since $\{a_{ij}(\lambda)\}$ is the inverse of $\{b_{ij}(\lambda)\}$, it follows (cf. VII.6.1) that a_{ij} depends continuously on λ for $\lambda \in \Lambda$.

Suppose that the present theorem is false. Then we may let $\{\rho_{ij}\}$ and $\{\mu_{ij}\}$ be two distinct positive matrix measures defined on Λ, having the properties stated in Theorem 13. For each Borel set e with compact closure contained in Λ_0, put

$$\hat{\rho}_{ij}(e) = \sum_{k,l=1}^{n} \int_e b_{ki}(\lambda)\overline{b_{lj}(\lambda)}\rho_{kl}(d\lambda)$$

and

$$\hat{\mu}_{ij}(e) = \sum_{k,l=1}^{n} \int_e b_{ki}(\lambda)\overline{b_{lj}(\lambda)}\mu_{kl}(d\lambda).$$

If $[\xi_1, \ldots, \xi_n]$ is an n-tuple of complex numbers, we have

$$\sum_{i,j=1}^{n} \hat{\rho}(e)_{ij}\xi_i\bar{\xi}_j = \int_e \{\sum_{k,l=1}^{n} f_k(\lambda)\overline{f_l(\lambda)}\rho_{kl}(d\lambda)\} \geqq 0,$$

where $f_k(\lambda) = \sum_{i=1}^{n}\xi_i b_{ki}(\lambda)$. Thus, $\{\hat{\rho}_{ij}\}$ is a positive matrix measure defined on Λ. In the same way we show that $\{\hat{\mu}_{ij}\}$ is a positive matrix measure defined on Λ. By Corollary III.10.6,

$$\begin{aligned}
&\sum_{i,j=1}^{n} \int_e a_{ik}(\lambda)\overline{a_{jl}(\lambda)}\hat{\rho}_{ij}(d\lambda) \\
&\qquad = \sum_{i,j,p,q=1}^{n} \int_e a_{ik}(\lambda)b_{pi}(\lambda)\overline{a_{jl}(\lambda)b_{qj}(\lambda)}\rho_{pq}(d\lambda) \\
&\qquad = \int_e \rho_{kl}(d\lambda) = \rho_{kl}(e).
\end{aligned}$$

In the same way, we show that

$$\sum_{i,j=1}^{n} \int_e a_{ik}(\lambda)\overline{a_{jl}(\lambda)}\hat{\mu}_{ij}(d\lambda) = \mu_{kl}(e).$$

Thus, since $\{\rho_{ij}\}$ and $\{\mu_{ij}\}$ are distinct, $\{\hat{\rho}_{ij}\}$ and $\{\hat{\mu}_{ij}\}$ are distinct.

Let ρ be a positive σ-finite Borel measure on Λ with respect to which all the measures ρ_{kl} are continuous, and let $\{\nu_{kl}\}$ be the corresponding matrix of densities. Then it is clear that all the measures $\hat{\rho}_{ij}$ are ρ-continuous, and that the corresponding matrix of densities is

$$\{\hat{\nu}_{ij}(\lambda)\} = \left\{\sum_{k,l=1}^{n} b_{ki}(\lambda)\overline{b_{lj}(\lambda)}\nu_{kl}(\lambda)\right\}.$$

Consequently, if we put $(Bf)_i(\lambda) = \sum_{k=1}^{n} b_{ik}(\lambda)f_k(\lambda)$ for each n-tuple $F = [f_1, \ldots, f_n]$ of Borel functions, B is an isometric isomorphism of $L_2(\Lambda, \{\hat{\rho}_{ij}\})$ into $L_2(\Lambda, \{\rho_{ij}\})$. In the same way, B is an isometric isomorphism of $L_2(\Lambda, \{\hat{\mu}_{ij}\})$ into $L_2(\Lambda, \{\mu_{ij}\})$.

Similarly, if we put $[AF]_i(\lambda) = \sum_{k=1}^{n} a_{ik}(\lambda)f_k(\lambda)$ for each n-tuple $F = [f_1, \ldots, f_n]$ of Borel functions, A is an isometric isomorphism

of $L_2(\Lambda, \{\rho_{ij}\})$ into $L_2(\Lambda, \{\hat{\rho}_{ij}\})$ and an isometric isomorphism of $L_2(\Lambda, \{\mu_{ij}\})$ into $L_2(\Lambda, \{\hat{\mu}_{ij}\})$. Since $\{a_{ij}(\lambda)\}$ and $\{b_{ij}(\lambda)\}$ are inverse matrices, it follows readily that $AB = BA = I$. Thus, A and B are isometric isomorphisms onto all of $L_2(\Lambda, \{\hat{\rho}_{ij}\})$ and $L_2(\Lambda, \{\mu_{ij}\})$, respectively.

Since

$$A\left[\int_c^d f(t)\overline{\sigma_i(t, \lambda)}dt\right] = \left[\int_c^d f(t)\overline{\hat{\sigma}_j(t, \lambda)}dt\right],$$

it follows from (i) of Theorem 13 that the limit

$$[(\hat{V}f)_i(\lambda)] = \lim_{\substack{c\to a\\ d\to b}} \left[\int_c^d f(t)\overline{\hat{\sigma}_i(t, \lambda)}dt\right]$$

exists in the topology of $L_2(\Lambda, \{\hat{\rho}_{ij}\})$ for each f in $L_2(I)$, and defines an isometric isomorphism $\hat{V}$ of $E(\Lambda)L_2(I)$ onto $L_2(\Lambda, \{\hat{\rho}_{ij}\})$. It follows similarly that this same limit exists in the topology of $L_2(\Lambda, \{\hat{\mu}_{ij}\})$ for each $f \in L_2(I)$, and defines an isometric isomorphism $\hat{V}$ of $E(\Lambda)L_2(I)$ onto $L_2(\Lambda, \{\hat{\mu}_{ij}\})$. Clearly, if V is as in Theorem 13(i), $\hat{V} = AV$.

It is clear from the definition of A that for each n-tuple $F = [f_1, \ldots, f_n]$ of Borel functions defined on Λ, $AF \in L_2(\Lambda, \{\hat{\rho}_{ij}\})$ if and only if $F \in L_2(\Lambda, \{\rho_{ij}\})$. Moreover, if G is the Borel function of Theorem 13(ii), and H is the n-tuple $[G(\cdot)f_i(\cdot)]$, then it is clear from the definition of A that $AH = [G(\cdot)(AF)_i(\cdot)]$. Thus, it follows from (ii) of Theorem 13 that for each Borel function G defined on the real line and vanishing outside Λ,

$$\hat{V}\mathfrak{D}(G(T)) = \{[f_i] \in L_2(\Lambda, \{\hat{\rho}_{ij}\}) | [G(\cdot)f_i(\cdot)] \in L_2(\Lambda, \{\hat{\rho}_{ij}\})\}$$

and

$$(\hat{V}G(T)f)_i(\lambda) = G(\lambda)(Vf)_i(\lambda), \quad i = 1, \ldots, n, \quad \lambda \in \Lambda, \quad f \in \mathfrak{D}(G(T)).$$

It follows in the same way that all these statements hold if $L_2(\Lambda, \{\hat{\rho}_{ij}\})$ is replaced by $L_2(\Lambda, \{\hat{\mu}_{ij}\})$.

Since the basis $\hat{\sigma}_1(\cdot, \lambda), \ldots, \hat{\sigma}_n(\cdot, \lambda)$ *satisfies the hypotheses of Theorem* 15, it follows that $\{\hat{\rho}_{ij}\} = \{\hat{\mu}_{ij}\}$. But, we saw above that $\{\hat{\rho}_{ij}\}$ and $\{\hat{\mu}_{ij}\}$ were distinct. Q.E.D.

On the basis of this corollary, we are able to make a number

of simplifications in Theorems 13 and 14 which are of practical importance.

The difficulty of making calculations with an $n \times n$ positive matrix measure is a rapidly increasing function of n. For this reason, we always strive to keep n as small as possible. In proving Theorem 13, i.e., in recasting Theorem 1 in terms of a set $\sigma_1(\cdot, \lambda), \ldots, \sigma_k(\cdot, \lambda)$ of linearly independent solutions of $\tau\sigma = \lambda\sigma$, it is not necessary that k be the same as the order n of τ; i.e., that $\sigma_1, \ldots, \sigma_k$ constitute a complete basis for the set of all solutions of $\tau\sigma = \lambda\sigma$. As the reader will readily see on reviewing the proof of Theorem 13, it is only necessary that each of the functions $W_1(\cdot, \lambda), \ldots, W_m(\cdot, \lambda)$ of Theorem 1 be in the linear span of $\sigma_1(\cdot, \lambda), \ldots, \sigma_k(\cdot, \lambda)$. In certain cases (cf. Theorem 4, Corollary 5) k can be significantly smaller than n.

For this reason, we make the following definition.

22 Definition. Let τ be a formally self adjoint formal differential operator defined on an interval I, and let T be a self adjoint extension of $T_0(\tau)$. Let Λ be an open subinterval of the real axis. For each λ in Λ, let $\sigma_1(\cdot, \lambda), \ldots, \sigma_k(\cdot, \lambda)$ be a linearly independent set of solutions of $\tau\sigma = \lambda\sigma$. Suppose that σ_i is continuous on $I \times \Lambda$ for $i = 1, \ldots, k$. Let μ and $W_1, \ldots, W_m$ be the measure and the kernels, respectively, of Theorem 1. If, for μ-almost all λ in Λ, the functions $W_1(\cdot, \lambda), \ldots, W_m(\cdot, \lambda)$ are in the linear span of $\sigma_1(\cdot, \lambda), \ldots, \sigma_k(\cdot, \lambda)$, then $\sigma_1, \ldots, \sigma_k$ will be called a *determining set for T on the interval Λ.*

A number of remarks concerning this definition are in order. Note that a determining set for T on the whole infinite interval $(-\infty, +\infty)$ always exists. For example, if n is the order of τ, we may take c in I and require that $\sigma_{i+1}^{(j)}(c, \lambda) = \delta_i^j$, $i = 0, \ldots, n-1$. Note also that Theorem 4 describes an important situation in which a determining set for T may consist of fewer than n functions. Of course, Theorem 4 is by no means the last word in this connection.

Finally, we assert that the concept of a determining set depends only on the operator T, and not on the specific choice of a measure μ and kernels W_i in Theorem 1.

To prove this, let $\hat{\mu}$ and $\hat{W}_i$, $i = 1, \ldots, m$, be a second measure and a second set of kernels having the properties described in Theorem

1. Suppose that $\sigma_1, \ldots, \sigma_k$ are as in Definition 22, and that the functions $W_i(\cdot, \lambda)$ are in the linear span of $\sigma_1(\cdot, \lambda), \ldots, \sigma_k(\cdot, \lambda)$, for μ-almost all λ. We will show that each point λ_0 in Λ has a neighborhood N such that the functions $\hat{W}_i(\cdot, \lambda)$ are in the linear span of $\sigma_1(\cdot, \lambda), \ldots, \sigma_k(\cdot, \lambda)$ for $\hat{\mu}$-almost all $\lambda \in N$. Since Λ can be covered by a countable family of such neighborhoods N, this will establish the desired conclusion.

Let λ_0 be in Λ and choose some point c in I. Let

$$v_i(\lambda) = [\sigma_i(c, \lambda), \sigma_i^1(c, \lambda), \ldots, \sigma_i^{(n-1)}(c, \lambda)], \qquad i = 1, \ldots, k$$

and find vectors $v_i = [v_i^{(0)}, \ldots, v_i^{(n-1)}]$, $i = k+1, \ldots, n$, in Euclidean n-space which are independent of the vectors $v_i(\lambda_0)$, $i = 1, \ldots, k$. Let $\sigma_{k+1}(\cdot, \lambda), \ldots, \sigma_n(\cdot, \lambda)$ be the unique solution of $\tau\sigma = \lambda\sigma$ determined by the initial conditions $\sigma_i^{(j)}(c, \lambda) = v_i^{(j)}$, $j = 0, \ldots, n-1$, $i = k+1, \ldots, n$. By Corollary 15, $v_i(\lambda)$ depends continuously on λ, $i = 1, \ldots, n$. Since the vectors $v_1(\lambda), \ldots, v_k(\lambda), v_{k+1}, \ldots, v_n$ are linearly independent for $\lambda = \lambda_0$ (so that the $n \times n$ determinant of their components is non vanishing), there exists a small subinterval N of Λ containing λ_0 such that $v_1(\lambda), \ldots, v_k(\lambda), v_{k+1}, \ldots, v_n$ are linearly independent for $\lambda \in N$. Consequently, $\sigma_1(\cdot, \lambda), \ldots, \sigma_n(\cdot, \lambda)$ are linearly independent for λ in N, so that, for λ in N, they form a basis for the space of solutions of $\tau\sigma = \lambda\sigma$. Consequently, we may write

$$W_i(\cdot, \lambda) = \sum_{j=1}^{n} a_{ij}(\lambda)\sigma_j(\cdot, \lambda), \qquad i = 1, \ldots, m,$$

$$\hat{W}_i(\cdot, \lambda) = \sum_{j=1}^{n} \hat{a}_{ij}(\lambda)\sigma_j(\cdot, \lambda), \qquad i = 1, \ldots, m.$$

By assumption, $a_{ij}(\lambda) = 0$ for μ-almost all $\lambda \in N$ if $j > k$. We wish to show that $\hat{W}_i(\cdot, \lambda)$ is in the linear span of $\sigma_1(\cdot, \lambda), \ldots, \sigma_k(\cdot, \lambda)$ for $\hat{\mu}$-almost all λ in N; i.e., we wish to show that $\hat{a}_{ij}(\lambda) = 0$ for μ-almost all λ in N if $j > k$. If this is false, there exists an i_0 and a $j_0 > k$ such that $\hat{a}_{i_0 j_0}(\lambda)$ fails to vanish for μ-almost all λ. As demonstrated in the course of the proof of Theorem 18, the matrix measure $\{\rho_{ij}\}$ of Theorem 18, which is unique by Corollary 20, is given by the formula

$$\rho_{ij}(e) = \sum_{p=1}^{m} \int_e a_{pj}(\lambda)\overline{a_{pi}(\lambda)}\mu(d\lambda)$$

for each Borel set e with compact closure contained in N. For the same reason,

$$\rho_{ij}(e) = \sum_{p=1}^{m} \int_e \hat{a}_{pj}(\lambda)\overline{\hat{a}_{pi}(\lambda)}\hat{\mu}(d\lambda).$$

This first equation shows that

$$\rho_{i_0 i_0}(e) = \int_e \sum_{p=1}^{m} |a_{pi_0}(\lambda)|^2 \mu(d\lambda) = 0$$

for every Borel set e with compact closure contained in N. On the other hand, since $\sum_{p=1}^{m} |\hat{a}_{pi_0}(\lambda)|^2$ fails to vanish for $\hat{\mu}$-almost all $\lambda \in N$, there exists a Borel set e with compact closure contained in N such that

$$\rho_{i_0 i_0}(e) = \int_e \sum_{p=1}^{m} |\hat{a}_{pi_0}(\lambda)|^2 \hat{\mu}(d\lambda) \neq 0.$$

This contradiction proves our assertion. Q.E.D.

In terms of the notion of a determining set for T, Theorems 13 and 14 may be restated as follows:

23 THEOREM. *Let τ be a formally self adjoint formal differential operator of order n defined on an interval I with end points a, b. Let T be a self adjoint extension of $T_0(\tau)$. Let Λ be an open interval of the real axis, and suppose that $\sigma_1, \ldots, \sigma_k$ is a determining set for T on Λ. Then there exists a positive $k \times k$ matrix measure $\{\hat{\rho}_{ij}\}$ defined on Λ, such that*

(i) *the limit*

$$[(Vf)_i(\lambda)] = \lim_{\substack{c \to a \\ d \to b}} \left[\int_c^d f(t)\overline{\sigma_i(t, \lambda)}dt \right]$$

exists in the topology of $L_2(\Lambda, \{\hat{\rho}_{ij}\})$ for each $f \in L_2(I)$ and defines an isometric isomorphism V of $E(\Lambda)L_2(I)$ onto all of $L_2(\Lambda, \{\hat{\rho}_{ij}\})$;

(ii) *for each Borel function G defined on the real line and vanishing outside Λ,*

$$V\mathfrak{D}(G(T)) = \{[f_i] \in L_2(\Lambda, \{\hat{\rho}_{ij}\}) | [Gf_i] \in L_2(\Lambda, \{\hat{\rho}_{ij}\})\}$$

and

$$(VG(T)f)_i(\lambda) = G(\lambda)(Vf)_i(\lambda), \quad i = 1, \ldots, k, \quad \lambda \in \Lambda, \quad f \in \mathfrak{D}(G(T)).$$

The proof is nearly identical with the proof of Theorem 13, and if the reader reviews the proof of that theorem he will easily be able to make the few modifications required. In the same way, by making a few obvious modifications in the proof of Theorem 14, we obtain the following result.

24 THEOREM. *Let T, Λ, $\{\hat{\rho}_{ij}\}$, etc., be as in Theorem 23. Let λ_0 and λ_1 be the end points of Λ. Then*

(i) *the inverse of the isometric isomorphism V of $E(\Lambda)L_2(I)$ onto $L_2(\Lambda, \{\hat{\rho}_{ij}\})$ is given by the formula*

$$(V^{-1}F)(t) = \lim_{\substack{\mu_0 \to \lambda_0 \\ \mu_1 \to \lambda_1}} \int_{\mu_0}^{\mu_1} \left(\sum_{i,j=1}^{k} F_i(\lambda)\sigma_j(t, \lambda)\hat{\rho}_{ij}(d\lambda) \right),$$

where $F = [F_1, \ldots, F_k] \in L_2(\Lambda, \{\hat{\rho}_{ij}\})$, the limit existing in the topology of $L_2(I)$;

(ii) *if G is a bounded Borel function vanishing outside a Borel set e whose closure is compact and contained in Λ, then $G(T)$ has the representation*

$$\big(G(T)f\big)(t) = \int_I f(s)K(G; t, s)ds,$$

where

$$K(G; t, s) = \sum_{i,j=1}^{k} \int_e G(\lambda)\sigma_j(t, \lambda)\overline{\sigma_i(s, \lambda)}\hat{\rho}_{ij}(d\lambda).$$

Moreover, given any compact interval $J \subseteq I$,

$$\sup_{t \in J} \int_I |K(G; t, s)|^2 ds < \infty.$$

We devote the remainder of the present section to stating and proving a number of results which make it easier to apply the main results proved up to now.

25 THEOREM. *Let τ, T, $\sigma_1, \ldots, \sigma_n$ be as in Theorem 13. Then a subset $\sigma_1, \ldots, \sigma_k$ of $\sigma_1, \ldots, \sigma_n$ is a determining set for T if and only if $\rho_{jj}(e) = 0$ for $j > k$ and for each Borel set e with compact closure contained in Λ. If the subset $\sigma_1, \ldots, \sigma_k$ is a determining set for T, and if $\{\rho_{ij}\}$, $i, j = 1, \ldots, n$ is the matrix measure of Theorem 13, then the*

matrix measure $\{\hat\rho_{ij}\}$, $i, j = 1, \ldots, k$ *of Theorem 23 is unique, and* $\hat\rho_{ij} = \rho_{ij}$, $i, j = 1, \ldots, k$; $\rho_{ij} = 0$, *if* $i > k$ *or* $j > k$.

PROOF. Suppose that $\sigma_1, \ldots, \sigma_k$ is a determining set for T. Then it is evident from Theorem 23 that if we define $\{\rho_{ij}\}$, $i, j = 1, \ldots, n$, by

$$[*] \qquad \begin{aligned} \rho_{ij}(e) &= \hat\rho_{ij}(e), \qquad i, j = 1, \ldots, k, \\ \rho_{ij}(e) &= 0 \text{ if } i > k \text{ or } j > k, \end{aligned}$$

we get a matrix measure $\{\rho_{ij}\}$ which by Corollary 21 must be the same as the matrix measure of Theorem 13. In particular, $\hat\rho_{ij}$ is unique. Thus, all that remains for us to prove is that if $\rho_{jj}(e) = 0$ for $j > k$, then $\sigma_1, \ldots, \sigma_k$ is a determining set for T. Suppose that this is not the case, and let μ and W_i, $i = 1, \ldots, m$, be as in Theorem 1. We have

$$W_i(\cdot, \lambda) = \sum_{j=1}^{n} a_{ij}(\lambda)\sigma_j(\cdot, \lambda), \qquad \lambda \in \Lambda, \qquad i = 1, \ldots, m,$$

in terms of certain coefficients a_{ij}. Since we are assuming that $\sigma_1, \ldots, \sigma_k$ is not a determining set, some coefficient $a_{i_0 j_0}$, with $j_0 > k$, fails to vanish μ-almost everywhere. Since, by the proof of Theorem 18, (and by the uniqueness of $\{\rho_{ij}\}$)

$$\rho_{j_0 j_0}(e) = \int_e \left\{ \sum_{i=1}^{m} |a_{ij_0}(\lambda)|^2 \right\} \mu(d\lambda),$$

it follows that $\rho_{j_0 j_0}$ is not zero for every Borel set with compact closure contained in Λ, contrary to assumption. Q.E.D.

26 COROLLARY. *Let* τ, T, *etc., be as in Definition* 22. *Then the matrix measure* $\{\hat\rho_{ij}\}$ *of Theorem 23 is unique.*

PROOF. If the determining set $\sigma_1, \ldots, \sigma_k$ of Theorem 23 were known to be part of a basis $\sigma_1, \ldots, \sigma_n$ with the properties of Theorem 18, then the uniqueness of $\{\hat\rho_{ij}\}$ would follow immediately from the preceding theorem. Moreover, in the course of the proof preceding the statement of Theorem 23, it was shown that if λ_0 is any point in Λ, there exists a small open subinterval N of Λ, containing λ_0, such that the set of restrictions $\hat\sigma_1, \ldots, \hat\sigma_k$ of $\sigma_1, \ldots, \sigma_k$ to $I \times N$ is a subset of just such a basis. Thus, it follows from the preceding theorem that if

$\{\hat{\rho}_{ij}\}$ is the matrix measure of Theorem 23, the values $\hat{\rho}_{ij}(e)$ are uniquely determined for each $e \subseteq N$. Since Λ is the union of a sequence of neighborhoods of the same type as N, the uniqueness of $\{\hat{\rho}_{ij}\}$ follows immediately. Q.E.D.

27 THEOREM. *Let τ, T, Λ, $\sigma_1, \ldots, \sigma_n$, etc., be as in Theorem* 18. *Then if, for $j > k$, the functions $\theta^+_{jj}(\lambda)$ of Theorem* 18 *(or, the functions $\theta^-_{jj}(\lambda)$ of Theorem* 18) *may be extended to analytic functions defined on the whole neighborhood U of Λ, $\sigma_1, \ldots, \sigma_k$ is a determining set for T.*

PROOF. If $\theta^+_{jj}(\cdot)$ is analytic for $j > k$, it follows from Theorem 18 that $\rho_{jj}(e) = 0$ for $j > k$ and each Borel set e with compact closure contained in Λ. Thus, by Theorem 25, $\sigma_1, \ldots, \sigma_k$ is a determining set for T. Q.E.D.

28 COROLLARY. *Let T, Λ, σ_i, etc., be as in Theorem* 18. *Suppose that for each λ in a neighborhood of Λ such that $\mathscr{I}\lambda \neq 0$, $\sigma_1, \ldots, \sigma_k$ span the space of all solutions of $\tau\sigma = \lambda\sigma$ which are square-integrable at an end point a of I and satisfy all the boundary conditions at a of the family of boundary conditions determining T. Then $\sigma_1, \ldots, \sigma_k$ is a determining set for T.*

PROOF. Suppose for the sake of definiteness that a is the left end point of I. By Corollary 3.12, the resolvent kernel $K(t, s; \lambda)$ of Theorem 18 has the form

$$K(t, s; \lambda) = \sum_{i=1}^{k} \sum_{j=1}^{n} \delta_{ij}(\lambda)\sigma_i(t, \lambda)\overline{\sigma_j(s, \bar{\lambda})}, \qquad s < t,$$

in terms of certain coefficients δ_{ij}. But this means that for $j > k$, the coefficients θ^+_{jj} of Theorem 18 are all zero. Our present assertion now follows immediately from the preceding theorem. Q.E.D.

29 COROLLARY. *Let τ, T, Λ, $\sigma_1, \ldots, \sigma_n$, etc., be as in Theorem* 18. *Then a point λ in Λ is in the resolvent set for T if the functions θ^+_{jj} (or equivalently, the functions θ^-_{jj}) of Theorem* 18 *can be extended to be analytic functions defined on a neighborhood of λ.*

PROOF. It is clear from Theorem 18 that if θ^+_{jj} can be extended to be analytic on a neighborhood N of λ, then $\rho_{jj}(e)$ vanishes for each $j = 1, \ldots, n$ and each Borel subset e of N. It then follows from

Theorem 25 that $\rho_{ij}(e) = 0$ for all $i, j = 1, \ldots, n$, and by Corollary 15 that λ is in the resolvent set of T.

Conversely, let λ be in the resolvent set of T. Then, by Theorem 18, θ_{ij}^{+} is analytic in the neighborhood of λ.

The proof for θ_{ij}^{-} is exactly similar. Q.E.D.

80 Corollary. *Let τ, T, Λ, σ_i, θ_{ij}^{+}, etc., be as in Theorem 18. Then an isolated point $\lambda_0 \in \Lambda\sigma(T)$ is an isolated singularity of θ_{ij}^{+} (or, equivalently, of θ_{ij}^{-}). Moreover, $\rho_{ij}(\{\lambda_0\})$ is the residue at λ_0 of θ_{ij}^{+} (of θ_{ij}^{-}).*

Proof. The first assertion follows immediately from the preceding theorem. It follows from Corollary 15 and Theorem 18 that if a and b are two points such that $(a, b) \cap \sigma(T) = \{\lambda_0\}$, then

$$[*] \qquad \rho_{ij}(\{\lambda_0\}) = \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\frac{1}{2\pi i}\int_{a+\varepsilon}^{b-\varepsilon}\big(\theta_{ij}^{+}(\lambda-i\delta)-\theta_{ij}^{+}(\lambda+i\delta)\big)d\lambda.$$

If $C_{\delta,\varepsilon}$ denotes the rectangle with corners $a+\varepsilon+i\delta$, $a+\varepsilon-i\delta$, $b-\varepsilon-i\delta$, $b-\varepsilon+i\delta$, then since θ_{ij}^{+} is continuous in the neighborhood of $b-\varepsilon$ and $a+\varepsilon$, it is clear that

$$\lim_{\delta\to 0}\int_{a+\varepsilon}^{a-\varepsilon}\big(\theta_{ij}^{+}(\lambda-i\delta)-\theta_{ij}^{+}(\lambda+i\delta)\big)d\lambda = \lim_{\delta\to 0}\int_{C_{\delta,\varepsilon}}\theta_{ij}^{+}(\zeta)d\zeta.$$

On the other hand, if C denotes any sufficiently small circle about λ_0, it is evident from Cauchy's theorem that

$$\int_C \theta_{ij}^{+}(\zeta)d\zeta = \int_{C_{\delta,\varepsilon}}\theta_{ij}^{+}(\zeta)d\zeta.$$

Thus it follows immediately from [*] that

$$\rho_{ij}(\{\lambda_0\}) = \frac{1}{2\pi i}\int_C \theta_{ij}^{+}(\zeta)d\zeta.$$

An exactly similar proof shows that

$$\rho_{ij}(\{\lambda_0\}) = \frac{1}{2\pi i}\int_C \theta_{ij}^{-}(\zeta)d\zeta.$$

Q.E.D.

The spectral theory developed in Sections one through four, and in the present section, enables us to establish the specific form of the spectral resolution of a great variety of differential operators.

To begin with the simplest possible case, consider the first order operator $\tau_1 = (1/i)(d/dt)$ on the interval $[0, 1]$. By the remark following Definition 2.29, the two linear functionals $f \to f(0)$ and $f \to f(1)$ form a complete set of boundary values for τ_1 and the most general self adjoint extension T_θ of $T_0(\tau)$ is defined by a boundary condition $f(0) = e^{i\theta}f(1)$. Since $[0,1]$ is a closed interval, it follows from Theorems 4.1 and 4.2 that the spectrum of T_θ consists entirely of isolated points, every such point being an eigenvalue of T_θ, and that T_θ has a complete set of orthonormal eigenfunctions. The (unique) solution of $\tau_1\sigma = \lambda\sigma$ is clearly $e^{i\lambda t}$; this function clearly satisfies the boundary condition $f(0) = e^{i\theta}f(1)$ if and only if λ is congruent to $-\theta$ modulo 2π. Thus, the eigenvalues of T_θ are the numbers $2\pi n-\theta$, n being an arbitrary positive or negative integer. The corresponding eigenfunctions are $e^{(2\pi i n - i\theta)t}$, and are already normalized. From Theorem 4.2(c) we learn that this set of functions is *complete.*

By the remarks following Definition 2.29, the formal differential operator $(1/i)(d/dt)$, if considered to be defined on the interval $[0, \infty)$, cannot lead to any self adjoint operator in Hilbert space. Consequently, the next case to engage our attention is the formal differential operator $\tau_2 = (1/i)(d/dt)$, defined on the interval $(-\infty, +\infty)$. By the remarks under "Case 3" following Definition 2.29, τ_2 has no boundary values, so that $T_0(\tau_2)$ has the unique self adjoint extension $T_1(\tau_2)$. A basis for the space of solutions of $\tau_2\sigma = \lambda\sigma$ is furnished by the single function $e^{i\lambda t}$. In the remarks following 3.16, we have expressed the resolvent of $T_1(\tau)$ in terms of this "basis", and found that the constants θ_{ij}^+ of the Titchmarsh-Kodaira theorem (18) are

$$\theta^+(\lambda) = \theta_{11}^+(\lambda) = -i, \qquad \mathscr{I}\lambda < 0,$$
$$\theta^+(\lambda) = \theta_{11}^+(\lambda) = 0, \qquad \mathscr{I}\lambda > 0.$$

Thus, by the Titchmarsh-Kodaira theorem,

$$\rho(\lambda_1, \lambda_2) = \rho_{11}(\lambda_1, \lambda_2) = \frac{1}{2\pi i}\int_{\lambda_1}^{\lambda_2} d\lambda = \frac{1}{2\pi}(\lambda_2-\lambda_1),$$

so that ρ is simply $1/2\pi$ times the Lebesgue measure on the λ-axis. We are consequently able to derive the following theorem immediately from Theorems 13 and 14.

31 THEOREM. *For each f in $L_2(-\infty, +\infty)$, the limit*

$$(Ff)(\lambda) = \lim_{A\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-A}^{+A} f(t)e^{-it\lambda}\,dt$$

exists in the topology of $L_2(-\infty, +\infty)$, and defines an isometric isomorphism of $L_2(-\infty, +\infty)$ into itself, whose inverse is given by the formula

$$(F^{-1}f)(t) = \lim_{A\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-A}^{+A} f(\lambda)e^{it\lambda}\,d\lambda,$$

the limit existing in the topology of $L_2(-\infty, +\infty)$. Moreover,

$$\int_{-\infty}^{+\infty} |\lambda|^2 |(Ff)(\lambda)|^2 d\lambda < \infty$$

if and only if f is absolutely continuous and f' is in $L_2(-\infty, +\infty)$, and in this case

$$(Ff')(\lambda) = i\lambda(Ff)(\lambda), \qquad -\infty < \lambda < +\infty.$$

The first part of this theorem is identical with the Plancherel theorem (XI.3.21); the second part gives a useful connection between the Fourier transform and the operation of differentiation. Of course, in stating the second part of the above theorem, we have by no means exhausted the content of Theorem 13(ii), which would immediately yield connections between the Fourier integral and higher derivatives, etc.

After first order formal differential operators, the next most complex class of formal differential operators are second order formal differential operators. Let us begin our acquaintance with this important class of operators by considering the formal differential operator $\tau_3 = -(d/dt)^2$, defined on the interval $[0, 1]$. Of the wide variety of possible sets of self adjoint boundary conditions which may be imposed on τ_3, let us confine our attention to four:

Set A:	$f(0) = 0,$	$f(1) = 0,$
Set B:	$f'(0) = 0,$	$f'(1) = 0,$
Set C:	$f(0) = 0,$	$f'(1) = 0,$
Set D:	$f(0) = f(1),$	$f'(0) = f'(1).$

All but set D are separated. (It is worth noting in passing that the most general self adjoint set of separated boundary conditions for τ_3 is described by Corollary 2.31). Again we are in the situation of Section 4, the interval being finite, the spectrum being discrete, and the set of eigenfunctions being complete. With boundary conditions A and C, the unique solution of $\tau_3\sigma = \lambda\sigma$ satisfying the boundary condition $\tau_3\sigma = \lambda\sigma$ is $\sin \sqrt{\lambda}t$. With boundary conditions A, the eigenvalues are consequently to be determined from the equation $\sin \sqrt{\lambda} = 0$. Consequently, in Case A, the eigenvalues λ are the numbers of the form $(n\pi)^2$, $n \geqq 1$; in Case C, the numbers $((n+\frac{1}{2})\pi)^2$, $n \geqq 0$. In Case A, the (normalized) eigenfunctions are $\{2^{-1/2} \sin n\pi t\}$; in Case C, they are $\{2^{-1/2} \sin (n+\frac{1}{2})\pi t\}$. Thus, using Theorem 4.2(c), we are able to establish the completeness of various collections of sine-functions.

In Case B, the unique solution of $\tau_3\sigma = \lambda\sigma$ satisfying the boundary condition $\sigma'(0) = 0$ is $\cos \sqrt{\lambda}x$. The eigenvalues are consequently to be determined from the equation $(\cos \sqrt{\lambda}x)'|_{x=1} = 0$; i.e., from the equation $\sin \sqrt{\lambda} = 0$. Consequently, in Case B, the eigenvalues are the numbers of the form $(n\pi)^2$. and the normalized eigenfunctions are

$$1, \quad \frac{1}{\sqrt{2}} \cos \pi x, \quad \frac{1}{\sqrt{2}} \cos 2\pi x, \quad \frac{1}{\sqrt{2}} \cos 3\pi x, \ldots .$$

Finally, we turn to Case D. The functions $\sin \sqrt{\lambda}x$ and $\cos \sqrt{\lambda}x$ together form a basis for the set of solutions of $\tau_3\sigma = \lambda\sigma$. In order that there exist a non-zero linear combination $f(x) = a \sin \sqrt{\lambda}x + b \cos \sqrt{\lambda}x$ of these functions satisfying $f(0) = f(1)$, $f'(0) = f'(1)$, i.e., in order that there exist a non-zero pair a, b such that

$$\begin{aligned} b - a \sin \sqrt{\lambda} - b \cos \sqrt{\lambda} &= 0, \\ a - a \cos \sqrt{\lambda} + b \sin \sqrt{\lambda} &= 0, \end{aligned}$$

it is necessary and sufficient that the determinant

$$\begin{vmatrix} -\sin \sqrt{\lambda} & 1 - \cos \sqrt{\lambda} \\ 1 - \cos \sqrt{\lambda} & \sin \sqrt{\lambda} \end{vmatrix} = 2 \cos \sqrt{\lambda} - 2$$

vanish. Consequently, the eigenvalues in Case D are the numbers of the form $(2\pi n)^2$, $n \geqq 0$. Associated with the eigenvalue zero is the

unique normalized eigenfunction $\varphi_0(t) = 1$. When $\lambda = (2\pi n)^2$, $n \geqq 1$, both solutions $\cos 2\pi nt$ and $\sin 2\pi nt$ of $\tau_3\sigma = \lambda\sigma$ satisfy $f(0) = f(1)$, $f'(0) = f'(1)$. Thus, in Case D, a two-dimensional space of eigenfunctions is associated with each eigenvalue $\lambda = (2\pi n)^2$, $n \geqq 1$. An orthonormal basis for this space is given by the functions

$$\frac{1}{\sqrt{2}} \sin 2\pi nx, \qquad \frac{1}{\sqrt{2}} \cos 2\pi nx.$$

Thus, we see that in Case D we are led to consider expansions in the complete orthonormal set of Fourier functions

$$1. \quad \frac{1}{\sqrt{2}} \sin 2\pi x, \quad \frac{1}{\sqrt{2}} \cos 2\pi x, \quad \frac{1}{\sqrt{2}} \sin 4\pi x, \quad \frac{1}{\sqrt{2}} \cos 4\pi x, \quad \text{etc.}$$

Let us now consider a number of singular examples. Suppose, for instance, that we study the formal differential operator $\tau_4 = -(d/dt)^2$ on the interval $[0, \infty)$. A basis for the space of the solutions of $\tau_4\sigma = \lambda\sigma$ is furnished by $e^{i\sqrt{\lambda}t}$ and $e^{-i\sqrt{\lambda}t}$. If $\lambda = i$, the first of these solutions is square-integrable, but the second is not. Consequently, (since τ_4 is real, so that its deficiency indices are equal) the deficiency indices of τ_4 are (1,1). By the remarks following Definition 2.21, τ_4 has no boundary values at infinity. Moreover, the most general self adjoint extension T_k of $T_0(\tau_4)$ is determined by the single boundary condition

$$f(0) + kf'(0) = 0, \qquad -\infty < k \leqq \infty.$$

We divide our study of these various possibilities into four cases:

Case (i) : $k = 0$. Boundary condition $f(0) = 0$.

Case (ii) : $k = \infty$. Boundary condition $f'(0) = 0$.

Case (iii): $-\infty < k < 0$.

Case (iv): $0 < k < \infty$.

Let us first find the point spectrum of T_k. Since no linear combination of $e^{i\sqrt{\lambda}x}$ and $e^{-i\sqrt{\lambda}x}$ is square-integrable if $\lambda > 0$, no point of the positive real axis can belong to the point spectrum of T_k. Since a basis for the space of the solutions of $\tau_4\sigma = 0$ is provided by 1 and x,

$\lambda = 0$ can never be in the point spectrum of T_k either. If λ is negative, we may write our two solutions $e^{i\sqrt{\lambda}x}$ and $e^{-i\sqrt{\lambda}x}$ as $e^{\sqrt{-\lambda}x}$ and $e^{-\sqrt{-\lambda}x}$. The first of these is not in $L_2(0, \infty)$, but the second is. The function $e^{-\sqrt{-\lambda}x}$ satisfies the boundary condition $f(0)+kf'(0)$ if and only if $1-k\sqrt{-\lambda} = 0$; i.e., if and only if k is positive and $\lambda = -1/k^2$. Thus, only in case (iv) does T_k have a non-void point spectrum, which consists of the single point $\lambda = -1/k^2$, with the associated normalized eigenfunction $2^{1/2}k^{-1/2}e^{-x/k}$.

Next we turn to an analysis of the continuous spectrum. First consider the interval $-\infty < \lambda < 0$. For λ in the left half plane, we can conveniently use the basis $e^{\sqrt{-\lambda}t}$, $e^{-\sqrt{-\lambda}t}$ for the set of all solutions of $\tau\sigma = \lambda\sigma$.

In case (i), the function $e^{\sqrt{-\lambda}t}-e^{-\sqrt{-\lambda}t}$ satisfies the boundary condition $f(0) = 0$. The function $e^{-\sqrt{-\lambda}t}$ is square-integrable at $t = \infty$. Consequently, by Theorem 3.16, if $\mathscr{I}\lambda \neq 0$, the resolvent $R(\lambda; T_0)$ is an integral operator with kernel

$$-\frac{(e^{\sqrt{-\lambda}s}-e^{-\sqrt{-\lambda}s})e^{-\sqrt{-\lambda}t}}{2\sqrt{-\lambda}}, \qquad s < t,$$

$$-\frac{(e^{\sqrt{-\lambda}t}-e^{-\sqrt{-\lambda}t})e^{-\sqrt{-\lambda}s}}{2\sqrt{-\lambda}}, \qquad t < s.$$

The matrix $\theta_{ij}^-(\lambda)$ of Theorem 18 is consequently

$$\begin{pmatrix} \dfrac{1}{2\sqrt{-\lambda}} & -\dfrac{1}{2\sqrt{-\lambda}} \\ 0 & 0 \end{pmatrix}.$$

Since all its elements are analytic on $-\infty < \lambda < 0$, it follows from Corollary 29 that the entire interval $-\infty < \lambda < 0$ belongs to the resolvent set of T_0.

The reader will have no difficulty in carrying out an exactly similar computation and deriving an exactly similar result in case (ii). In cases (iii) and (iv) the function

$$(k\sqrt{-\lambda}-1)e^{\sqrt{-\lambda}t}+(k\sqrt{-\lambda}+1)e^{-\sqrt{-\lambda}t}$$

satisfies $f(0)+kf'(0) = 0$. Consequently, by Theorem 3.16, the resolvent $R(\lambda; T_0)$ is an integral operator with the kernel

$$\frac{\{(k\sqrt{-\lambda}-1)e^{\sqrt{-\lambda}s}+(k\sqrt{-\lambda}+1)e^{-\sqrt{-\lambda}s}\}e^{-\sqrt{-\lambda}t}}{2k\lambda+2\sqrt{-\lambda}}, \qquad s < t,$$

$$\frac{\{k(\sqrt{-\lambda}-1)e^{\sqrt{-\lambda}t}+k(\sqrt{-\lambda}+1)e^{-\sqrt{-\lambda}t}\}e^{-\sqrt{-\lambda}s}}{2k\lambda+2\sqrt{-\lambda}}, \qquad t < s.$$

The matrix $\theta^-_{ij}(\lambda)$ is consequently

$$\begin{pmatrix} \dfrac{k\sqrt{-\lambda}+1}{2(k\lambda+\sqrt{-\lambda})} & \dfrac{k\sqrt{-\lambda}-1}{2(k\lambda+\sqrt{-\lambda})} \\ 0 & 0 \end{pmatrix}.$$

In case (iii) k is negative, so all the elements of this matrix are analytic for negative λ. Thus, in case (iii), the entire negative real axis belongs to the resolvent set of T_k. In case (iv), the first element of the above matrix has a pole at $\lambda = -1/k^2$. (Note that $[k\sqrt{-\lambda}-1]/[2(k\lambda+\sqrt{-\lambda})]$ is analytic at this point also.) The residue of $[k\sqrt{-\lambda}+1]/2(k\lambda+\sqrt{-\lambda})$ at this point is $2k^{-1}$. Thus, by Corollary 30 and the remarks following Theorem 16, the orthonormal eigenfunction associated with the eigenvalue $\lambda = -1/k^2$ is $2^{1/2}k^{-1/2}e^{x/k}$. This fact has already been noted; but the present derivation is of particular interest since it serves to emphasize the fact that the normalization factors for the orthonormal eigenfunctions of a differential operator can be obtained directly from the Titchmarsh-Kodaira theorem. In the present case, e^{-kx} is easy enough to normalize directly, but in those cases to be studied below, in which the eigenfunctions are, say, Laguerre polynomials, we will find considerable use for the general method of normalization based on the Titchmarsh-Kodaira theorem.

Since we have seen that $\lambda = 0$ is never in the point spectrum of our operator, it only remains for us to investigate that part of the spectrum lying in the region $0 < \lambda < \infty$. First consider case (i). If λ is in the right half plane, a convenient basis for the set of solutions of $\tau\sigma = \lambda\sigma$ is furnished by the pair of functions $\sin\sqrt{\lambda}t$ and $\cos\sqrt{\lambda}t$.

The first of these satisfies the boundary condition $f(0) = 0$. If $\mathscr{I}\lambda > 0$, the linear combination

$$e^{i\sqrt{\lambda}t} = \cos\sqrt{\lambda}t + i\sin\sqrt{\lambda}t$$

belongs to $L_2(0, \infty)$; if $\mathscr{I}\lambda < 0$, the linear combination

$$e^{-i\sqrt{\lambda}t} = \cos\sqrt{\lambda}t - i\sin\sqrt{\lambda}t$$

belongs to $L_2(0, \infty)$. Consequently, by Theorem 3.16, the resolvent $R(\lambda; T)$ is an integral operator with the kernel

$$\begin{array}{lll} \dfrac{\sin\sqrt{\lambda}s(\cos\sqrt{\lambda}t + i\sin\sqrt{\lambda}t)}{\sqrt{\lambda}}, & s < t, & \mathscr{I}\lambda > 0, \\[2ex] \dfrac{\sin\sqrt{\lambda}t(\cos\sqrt{\lambda}s + i\sin\sqrt{\lambda}s)}{\sqrt{\lambda}}, & t < s, & \mathscr{I}\lambda > 0, \\[2ex] \dfrac{\sin\sqrt{\lambda}s(\cos\sqrt{\lambda}t - i\sin\sqrt{\lambda}t)}{\sqrt{\lambda}}, & s < t, & \mathscr{I}\lambda < 0, \\[2ex] \dfrac{\sin\sqrt{\lambda}t(\cos\sqrt{\lambda}s - i\sin\sqrt{\lambda}s)}{\sqrt{\lambda}}, & t < s, & \mathscr{I}\lambda < 0. \end{array}$$

Consequently, the matrix $\theta_{ij}^{+}(\lambda)$ of Theorem 18 is

$$\begin{pmatrix} +\dfrac{i}{\sqrt{\lambda}} & -\dfrac{1}{\sqrt{\lambda}} \\ 0 & 0 \end{pmatrix}, \qquad \mathscr{I}\lambda > 0,$$

and

$$\begin{pmatrix} -\dfrac{i}{\sqrt{\lambda}} & -\dfrac{1}{\sqrt{\lambda}} \\ 0 & 0 \end{pmatrix}, \qquad \mathscr{I}\lambda < 0.$$

Thus, only the measure ρ_{11} of Theorem 18 is non zero, and using Theorem 18, we see that this measure is given by the formula

$$\rho_{11} = \frac{1}{2\pi i}\int_a^b \frac{d\lambda}{\sqrt{\lambda}}.$$

Consequently, we learn from Theorems 13 and 14 that

$$\lim_{A\to\infty} \frac{1}{\sqrt{\pi}} \int_0^A (\sin \sqrt{\lambda} t) f(t)\, dt$$

defines an isometric isomorphism of $L_2(0, \infty)$ onto the space of all functions g such that

$$\int_0^\infty |g(\lambda)|^2 \frac{d\lambda}{\sqrt{\lambda}} < \infty,$$

and that the inverse of this isomorphism is given by the formula

$$\lim_{A\to\infty} \frac{1}{\sqrt{\pi}} \int_0^A (\sin \sqrt{\lambda} t) g(\lambda) \frac{d\lambda}{\sqrt{\lambda}}.$$

If we make the change of variable $\sqrt{\lambda} = \mu$, this result takes on the following more symmetric form.

82 THEOREM. (*Fourier Sine Theorem*) *Let* $f \in L_2(0, \infty)$. *The limit*

$$(\mathscr{S} f)(\mu) = \lim_{A\to\infty} \sqrt{\frac{2}{\pi}} \int_0^A (\sin \mu t) f(t)\, dt$$

exists in the norm of $L_2(0, \infty)$, *and defines an isometric isomorphism of* $L_2(0, \infty)$ *onto itself which is self inverse:*

$$\mathscr{S}^2 = I, \qquad \mathscr{S} = \mathscr{S}^{-1}.$$

We have

$$\int_0^\infty \mu^4 |(\mathscr{S} f)(\mu)|^2 d\mu < \infty$$

if and only if f *has an absolutely continuous first derivative,*

$$\int_0^\infty |f''(t)|^2 dt < \infty,$$

and $f(0) = 0$; *in this case,*

$$\mu^2 (\mathscr{S} f)(\mu) = -(\mathscr{S} f'')(\mu).$$

The reader will have no difficulty in verifying that the corresponding calculations in case (ii) lead to the following result.

83 THEOREM. (*Fourier Cosine Theorem*) *Let* $f \in L_2(0, \infty)$. *The limit*

$$(\mathscr{C} f)(\mu) = \lim_{A\to\infty} \sqrt{\frac{2}{\pi}} \int_0^A (\cos \mu t) f(t)\, dt$$

exists in the norm of $L_2(0, \infty)$, *and defines an isometric isomorphism of* $L_2(0, \infty)$ *onto itself which is self inverse:*

$$\mathscr{C}^2 = I, \qquad \mathscr{C} = \mathscr{C}^{-1}.$$

We have

$$\int_0^\infty \mu^4 |(\mathscr{C}f)(\mu)|^2 d\mu < \infty$$

if and only if f has an absolutely continuous first derivative,

$$\int_0^\infty |f''(t)|^2 dt < \infty,$$

and $f'(0) = 0$; *in this case,*

$$\mu^2(\mathscr{C}f)(\mu) = -(\mathscr{C}f'')(\mu).$$

Next, consider cases (iii) and (iv). The function $\sin\sqrt{\lambda}t - k\sqrt{\lambda}\cos\sqrt{\lambda}t$ satisfies the boundary condition $f(0)+kf'(0) = 0$. This function and the function $\cos\sqrt{\lambda}t$ together furnish a basis for the set of solutions of $\tau\sigma = \lambda\sigma$. If $\mathscr{I}\lambda > 0$, the function

$$\begin{aligned} e^{i\sqrt{\lambda}t} = \cos\sqrt{\lambda}t + i\sin\sqrt{\lambda}t &= i(\sin\sqrt{\lambda}t - k\sqrt{\lambda}\cos\sqrt{\lambda}t) \\ &\quad + (1+ik\sqrt{\lambda})\cos\sqrt{\lambda}t \end{aligned}$$

belongs to $L_2(0, \infty)$; if $\mathscr{I}\lambda < 0$, the function

$$\begin{aligned} e^{-i\sqrt{\lambda}t} &= \cos\sqrt{\lambda}t - i\sin\sqrt{\lambda}t \\ &= -i(\sin\sqrt{\lambda}t - k\sqrt{\lambda}\cos\sqrt{\lambda}t) + (1-ik\sqrt{\lambda})\cos\sqrt{\lambda}t \end{aligned}$$

belongs to $L_2(0, \infty)$. Consequently, by Theorem 3.16, the resolvent $R(\lambda; T_k)$ is an integral operator with the kernel

$K(\lambda; s, t)$

$$= \frac{(\sin\sqrt{\lambda}s - k\sqrt{\lambda}\cos\sqrt{\lambda}s)\{i(\sin\sqrt{\lambda}t - k\cos\sqrt{\lambda}t) + (1+ik\sqrt{\lambda})\cos\sqrt{\lambda}t\}}{\sqrt{\lambda}(1+ik\sqrt{\lambda})}$$

$s < t. \qquad \mathscr{I}\lambda > 0.$

$$= \frac{(\sin\sqrt{\lambda}s - k\sqrt{\lambda}\cos\sqrt{\lambda}s)\{\;\; i(\sin\sqrt{\lambda}t - k\cos\sqrt{\lambda}t) + (1-ik\sqrt{\lambda})\cos\sqrt{\lambda}t\}}{\sqrt{\lambda}(1-ik\sqrt{\lambda})}$$

$s < t, \qquad \mathscr{I}\lambda < 0.$

The matrix $\theta_{ij}^{+}(\lambda)$ of Theorem 18 is consequently

$$\begin{pmatrix} \dfrac{i}{\sqrt{\lambda}(1+ik\sqrt{\lambda})} & \dfrac{1}{\sqrt{\lambda}} \\ 0 & 0 \end{pmatrix}, \qquad \mathscr{I}\lambda > 0,$$

$$\begin{pmatrix} \dfrac{-i}{\sqrt{\lambda}(1-ik\sqrt{\lambda})} & \dfrac{1}{\sqrt{\lambda}} \\ 0 & 0 \end{pmatrix}, \qquad \mathscr{I}\lambda < 0.$$

It follows that the only non-zero measure ρ_{ij} of Theorem 18 is ρ_{11}, which is given by the formula

$$\rho_{11}(a, b) = \frac{1}{\pi}\int_a^b \frac{d\lambda}{\sqrt{\lambda}(1+k^2\lambda)}, \qquad b > a > 0.$$

Thus, putting $\mu^2 = \lambda$, we obtain the following two theorems, which correspond to cases (iii) and (iv) above respectively.

34 THEOREM. *Let* $0 < k < \infty$. *Let*

$$\varphi(t, \mu) = \sqrt{\frac{2}{\pi}}\,(1+k^2\mu^2)^{-1/2}(\sin \mu t + k\mu \cos \mu t).$$

Then, if $f \in L_2(0, \infty)$, *the limit*

$$(U_k f)(\mu) = \lim_{A\to\infty}\int_0^A f(t)\varphi(t, \mu)dt$$

exists in the topology of $L_2(0, \infty)$, *and defines an isometric isomorphism of* $L_2(0, \infty)$ *onto itself, whose inverse is given by the formula*

$$(U_k^{-1}g)(t) = \lim_{A\to\infty}\int_0^A g(\mu)\varphi(t, \mu)d\mu.$$

We have

$$\int_0^\infty \mu^4|(U_k f)(\mu)|^2 d\mu < \infty$$

if and only if f has an absolutely continuous first derivative,

$$\int_0^\infty |f''(t)|^2 dt < \infty,$$

and $f(0) = kf'(0)$; *in this case*

$$\mu^2(U_k f)(\mu) = -(U_k f'')(\mu).$$

35 THEOREM. *Let* $0 < k < \infty$, *and*

$$\psi(t, \mu) = \sqrt{\frac{2}{\pi}}\,(1+k^2\mu^2)^{-1/2}(\sin \mu t - k\mu \cos \mu t).$$

Let E^1 *denote one-dimensional unitary space. If* $f \in L_2(0, \infty)$, *the limit*

$$(V_k^{(0)} f)(\mu) = \lim_{A\to\infty} \int_0^A f(t)\psi(t, \mu)dt$$

exists in the topology of $L_2(0, \infty)$. *If we put*

$$V_k^{(1)} f = \sqrt{\frac{2}{k}} \int_0^\infty f(t)e^{-t/k}dt,$$

then the formula

$$V_k f = [V_k^{(0)} f, V_k^{(1)} f]$$

defines an isometric isomorphism of $L_2(0, \infty)$ *onto the direct sum* $L_2(0, \infty) \oplus E^1$. *The inverse of this isometric isomorphism is given by the formula*

$$(V_k^{-1}[g, \alpha])(t) = \alpha \sqrt{\frac{2}{k}}\, e^{-t/k} + \lim_{A\to\infty} \int_0^A g(\mu)\psi(t, \mu)d\mu,$$

the limit on the right existing in the topology of $L_2(0, \infty)$. *We have*

$$\int_0^\infty \mu^4 |(V_k^{(0)} f)(\mu)|^2 d\mu < \infty$$

if and only if f *has an absolutely continuous first derivative,*

$$\int_0^\infty |f''(t)|^2 dt < \infty,$$

and $f(0) + kf'(0) = 0$; *in this case,*

$$(V_k^{(0)} f'')(\mu) = -\mu^2 (V_k^{(0)} f)(\mu),$$

$$(V_k^{(1)} f'') = \frac{1}{k^2}\,(V_k^{(1)} f).$$

It is clear that by considering other formal differential operators with constant coefficients, we can construct other series and integral expansions involving trigonometric and exponential functions. A more interesting possibility, however, is to study eigenfunction expansions arising from formal differential operators with non-constant coefficients. In this case, however, the problem of choosing a basis for the solutions of $\tau\sigma = \lambda\sigma$ is by no means trivial. If, for instance, we take $\tau = -(d/dt)^2 + t^2$, then the solutions of $\tau\sigma = \lambda\sigma$ are particular confluent hypergeometric functions. Because of the difficulty of dealing with the various functions which may arise as solutions of the equation $\tau\sigma = \lambda\sigma$, we prefer to postpone any further examination of special eigenfunction expansions to Section 8 below. In that section we shall first develop a part of the theory of "special functions," and on the basis of this theory, will discuss a number of famous complete orthonormal sets, unitary integral transformations, etc.

6. Qualitative Theory of the Deficiency Index

The methods developed in Section 5, especially the Titchmarsh-Kodaira theorem (5.18), enable us to calculate the spectral resolution of a self adjoint operator T which is derived from a formal differential operator τ by the imposition of a specific set of boundary conditions. From the spectral resolution of T, the spectrum of $\sigma(T)$ can be determined immediately. However. it is sometimes not very easy to make the calculations called for by the methods of Section 5. For instance, for the operator $\tau_1 = -(d/dt)^2 + t^2$ one finds that the solutions of $(\tau_1 - \lambda)\sigma = 0$ are expressed in terms of confluent hypergeometric functions, and hence that the spectral analysis of τ_1 necessarily involves a knowledge of the properties of confluent hypergeometric functions. If one attempted to study $\tau_2 = -(d/dt)^2 + x^6$, in which case the solutions of $(\tau_2 - \lambda)\sigma = 0$ would bave to be expressed in terms of even more unfamiliar transcendental functions, the calculations involved by Theorem 5.18 would be rather complex. Nevertheless, it will be seen in the present section and the section to follow that in these cases and in others, substantial amounts of information about $\sigma(T)$, the deficiency indices of T, etc., can be obtained by an almost direct inspection of the coefficients of τ. We

shall, for example, develop results that enable us to state that no self adjoint operator derived from either τ_1 or τ_2 has a continuous spectrum, and that, in fact, the spectrum of each such operator consists of an infinite sequence of points approaching $+\infty$.

We begin by defining a certain type of "spectrum" for the formal differential operator τ.

1 Definition. Let T be a closed operator in Hilbert space, Then the set of complex numbers λ such that the range of $\lambda I - T$ is not closed is called the ***essential spectrum*** of T and is denoted by $\sigma_e(T)$.

It is clear that $\sigma_e(T) \subseteq \sigma(T)$. If τ is a formal differential operator defined on the interval I, then the essential spectrum of the closed operator $T_1(\tau)$ in $L_2(I)$ is called the ***essential spectrum*** $\sigma_e(\tau)$ of τ.

2 Lemma. *Let $\mathfrak{X}$ be a Banach space, and suppose that $\mathfrak{X} = \mathfrak{Y} + \mathfrak{N}$, where $\mathfrak{N}$ is a finite dimensional space and $\mathfrak{Y}$ is a closed subspace. Let T be a bounded linear operator from $\mathfrak{X}$ to a second Banach space $\mathfrak{X}_1$, Then $T\mathfrak{Y}$ is closed in $\mathfrak{X}_1$ if and only if $T\mathfrak{X}$ is closed.*

Proof. To prove that $T\mathfrak{X}$ is closed if $T\mathfrak{Y}$ is closed, we shall prove more generally that the sum of a closed subspace $\mathfrak{Z}$ of a B-space, and of a finite dimensional space $\hat{\mathfrak{N}}$, is closed. It is clear that proceeding inductively we may assume without loss of generality that $\hat{\mathfrak{N}}$ is one-dimensional. Thus $\hat{\mathfrak{N}}$ is identical with the set $\{\alpha x\}$ of all multiples of a non-zero vector x. If $x \in \mathfrak{Z}$, we have nothing to prove; hence, suppose that $x \notin \mathfrak{Z}$. Then every $y \in \mathfrak{Z} + \hat{\mathfrak{N}}$ can be written uniquely as $y = z + \alpha x$, where $z \in \mathfrak{Z}$. Let $y_n \in \mathfrak{Z} + \hat{\mathfrak{N}}$, and let $y_n \to y_\infty$. Then $y_n = z_n + \alpha_n x$. If $\{\alpha_n\}$ is bounded, we may suppose, on passing to a subsequence, that $\alpha_n \to \alpha$. In this case, $z_n \to y_\infty - \alpha x$, so that $y_\infty - \alpha x \in \mathfrak{Z}$, and thus $y_\infty \in \mathfrak{Z} + \hat{\mathfrak{N}}$. On the other hand, $\{\alpha_n\}$ must be bounded. Indeed, if it were unbounded, we could suppose on passing to a subsequence that $|\alpha_n| \to \infty$. Then, putting $\hat{z}_n = \alpha_n^{-1} z_n$, we would have $\hat{z}_n \to -x$, so that $x \in \mathfrak{Z}$, contrary to assumption. This shows that $\mathfrak{Z} + \hat{\mathfrak{N}}$ is closed, and proves the first half of the present lemma.

To prove the second half of the present lemma suppose that $T(\mathfrak{Y} + \mathfrak{N})$ is closed. Let n be the dimension of $\mathfrak{N}$, and $\mathfrak{N}_1, \ldots, \mathfrak{N}_n = \mathfrak{N}$ an increasing sequence of subspaces of $\mathfrak{N}$, such that $\dim \mathfrak{N}_i = i$. We shall prove by induction on m that $T(\mathfrak{Y} + \mathfrak{N}_{n-m})$ is closed. Since

by what has been shown above $\mathfrak{Y}+\mathfrak{N}_{n-m}$ is closed, it is sufficient for this purpose to establish the converse part of the present lemma under the additional hypothesis that $\mathfrak{N}$ is one-dimensional, i.e., that $\mathfrak{N}=\{\alpha x\}$. If $Tx\in T\mathfrak{Y}$, then $T(\mathfrak{Y}+\mathfrak{N})=T(\mathfrak{Y})$, so $T\mathfrak{Y}$ is closed. Hence we may assume that $Tx\notin T\mathfrak{Y}$. Suppose that $T\mathfrak{Y}$ is not closed, and let $z\notin T\mathfrak{Y}$. $z\in\overline{T\mathfrak{Y}}$, so that there exists a sequence of elements $y_n\in\mathfrak{Y}$ such that $Ty_n\to z$. Since $T(\mathfrak{Y}+\mathfrak{N})$ is closed, there exists an element $y+\alpha x$, $y\in\mathfrak{Y}$, such that $z=T(y+\alpha x)$. We must have $\alpha\neq 0$, since $z\notin T\mathfrak{Y}$. Now $T\big((y-y_n)+\alpha x\big)\to 0$. Since $T(\mathfrak{Y}+\mathfrak{N})$ is closed, it follows from Lemma VI.6.1 that there exists a sequence $\{\hat{y}_n+\alpha_n x\}$ such that $(\hat{y}_n+\alpha_n x)\to 0$, while

$$T\big((y-y_n)+\alpha x\big)=T(\hat{y}_n+\alpha_n x),$$

that is,

$$(\alpha-\alpha_n)Tx=T(y_n+\hat{y}_n-y).$$

Since Tx is not in $T\mathfrak{Y}$, $\alpha=\alpha_n$ and hence $y_n\to -\alpha x$ contradicting the assumption that $\mathfrak{Y}$ is closed. Q.E.D.

3 Corollary. *Let τ be a formally symmetric formal differential operator, and let T be any closed symmetric extension of $T_0(\tau)$. Then the essential spectrum of τ coincides with the essential spectrum of T.*

Proof. By XII.4.8(c) and 2.10, $T_0^*\supseteq T\supseteq T_0$. Introduce into $\mathfrak{D}\big(T_1(\tau)\big)=\mathfrak{D}\big(T_0(\tau)^*\big)$ the inner product $(f,g)^*=(f,g)+(T_1f,T_1g)$. Then, by XII.4.10 the Hilbert space $\mathfrak{D}(T_1)=\mathfrak{D}(\overline{T_0^*})$ is a direct sum of the form $\mathfrak{D}(\overline{T}_0)\oplus\mathfrak{D}_+\oplus\mathfrak{D}_-$, where $\mathfrak{D}_+$ and $\mathfrak{D}_-$ are finite-dimensional (cf. 1.3). Thus, since $\mathfrak{D}(T_0^*)\supseteq\mathfrak{D}(T)\supseteqq\mathfrak{D}(T_0)$, it follows that $\mathfrak{D}(T_1)=\mathfrak{D}(T)+\mathfrak{N}$, where $\mathfrak{N}$ is a subspace of $\mathfrak{D}_+\oplus\mathfrak{D}_-$, and is consequently finite dimensional. Since T is closed, $\mathfrak{D}(T)$ is a closed subspace of $\mathfrak{D}(T_1)$ (cf. XII.4.5). If T_1 is regarded as an operator mapping the Hilbert space $\mathfrak{D}(T_1)$ into $\mathfrak{H}$, it is clearly bounded. Thus, the result follows immediately from the preceding lemma and from the definition of the essential spectrum. Q.E.D.

4 Corollary. *Let T be a self adjoint extension of $T_0(\tau)$. Then $\sigma_e(T)=\sigma_e(\tau)$*

Proof. This is an immediate consequence of Lemma XII.4.8(e) and the preceding corollary. Q.E.D.

5 THEOREM. *The essential spectrum of a self adjoint operator T is the set of non-isolated points of $\sigma(T)$.*

PROOF. Suppose λ is an isolated point in the spectrum of T. For simplicity, we shall write U for the closed operator $\lambda I - T$. Notice that $\mathfrak{D}(U) = \mathfrak{D}(T)$. Let E be the resolution of the identity for T (cf. XII.2); then, by XII.2.7(c), we have

$$E(\{\lambda\})Ux = 0, \qquad x \in \mathfrak{D}(T).$$

Let $\sigma_1 = \sigma(T) - \{\lambda\}$. Then

$$\big(E(\sigma_1)U\big)x = \big(I - E(\{\lambda\})(\lambda I - T)\big)x = (\lambda I - T)x$$

which shows that the range of the projection $E(\sigma_1)$ contains the range of T.

Choose a neighborhood V of λ which is disjoint from σ_1, and let $f(\mu) = (\lambda - \mu)^{-1}$ if $\mu \notin V$ and $f(\mu) = 0$ if $\mu \in V$. Suppose that y is in the range of $E(\sigma_1)$; then by Theorem XII.2.6 $f(T)y$ is in $\mathfrak{D}(T)$, and $[Uf(T)]y = E(\sigma_1)y = y$. By this remark and the preceding paragraph, it follows that the range of $E(\sigma_1)$ (which is obviously closed) coincides with the range of U, which shows that λ is not in the essential spectrum of T.

To complete the proof it will be shown that any point λ in the spectrum of T such that the range of $\lambda I \quad T$ is closed is an isolated point of the spectrum.

Let λ be such a point, and let $\mathfrak{N}$ be the null space of U, that is, the set of all $x \in \mathfrak{D}(U)$ such that $Ux = 0$. Then the restriction U_1 of U to $\mathfrak{D}(U) \cap \mathfrak{N}^{\perp}$ has the same range as U. Moreover, the graph of U_1 is evidently the orthocomplement of the set $\{[x, 0], x \in \mathfrak{N}\}$ in the graph of U. and is therefore closed. Thus U_1 is a closed one-to-one map with a closed range. By the closed graph theorem and by Theorem II.2.2, U_1 has a bounded inverse, that is, there is a constant k such that if $|U_1x| \leq 1$, then $|x| \leq k/2$. Now, by XII.2.6(c), $\mathfrak{N}$ is the range of the projection $E(\{\lambda\})$. Hence $\mathfrak{N}^{\perp}$ is the range of $E(\sigma_1)$. Therefore if x is in $\mathfrak{D}(T) = \mathfrak{D}(U)$, if $E(\sigma_1)x = x$, and if $|(\lambda I - T)x| \leqq 1$, then $|x| \leq k/2$.

Let $A = \{\mu | \mu \neq \lambda, |\mu - \lambda| < 1/k\}$. It suffices to show that $E(A) = 0$, for, by XII.2.9, this implies that the set A is disjoint from $\sigma(T)$. that is, that λ is isolated. Suppose there exists a vector x

in $\mathfrak{H}$ such that $E(A)x = x$. We can assume that $|x| = k$. Then, since $A \subset \sigma_1$, $E(\sigma_1)x = E(\sigma_1)E(A)x = E(A)x = x$. By Theorem XII.2.6 (c) we have

$$|(\lambda I - T)x|^2 = \int_A |\mu-\lambda|^2 (E(d\mu)x, x)$$

$$\leq \frac{1}{k^2}\int_A (E(d\mu)x, x) = \frac{1}{k^2}|x|^2 = 1,$$

that is, $|(\lambda I-T)x| \leq 1$ while $|x| = k$, which is a contradiction. Q.E.D.

6 THEOREM. *Let τ be a formally self adjoint formal differential operator defined on an interval I. Let there exist a point λ on the real axis not belonging to the essential spectrum of τ. Then both deficiency indices of τ are equal. Moreover, all the self adjoint extensions of $T_0(\tau)$ have the same set of non-isolated points, and this set is equal to $\sigma_e(\tau)$.*

PROOF. The second assertion follows immediately from Theorem 5 and Corollary 4. In proving the first assertion, it may be assumed without loss of generality (cf. XII.2.2 and XII.4.19) that $\lambda = 0$. Let $\mathfrak{N} = \{f | T_1(\tau)f = 0\}$. We shall construct a self adjoint extension T of the closure $\overline{T}_0$ of $T_0(\tau)$. By Corollary XII.4.13 and Lemma XII.4.8(b), this will show that the deficiency indices of τ are equal. It is evident (since $\overline{T}_0$ is symmetric by Lemma XII.4.6) that the restriction T_2 of $T_1(\tau)$ to $\mathfrak{D}(\overline{T}_0)+\mathfrak{N}$ is symmetric. Indeed, if $x_i \in \mathfrak{D}(\overline{T}_0)+\mathfrak{N}$, $i = 1,2$, we can write $x_i = y_i+z_i$, $i = 1,2$, where $y_i \in \mathfrak{D}(\overline{T}_0)$, $z_i \in \mathfrak{N}$, $i = 1,2$. Then by Theorem 2.10,

$$(T_1(\tau)x_1, x_2) = (\overline{T}_0 y_1, x_2) = (y_1, T_1(\tau)x_2) = (y_1, \overline{T}_0 y_2).$$

By symmetry $(T_1(\tau)x_2, x_1) = (y_2, \overline{T}_0 y_1)$, so, since $\overline{T}_0$ is symmetric, it follows that $(T_1(\tau)x_1, x_2) = (x_1, T_1(\tau)x_2)$. We assert, moreover, that T_2 is self adjoint. Indeed, the assumption that $\lambda \notin \sigma_e(\overline{T}_0)$ $(= \sigma_e(\tau)$ by Corollary 3) implies that the range $\mathfrak{R}(\overline{T}_0)$ of $\overline{T}_0$ is closed. Since by Theorem 2.10 and XII.1.6(d), $\mathfrak{N} = [\mathfrak{R}(\overline{T}_0)]^\perp$, we have $L_2(I) = \mathfrak{R}(\overline{T}_0) \oplus \mathfrak{N}$. Let x be in $\mathfrak{D}(T_2^*)$. Then for all $y \in \mathfrak{N}$,

$$(T_2^* x, y) = (x, T_2 y) = 0,$$

which shows that $T_2^* x \in \mathfrak{R}(\overline{T}_0)$. Hence there is an element x_1 in $\mathfrak{D}(T_2)$

such that $T_0 x_1 = T_2^* x_1 = T_2^* x$. Hence $x - x_1 \in \mathfrak{N}$, and $x = (x - x_1) + x_1$ is in $\mathfrak{D}(T_2)$, proving that T_2 is self adjoint. Q.E.D.

It follows from Theorem 5 and Corollary 4 that the set of non-isolated points of the spectrum of a self adjoint extension T of $T_0(\tau)$ is independent of the particular extension chosen, i.e., is independent of the particular set of boundary conditions defining this extension. We shall now show that the isolated points of $\sigma(T)$ depend quite strongly on the set of boundary conditions defining T, at least in the case in which τ is defined on an interval with at least one fixed end point.

7 Lemma. *Let T be a symmetric operator in Hilbert space $\mathfrak{H}$, the minimum of whose deficiency indices is k. If λ does not belong to the essential spectrum of T, then the equation $T^*x = \lambda x$ has at least k linearly independent solutions.*

Proof. In the case $\mathscr{I}\lambda \neq 0$ the statement has been proved in Theorem XII.4.19. If λ is real, then we can replace T by the operator $T - \lambda I$, which is still symmetric and has the same deficiency indices as T (cf. XII.4.19); we can therefore assume that $\lambda = 0$. The assertion then is that the null space $\mathfrak{N}$ of T^* is at least k-dimensional.

The method of proof is the following: it will be shown that if the theorem is false, then a proper symmetric extension T_2 of T can be constructed whose domain properly contains both $\mathfrak{D}(T)$ and the null-space of T^*. This readily yields a contradiction as follows: the assumption that $0 \notin \sigma_e(T)$ implies that the range $\mathfrak{R}(T)$ of T is closed. Let T_1 be the extension which is easily seen to be symmetric obtained by restricting T^* to $\mathfrak{D}(T) + \mathfrak{N}$. Then the range $\mathfrak{R}(T_1)$ of T_1 coincides with the range of T and is therefore closed. Moreover, the orthocomplement of $\mathfrak{R}(T)$ is $\mathfrak{N}$; (cf. XII.1.6) hence

$$[*] \qquad \mathfrak{H} = \mathfrak{R}(T) \oplus \mathfrak{N} = \mathfrak{R}(T_1) \oplus \mathfrak{N}.$$

Suppose now that T_2 is a proper symmetric extension of T_1. By XII.4.1, $T_2 \subseteq T^*$. If the range of T_2 properly contains the range of T_1, then, since $\mathfrak{R}(T_2)$ is a linear space, $\mathfrak{R}(T_2)$ contains an element orthogonal to $\mathfrak{R}(T_1)$. Thus $\mathfrak{D}(T_2)$ contains an element y such that $0 \neq T_2 y \in \mathfrak{N}$. But this is impossible, since if $T_2 y \in \mathfrak{N}$, then since $T_2 \supseteq T_1$ and $T_1 \mathfrak{N} = 0$, it follows that $(T_2 y, T_2 y) = (T_2 T_2 y, y) = 0$,

and therefore that $T_2 y = 0$. Thus $\mathfrak{R}(T_2) = \mathfrak{R}(T_1)$; but this is equally impossible, for it implies that there is an element y in $\mathfrak{D}(T_2)$ but not in $\mathfrak{D}(T_1)$ and an element x in $\mathfrak{D}(T_1)$ such that $T_2 y = T_1 x = T_2 x$. From this it follows that $y - x$ is in $\mathfrak{N} \subseteq \mathfrak{D}(T_1)$, so that y is in $\mathfrak{D}(T_1)$, a contradiction.

It remains to show that if the conclusion of the present lemma fails, then T_1 has a proper symmetric extension. By Lemma XII.4.11 and Theorem XII.4.12 it suffices to verify that neither of the deficiency spaces $\mathfrak{D}_+^{(1)}$ and $\mathfrak{D}_-^{(1)}$ of T_1 is $\{0\}$. Let $\mathfrak{D}_+ = \{(T+iI)\mathfrak{D}(T)\}^\perp$ be the positive deficiency space of T. Then (cf. XII.1.6(d))

$$\mathfrak{D}_+^{(1)} = \{(T_1+iI)\mathfrak{D}(T_1)\}^\perp = \{(T_1+iI)\mathfrak{D}(T)+\mathfrak{N}\}^\perp \supseteq \mathfrak{D}_+ \cap \{(T^*+iI)\mathfrak{N}\}$$
$$= \mathfrak{D}_+ \cap \mathfrak{N}^\perp.$$

Since we are assuming that $\dim \mathfrak{N} < k$ and $\dim \mathfrak{D}_+ \geqq k$, it follows that $\dim \mathfrak{D}_+^{(1)} \geqq 1$. Similarly $\dim \mathfrak{D}_-^{(1)} \geqq 1$. Therefore T_1 has a proper symmetric extension T_2, and the proof is complete. Q.E.D.

8 COROLLARY. *Let τ be a formally self adjoint formal differential operator defined on an interval I. If the minimum of the deficiency indices of $T_0(\tau)$ is k, then for $\lambda \notin \sigma_e(\tau)$ the equation $\tau\sigma = \lambda\sigma$ has at least k linearly independent solutions in $L_2(I)$.*

PROOF. By Theorem 2.10 and XII.4.7(c), the adjoint of $\overline{T_0(\tau)}$ is $T_1(\tau)$. The desired result thus follows immediately from the preceding lemma, Theorem 5, and Corollary 4. Q.E.D.

REMARK. The assumption that λ does not belong to the essential spectrum in Corollary 8 is necessary. For example, if $\tau = -(d/dt)^2$ on the interval $[0, \infty)$, both deficiency indices of τ may readily be seen to be 1. On the other hand, if $\lambda > 0$, the most general solution of $\tau\sigma = \lambda\sigma$ is easily seen to be $a \cos t\sqrt{\lambda} + b \sin t\sqrt{\lambda}$, and this function is never in $L_2(0, \infty)$ unless $a = b = 0$.

9 LEMMA. *Let τ be a formally symmetric formal differential operator on an interval I, and suppose that I has at least one fixed end point. Let the minimum of the deficiency indices of $T_0(\tau)$ be ν. Then, for each real λ, the number of linearly independent solutions of $\tau\sigma = \lambda\sigma$ in $L_2(I)$ is at most ν.*

PROOF. Since τ and $\tau - \lambda$ have the same deficiency indices by

Theorem XII.4.19, we can take $\lambda = 0$. We then wish to show that $\tau\sigma = 0$ has at most ν linearly independent solutions in $L_2(I)$. Let $\overline{T_0}$ be the closure of $T_0(\tau)$. Then by Lemma XII.4.7(c) and Theorem 2.10, $T_1(\tau) = (\overline{T_0})^*$. Recall from Section XII.4 that the linear space $\mathfrak{D}(T_1(\tau))$ becomes a Hilbert space upon the introduction of the inner product $(x, y)^*$ defined in Definition XII.4.2(a). In the following discussion we will deal with this inner product wherever the contrary is not explicitly indicated. By Lemma XII.4.10, $\mathfrak{D}(T_1(\tau)) = \mathfrak{D}(\overline{T_0}) \oplus \mathfrak{D}_+ \oplus \mathfrak{D}_-$, all three spaces on the right being closed. Suppose that the conclusion of the present lemma fails. Then $\mathfrak{N} = \{x | T_1(\tau)x = 0\}$ is more than ν-dimensional. Assume for the sake of definiteness that $\dim \mathfrak{D}_+ \leq \dim \mathfrak{D}_-$ so that $\dim \mathfrak{D}_+ = \nu$. Then the projection $y \to y_+$ of $\mathfrak{D}(T_1(\tau))$ on $\mathfrak{D}_+$ cannot map $\mathfrak{N}$ in a one-to-one way. Hence there exists an element y in $\mathfrak{N}$ such that $y_+ = 0$, that is, $y = y_0 + y_-$ where $y_0 \in \mathfrak{D}(\overline{T_0})$, $y_- \in \mathfrak{D}_-$. It is readily seen that the restriction T_2 of $T_1(\tau) = T_0(\tau)^*$ to $\mathfrak{D}(\overline{T_0}) + \mathfrak{N}$ is symmetric. Moreover, $y_- \in \mathfrak{D}(T_2)$. This means, however, that

$$
\begin{aligned}
i(y_-, y_-) = (T_1 y_-, y_-) = (T_2 y_-, y_-) = (y_-, T_2 y_-) &= (y_-, T_1 y_-) \\
&= (y_-, i y_-) = -i(y_-, y_-).
\end{aligned}
$$

Thus $y_- = 0$ and hence y is in $\mathfrak{D}(\overline{T_0})$. Since every boundary value for τ vanishes on $\mathfrak{D}(\overline{T_0(\tau)})$, it follows from Corollary 2.23 that the first $n-1$ derivatives of y vanish at the fixed end point of I. Since $T_1(\tau)y = 0$, it follows from Theorem 1.3 that y vanishes identically. This contradiction completes the proof. Q.E.D.

Remark. The hypothesis that I has a fixed end point is necessary in Lemma 9. Consider, for instance, the formal differential operator $\tau = -(d/dt)^2 + t^2$ on the interval $(-\infty, +\infty)$. If $f_1(t) = e^{t^2/2}$, $f_2(t) = e^{-t^2/2}$, we have $\tau f_1 = -f_1$, $\tau f_2 = f_2$. The deficiency indices of τ are equal by Corollary 2.14. They cannot be (2,2), since by Theorem 4.1, Theorem 5 and Corollary 4 this would imply that the essential spectrum of τ is vacuous, and by Corollary 8, all solutions of $\tau\sigma + \sigma = 0$ would lie in $L_2(-\infty, +\infty)$, which is not the case. Hence by Theorem 2.19, τ either has no boundary values at ∞ or none at $-\infty$. Since the change of variable $t \to -t$ sends τ into itself, if one of these possibilities holds, the other holds also. Thus τ has no boundary values, so that by

Lemma XII.4.21 the deficiency indices of τ are $(0, 0)$. Nevertheless, the equation $\tau\sigma = \sigma$ has the non-zero square-integrable solution f_2.

Taking together Lemmas 7, 9, and Corollary 8, we obtain the following theorem, which shows the extent to which the spectrum of a self adjoint operator derived from a formal differential operator depends on the boundary conditions involved.

10 THEOREM. *Let τ be a formally self adjoint formal differential operator defined on an interval I with at least one fixed end point. Let λ be an arbitrary real point not belonging to the essential spectrum of τ. Then the deficiency indices of τ are both equal to an integer k and*

(a) *for every self adjoint extension T of $T_0(\tau)$, the dimension of the null-space $\{f|Tf = \lambda f\}$ is at most k;*

(b) *there exist self adjoint extensions T of $T_0(\tau)$ such that $\lambda \notin \sigma(T)$;*

(c) *there exist self adjoint extensions T of $T_0(\tau)$ such that $\{f|Tf = \lambda f\}$ has any preassigned dimension between 1 and k.*

PROOF. The equality of the deficiency indices is given by Theorem 6. By Corollary 8 and Lemma 9, the equation $T_1(\tau)f = \lambda f$ has exactly k linearly independent solutions for every λ not in $\sigma_e(\tau)$. If T is a self adjoint extension of $T_0(\tau)$, then $T = T^* \subseteq T_0(\tau)^* = T_1(\tau)$ by Theorem 2.10. Thus (a) is evident.

In proving (b) and (c), we may assume without loss of generality (cf. XII.1.6(c) and XII.4.19) that $\lambda = 0$. Let $\mathfrak{N} = \{f|T_1(\tau)f = 0\}$. Let $\overline{T}_0$ be the closure of $T_0(\tau)$. It was shown in the course of the proof of Theorem 6 that the restriction T_2 of $T_1(\tau)$ to $\mathfrak{D}(\overline{T}_0)+\mathfrak{N}$ is self adjoint. Thus by Theorem XII.4.12(b), $\mathfrak{D}(\overline{T}_0)+\mathfrak{N}$ may be written in the form $\mathfrak{D}(\overline{T}_0) \oplus \Gamma$, where Γ is the graph of an isometric mapping U of the positive deficiency space $\mathfrak{D}_+$ onto the negative deficiency space $\mathfrak{D}_-$ of $\overline{T}_0$, and hence is exactly k-dimensional.

If T is a self adjoint extension of $T_0(\tau)$, then by hypothesis, 0 is not in the essential spectrum of τ, and hence by Corollary 4 and Theorem 5, the number 0 is either not in the spectrum of T, or is an isolated point of $\sigma(T)$. In this last case it follows from Theorem XII.2.9(b) that $\mathfrak{N}(T) = \{x|Tx = 0\}$ is non vacuous.

We shall complete the proof of the theorem by constructing self adjoint extensions T of $T_0(\tau)$ such that the dimension j of $\mathfrak{N}(T)$ takes on every value between zero and k. For j between zero and k, choose a

j-dimensional subspace $\mathfrak{S}_j$ of $\mathfrak{D}_+$, and let $\mathfrak{Q}_j$ be its orthocomplement in $\mathfrak{D}_+$. Define an isometric mapping U_j of $\mathfrak{D}_+$ onto $\mathfrak{D}_-$ as follows:

$$U_j x = Ux, \qquad x \in \mathfrak{S}_j,$$
$$U_j x = -Ux, \qquad x \in \mathfrak{Q}_j.$$

Let Γ_j be the graph of U_j: By Theorem XII.4.12(b), $\mathfrak{D}(T_0) \oplus \Gamma_j$ is the domain of a self adjoint extension T_j of $T_0(\tau)$. We shall prove that the dimension of

$$\mathfrak{N}(T_j) = \mathfrak{N} \cap \mathfrak{D}(T_j)$$

is exactly j.

Since $\mathfrak{D}(T_0)+\mathfrak{N} = \mathfrak{D}(T_0) \oplus \Gamma \subseteq \mathfrak{D}(T_0) \oplus \mathfrak{D}_+ \oplus \mathfrak{D}_-$, each x in $\mathfrak{D}(T_2)$ can be written uniquely in either of the forms x_0+x_1 or $x_0+x_++x_-$, where $x_0 \in \mathfrak{D}(T_0)$, $x_1 \in \Gamma$, $x_+ \in \mathfrak{D}_+$, $x_- \in \mathfrak{D}_-$. If $x \in \mathfrak{D}(T_0)$ then $x_1 = 0$; therefore the mapping $x \to x_1$ of $\mathfrak{N}$ into Γ maps $\mathfrak{N}$ onto all of Γ. By the two preceding lemmas, $\mathfrak{N}$ is exactly k-dimensional; because Γ is also k-dimensional, the mapping $x \to x_1$ is one-to-one. Furthermore, Γ is the graph of an isometry between $\mathfrak{D}_+$ and $\mathfrak{D}_-$. It follows that for x in $\mathfrak{N}$ the mapping $x \to x_+$ is one-to-one and onto all of $\mathfrak{D}_+$.

We have $x \in \mathfrak{D}(T_j)$ if and only if $x = x_0+x_++x_-$, where $x_- = U_j x_+$, and $x \in \mathfrak{N}$ only if $x \in \mathfrak{D}(T_2)$, that is, only if $x_- = Ux_+$. Consequently $x \in \mathfrak{D}(T_j) \cap \mathfrak{N}$ if and only if $x \in \mathfrak{N}$ and $U_j x_+ = Ux_+$, that is, if and only if $x \in \mathfrak{N}$ and $x_+ \in \mathfrak{S}_j$. Because the mapping $x \to x_+$ is one-to-one, the set of all $x \in \mathfrak{N}$ such that $x \in \mathfrak{S}_j$ is exactly j-dimensional. That is, $\mathfrak{D}(T_j) \cap \mathfrak{N}$ is exactly j-dimensional. Q.E.D.

11 THEOREM. *Let τ be a formally self adjoint differential operator of order n defined on an interval I. The following three conditions are equivalent:*

(a) *for some real λ_0, all the solutions of $\tau\sigma = \lambda_0\sigma$ lie in $L_2(I)$;*
(b) *the deficiency indices of τ are (n, n);*
(c) *for every real λ, all the solutions of $\tau\sigma = \lambda\sigma$ lie in $L_2(I)$.*

PROOF. Let a and b be the end points of I. Choose c in the interior of I; let $I' = (a, c] \cap I$ and $I'' = [c, b) \cap I$. Then all the solutions of $\tau\sigma = \lambda\sigma$ lie in $L_2(I)$ if and only if all the solutions of $\tau'\sigma = \lambda\sigma$ lie in $L_2(I')$ and all the solutions of $\tau''\sigma = \lambda\sigma$ lie in $L_2(I'')$. Since by

Corollary 2.26 the deficiency indices of τ are (n, n) if and only if the deficiency indices both of τ' and τ'' are (n, n), it is clear that without loss of generality, we may confine ourselves to the case when I has a fixed end point. In this case (a) implies (b) by Lemma 9; if (b) holds, then by Theorem 4.1, the resolvent $R(\lambda; T)$ of every self adjoint extension of $T_0(\tau)$ is compact. Then, by Theorem 4.2, the spectrum of T consists of a sequence of isolated points on the real axis, that is, the essential spectrum of T and hence (cf. Corollary 4 and Theorem 5) also that of τ, is vacuous. It follows from Lemma 7 that (b) implies (c). It is clear that (c) implies (a). Thus the proof is complete. Q.E.D.

The following corollary is contained in the preceding proof:

12 COROLLARY. *If the deficiency indices of τ are (n, n), then the essential spectrum of τ is vacuous.*

The next theorem gives a useful extension of Theorem 5.4.

13 THEOREM. *Let τ be a formally self adjoint formal differential operator on an interval I with end points a, b, and let T be a self adjoint extension of $T_0(\tau)$. Let U be an ordered representation of $L_2(I)$ relative to T, and let μ_i, e_i, m, W_i, etc., be defined as in Theorem 5.1. Let $a < c < b$, and let τ_1 and τ_2 denote the restrictions of τ to $I \cap [a, c]$ and $I \cap [c, b]$ respectively. Suppose that Λ is an interval of the real axis such that $\Lambda\sigma_e(\tau_1) = \phi$. Then for μ_i-almost all λ in Λ,*

$$W_i(\cdot, \lambda) \in L_2[a, c], \qquad i = 1, \ldots, m.$$

Moreover, if $B(f) = 0$ is a boundary condition at a satisfied by all f in $\mathfrak{D}(T)$, then, for μ_i-almost all λ in Λ, $B(W_i(\cdot, \lambda)) = 0$, $i = 1, \ldots, m$. If $\Lambda\sigma_e(\tau_2) = \phi$, similar remarks may be made about the behavior of the kernels W_i at the end point b.

REMARK. If both $\Lambda\sigma_e(\tau_1)$ and $\Lambda\sigma_e(\tau_2)$ are void, then $\Lambda\sigma_e(\tau)$ is void, and it follows from Theorem 6.18 that $W_i(\cdot, \lambda) \in L_2(a, b)$ μ_i-almost everywhere in Λ. The proof of Theorem 5.4 will then apply with evident slight modifications to show that if $B(f) = 0$ is a boundary condition satisfied by all $f \in \mathfrak{D}(T)$, we have $B(W_i(\cdot, \lambda)) = 0$ μ_i-almost everywhere in Λ. Consequently, in this case we have $W_i(\cdot, \lambda) \in \mathfrak{D}(T)$ μ_i-almost everywhere in λ. Of course, in case $\Lambda\sigma_e(\tau)$ is void it follows from Theorem 5 and Corollary 4 that $\Lambda\sigma(T)$ is a set of isolated points,

so that we are dealing with an isolated subset of the point spectrum of T.

PROOF. Once it is established that $W_i(\cdot, \lambda) \in L_2(a, c)$ for μ_{ii}-almost all $\lambda \in \Lambda$, the proof of Theorem 5.4 will apply word for word, and will yield the second assertion of our theorem. Since $\sigma_e(\tau_1)$ is closed by Theorem 5 and Corollary 4, we may suppose without loss of generality that Λ is open. It is then sufficient to show that each $\lambda \in \Lambda$ has a neighborhood Λ_0 such that $W_i(\cdot, \lambda) \in L_2(a, c)$ for μ_{ii}-almost all $\lambda \in \Lambda_0$, since Λ may then be written as a countable union of such neighborhoods Λ_0. We shall show below that for each $\lambda \in \Lambda$ there exists a neighborhood Λ_0 of λ, an integer k, and a basis $\sigma_1(\tau, \lambda), \ldots, \sigma_n(\tau, \lambda)$ for the set of solutions of $\tau\sigma = \lambda\sigma$, such that

(a) there exists in the complex λ-plane a neighborhood U of Λ_0 such that $\sigma_1(\cdot, \lambda), \ldots, \sigma_n(\cdot, \lambda)$ are analytic for $\lambda \in U$;

(b) for each $\lambda \in U$, $\sigma_1(\cdot, \lambda), \ldots, \sigma_k(\cdot, \lambda)$ all lie in $L_2(a, c)$;

(c) for no $\lambda \in U$ does there exist a non-trivial linear combination of $\sigma_{k+1}(\cdot, \lambda), \ldots, \sigma_n(\cdot, \lambda)$ which is in $L_2(a, c)$.

Using this information, the theorem follows from Corollary 5.28, Hence it is sufficient to verify the existence of a basis $\sigma_1, \ldots, \sigma_n$ satisfying (a), (b), and (c). This may be done as follows. By Theorem 9, the deficiency indices of τ_1 are both equal to an integer k, and for all $\lambda_0 \in \Lambda$ the equation $\tau_1\sigma = \lambda_0\sigma$ has exactly k linearly independent square-integrable solutions. For each $\lambda_0 \in \Lambda$, Theorem 10 guarantees the existence of a self adjoint extension $\hat{T}$ of $T_0(\tau)$ such that $\lambda_0 \notin \sigma(T)$. Then since the resolvent set $\rho(\hat{T})$ is open, a neighborhood U_1 of λ is included in $\rho(\hat{T})$. We shall also suppose that the intersection of U_1 with the real axis is included in Λ. For $\mu \in U_1$ let $A(\mu)$ denote the everywhere defined bounded operator $(\hat{T}-\mu I)^{-1}$. Writing $K(\alpha, \beta) = (\hat{T}-\alpha I)A(\beta)$, we have $K(\alpha, \alpha) = I$ and $K(\alpha, \beta)K(\beta, \gamma) = K(\alpha, \gamma)$. Moreover, $K(\alpha, \beta) = (\hat{T}-\beta I)A(\beta)+(\beta-\alpha)A(\beta) = I+(\beta-\alpha)A(\beta)$. Thus $K(\alpha, \beta)$ is bounded and depends analytically on α and β for $\alpha, \beta \in U_1$. Let $\mathfrak{N}_\alpha = \{f \mid T_0(\tau_1)^*f = \alpha f\}$ for $\alpha \in U_1$. Then if $f \in \mathfrak{N}_\alpha$, we have

$$
\begin{aligned}
(T_1(\tau_1)-\beta I)K(\alpha, \beta)f &= (T_1(\tau_1)-\beta I)f+(\beta-\alpha)(T_1(\tau_1)-\beta I)A(\beta)f \\
&= (T_1(\tau_1)-\beta I)f+(\beta-\alpha)f \\
&= (T_1(\tau_1)-\alpha I)f = 0.
\end{aligned}
$$

Thus $K(\alpha, \beta)$ defines a one-to-one map of $\mathfrak{N}_\alpha$ into $\mathfrak{N}_\beta$. In the same way, the inverse $K(\beta, \alpha)$ of $K(\alpha, \beta)$ defines a one-to-one map of $\mathfrak{N}_\beta$ onto $\mathfrak{N}_\alpha$. Thus, $K(\alpha, \beta)$ defines a one-to-one map of $\mathfrak{N}_\beta$ onto $\mathfrak{N}_\alpha$. Let $\hat{\sigma}_1(\cdot, \lambda_0), \ldots, \hat{\sigma}_n(\cdot, \lambda_0)$ be a basis for the space Σ of solutions of $\tau\sigma = \lambda_0\sigma$, and suppose this to be chosen in such a way that $\hat{\sigma}_1, \ldots, \hat{\sigma}_k$ is a basis for the k-dimensional subspace $\mathfrak{N}_{\lambda_0} = L_2(a, c) \cap \Sigma$ of Σ. Put

$$\sigma_i(\cdot, \lambda) = K(\lambda, \lambda_0)\hat{\sigma}_i(\cdot, \lambda_0), \qquad \lambda \in U_1, \qquad i \leqq k,$$

and determine $\sigma_i(\cdot, \lambda)$ for $\lambda \in U_1$ and $i > k$ by the equations

$$\tau_1\sigma_i = \lambda\sigma_i, \quad \sigma_i^{(j)}(c, \lambda) = \hat{\sigma}_i^{(j)}(c, \lambda_0), \quad i > k, \quad \lambda \in U_1, \quad j = 0, \ldots, n.$$

Then, by Corollary 1.5, $\sigma_i(t, \lambda)$ and its first $n-1$ derivatives depend continuously on t and analytically on λ for $t \in (a, c]$ and $\lambda \in U_1$. Of $\sigma_i(\cdot, \lambda)$ for $i \leq k$ we know to begin with only that $\tau_1\sigma_i(\cdot, \lambda) = \lambda\sigma_i(\cdot, \lambda)$ and that $\sigma_i(\cdot, \lambda)$ varies analytically with λ when regarded as a vector in $L_2(a, c)$. But, by Lemma 2.16 it follows immediately that $\sigma_i(\cdot, \lambda)$, $i \leq k$ and the first $n-1$ derivatives of these functions also depend continuously on t and analytically on λ for $\lambda \in U_1$. Therefore the Wronskian determinant

$$\mathscr{W}(\lambda) = \det \begin{vmatrix} \sigma_1(c, \lambda) & \sigma_1'(c, \lambda) & & \sigma_1^{(n-1)}(c, \lambda) \\ \sigma_2(c, \lambda) & \sigma_2'(c, \lambda) & & \sigma_2^{(n-1)}(c, \lambda) \\ \vdots & \cdots & & \cdots \\ \sigma_n(c, \lambda) & \sigma_n'(c, \lambda) & & \sigma_n^{(n-1)}(c, \lambda) \end{vmatrix}$$

is analytic for $\lambda \in U_1$. Since the n vectors $[\sigma_i(c, \lambda), \ldots, \sigma_i^{(n-1)}(c, \lambda)]$, $i = 1, \ldots, n$, are linearly independent for $\lambda = \lambda_0$ by Theorem 1.3, it follows that $W(\lambda_0) \neq 0$, and hence there exists a neighborhood $U \subsetneqq U_1$ of λ_0 such that $\mathscr{W}(\lambda) \neq 0$ for $\lambda \in U$. But then the n vectors $[\sigma_i(c, \lambda), \ldots, \sigma_i^{(n-1)}(c, \lambda)]$, $i = 1, \ldots, n$, are linearly independent for $\lambda \in U$, and consequently $\sigma_1(\cdot, \lambda), \ldots, \sigma_n(\cdot, \lambda)$ are linearly independent for $\lambda \in U$. Since for each $\lambda \in U$, $\sigma_1(\cdot, \lambda), \ldots \sigma_k(\cdot, \lambda)$ form a basis for the set of solutions of $\tau\sigma = \lambda\sigma$ lying in $L_2(a, c)$, it follows that if we put $\Lambda_0 = \Lambda \cap U$, we have constructed a basis $\sigma_1(\cdot, \lambda), \ldots, \sigma_n(\cdot, \lambda)$ satisfying (a), (b), and (c), and the theorem is proved. Q.E.D.

From the theorems given above, several interesting results on the deficiency indices of second order real operators follow.

14 THEOREM. *Let $\tau = -(d/dt)p(t)(d/dt)+q(t)$ be a real formally self adjoint second order formal differential operator defined on the interval $I = [a, \infty)$. Let $p(t) > 0$ for $t \in I$, and let $q(t)$ be bounded below. Then τ has no boundary values at infinity; i.e., the deficiency indices of τ are (1,1).*

PROOF. The equivalence of the two formulations given for the above theorem follows from the discussion in Section 2; in particular it follows from Corollaries 2.14 and 2.28 that the deficiency indices of τ are either (1,1) or (2,2); we wish to exclude the latter case. Let a real λ be chosen so large that $q(t)+\lambda \geqq 1$. By Theorem 11 we have only to show that not every solution of $(\tau+\lambda)\sigma = 0$ is square-integrable. Let f be a non trivial real solution of this equation such that $f(a) = 0$. By partial integration we find

$$\int_a^s \left[f(t)\frac{d}{dt}(p(t)f'(t))\right]dt = [f(t)p(t)f'(t)]_a^s - \int_a^s p(t)[f'(t)]^2 dt$$

$$= \tfrac{1}{2}p(s)\frac{d}{ds}[f(s)]^2 - \int_a^s p(t)[f'(t)]^2 dt$$

and hence

$$\begin{aligned} 0 &= \int_a^s ((\tau+\lambda)f)(t)f(t)dt \\ &= \int_a^s [p(t)(f'(t))^2 + (q(t)+\lambda)(f(t))^2]dt - \tfrac{1}{2}p(s)\frac{d}{ds}[f(s)]^2. \end{aligned}$$

If f is square-integrable, then f^2 cannot be monotone increasing, and hence the derivative of $(f(s))^2$ must take non-positive values for large s. That is, there exists a sequence $s_n \to \infty$ for which

$$0 \geqq \tfrac{1}{2}p(s_n)[f(s_n)^2]' = \int_a^{s_n} p(t)(f'(t))^2 + (q(t)+\lambda)f(t)^2 dt.$$

Thus

$$\int_a^\infty p(t)(f'(t))^2 + (q(t)+\lambda)f(t)^2 dt \leqq 0.$$

But both terms in the integral are non-negative; therefore it follows that

$$\int_0^\infty (f(t))^2 dt \leqq \int_0^\infty (q(t)+\lambda)(f(t))^2 dt \leqq 0$$

and therefore f vanishes identically. This contradiction completes the proof. Q.E.D.

15 COROLLARY. *Let $\tau = -(d/dt)p(t)(d/dt)+q(t)$ be any real formally self adjoint second order formal differential operator defined on the interval $(-\infty, +\infty)$. Let $p(t) > 0$ for $t \in I$, and let $q(t)$ be bounded below. Then τ has no boundary values either at $+\infty$ or $-\infty$; i.e., the deficiency indices of τ are $(0, 0)$.*

PROOF. This follows from Theorem 14 and Corollary 2.21. Q.E.D.

An improvement of the argument used in the proof of Theorem 14 yields an extension of this result.

16 THEOREM. *Let $\tau = -(d/dt)p(t)(d/dt)+q(t)$ be a real formally self adjoint formal differential operator defined on the interval $I = [a, \infty)$. Let $p(t) > 0$ for $t \in I$. Suppose that there exists a positive continuously differentiable function M defined in I such that*

(a) *$p(t)^{1/2}M'(t)M(t)^{-3/2}$ is bounded above;*

(b) *$\int_a^\infty (p(t)M(t))^{-1/2}dt = \infty$;*

(c) *$q(t)M(t)^{-1}$ is bounded below.*

Then τ has no boundary values at infinity; i.e., the deficiency indices of τ are $(1, 1)$.

PROOF. As in the proof of Theorem 14, it suffices to show that not every real solution of $\tau\sigma = 0$ can lie in $L_2(I)$. We shall proceed by contradiction.

Let f_1 be a real solution of $\tau\sigma = 0$ which satisfies the boundary condition $f_1(a) = 0$, $f_1'(a) = 1$, and let f_2 be a real solution of $\tau\sigma = 0$ which satisfies the boundary condition $f_2(a) = 1/p(a)$, $f_2'(a) = 0$.

Then, integrating by parts, we have

$$0 = \int_a^s \tau f_1(t)\frac{f_1(t)}{M(t)}dt = -\tfrac{1}{2}p(s)M(s)^{-1}(f_1^2(s))' + \int_a^s p(t)(f_1'(t))^2M(t)^{-1}dt$$

$$+\int_a^s q(t)(M(t))^{-1}(f_1(t))^2dt$$

$$-\int_a^s p(t)f_1'(t)f_1(t)M'(t)M(t)^{-2}dt.$$

Let $-k_1$ be a lower bound for $q(t)M(t)^{-1}$, where $k_1 > 0$, and let $k_2 > 0$ be an upper bound for $p(t)^{1/2}M'(t)M(t)^{-3/2}$. Then we have

$$0 \geqq -\tfrac{1}{2}p(s)M(s)^{-1}(f_1^2(s))' - k_1\int_a^s (f_1(t))^2 dt$$
$$- k_2\int_a^s p(t)^{1/2}M(t)^{-1/2}|f_1'(t)f_1(t)|dt$$
$$+ \int_a^s p(t)M(t)^{-1}(f_1'(t))^2 dt.$$

Now since f_1 is in $L_2(I)$ the derivative $(f_1^2(s))'$ cannot be positive for all sufficiently large s and so there is a sequence $\{s_n\}$ approaching ∞ such that $(f_1^2(s_n))' \leqq 0$ for all n. Thus, applying Schwarz's inequality to the third term of the formula displayed above we find

$$k_1\int_a^\infty (f_1(t))^2 dt + k_2\left\{\int_a^{s_n} p(t)M(t)^{-1}(f_1'(t))^2 dt\right\}^{1/2}\left\{\int_a^\infty (f_1(t))^2 dt\right\}^{1/2}$$
$$\geqq \int_a^{s_n} p(t)M(t)^{-1}(f_1'(t))^2 dt.$$

Letting $n \to \infty$, it follows that

$$\int_a^\infty p(t)M(t)^{-1}(f_1'(t))^2 dt < \infty.$$

In the same way it may be shown that

$$\int_a^\infty p(t)M(t)^{-1}(f_2'(t))^2 dt < \infty.$$

We have

$$p(t)(f_1'(t)f_2(t) - f_1(t)f_2'(t)) = 1,$$

since this is evident for $t = a$, while the derivative of the left hand side is clearly zero. Thus

$$(p(t)^{1/2}M(t)^{-1/2}f_1'(t))f_2(t) - (p(t)^{1/2}M(t)^{-1/2}f_2'(t))f_1(t)$$
$$= (p(t)M(t))^{-1/2}.$$

Since each of the four factors on the left is in $L_2(I)$, it follows that $(p(t)M(t))^{-1/2} \in L_1(I)$. But this contradicts hypothesis (b). Q.E.D.

17 Corollary. *Let $\tau = -(d/dt)^2+q(t)$ be a real formally self adjoint formal differential operator defined on the interval $I = [a, \infty)$. Let $t^{-2}q(t)$ be bounded below in the neighborhood of ∞. Then τ has no boundary values at infinity.*

PROOF. To prove the corollary it suffices to take $M(t) = t^2$ in Theorem 16. Q.E.D.

REMARK. The factor t^{-2} in the preceding theorem can be replaced by $(t \log t)^{-2}$, or $(t \log t \log \log t)^{-2}$, etc. Theorem 16 and Corollary 17 have "two-sided" consequences analogous to Corollary 15; we leave the formulation of these consequences to the reader.

The theorems stated above cover the cases in which $q(t)$ is bounded below at infinity, and also certain cases in which $q(t)$ approaches $-\infty$ as $t \to \infty$, but not "too rapidly." We now give a theorem which shows what happens when $q(t)$ approaches $-\infty$ rapidly.

18 LEMMA. *Suppose q is integrable in an interval of the form (a, b) $(-\infty < b \leqq \infty)$. Then the equation*

$$f''(t)+f(t)+q(t)f(t) = 0$$

has two solutions which in the neighborhood of b have the forms

$$e^{it}+o(1) \text{ and } e^{-it}+o(1).$$

PROOF. Let x be chosen so large that

$$\int_x^b |q(t)|dt < 1.$$

Consider the B-space $CB[x, b)$ of all bounded continuous functions defined on $[x, b)$, with norm $|f| = \text{lub}_{x \leq t < b}|f(t)|$. In this space consider the following linear transformation:

$$(Mf)(t) = \int_t^b \sin (s-t)q(s)f(s)ds.$$

Since $|(Mf)(t)| \leqq |f| \int_t^b |q(s)|ds$, we have $|M| < 1$. Moreover, $(Mf)(t) = o(1)$ as $t \to b$ for each $f \in CB(x, b)$. By Lemma VII.3.4, the map $I+M$ has an inverse, so that, in particular, the equation

$$e^{\pm it} = f(t)+(Mf)(t) = f(t) + \int_t^b \sin (s-t)q(s)f(s)ds$$

has a unique solution f in $CB[x, b)$. Since $(Mf)(t) = o(1)$, it is clear that $f(t) = e^{it}+o(1)$ as $t \to \infty$. Differentiating the above equation we find

$$\pm ie^{\pm it} = f'(t) - \int_t^b \cos (s-t)q(s)f(s)ds,$$

$$e^{\pm it} = f''(t)+q(t)f(t) - \int_t^b \sin (s-t)q(s)f(s)ds$$

whence

$$f''(t)-q(t)f(t)+f(t) = 0.$$

Q.E.D.

19 Corollary. *Suppose that the function $b(\cdot)$ is such that*

$$\int_a^b |2b'(t)+(b(t))^2|dt < \infty \quad (-\infty < a < b \leqq \infty).$$

Then the equation

$$f''(t)+b(t)f'(t)+f(t) = 0$$

has two solutions in the neighborhood of b, of the forms

$$f(t) = e^{it}\left[\exp\left(-\tfrac{1}{2}\int_a^t b(s)ds\right)\right](1+o(1)),$$
$$f(t) = e^{-it}\left[\exp\left(-\tfrac{1}{2}\int_a^t b(s)ds\right)\right](1+o(1))$$

respectively.

Proof. Let f be a solution of the above equation. Write f in the form

$$f(t) = g(t)\exp\left(-\tfrac{1}{2}\int_a^t b(s)ds\right).$$

Then

$$f'(t) = \left[\exp\left(-\tfrac{1}{2}\int_a^t b(s)ds\right)\right](g'(t)-\tfrac{1}{2}b(t)g(t))$$
$$f''(t) = \left[\exp\left(-\tfrac{1}{2}\int_a^t b(s)ds\right)\right](g''(t)-b(t)g'(t)$$
$$-\tfrac{1}{2}b'(t)g(t)+\tfrac{1}{4}b(t)^2g(t)).$$

Thus g satisfies the differential equation

$$g''(t)-(\tfrac{1}{2}b'(t)+\tfrac{1}{4}b(t)^2)g(t)+g(t) = 0,$$

so that the conclusion follows immediately from the preceding lemma. Q.E.D.

20 Theorem. *Let $\tau = -(d/dt)p(t)(d/dt)+q(t)$ be a second order formal differential operator defined on the interval $[a, b)$ $(a < b \leq \infty)$. Assume that*

(a) $p(t) > 0$ *and* $q(t) > 0$ *for* t *sufficiently near to* b;

(b) $$\int_a^b \left| \left[\frac{(q(t)p(t))'}{q(t)^{3/2}p(t)^{1/2}} \right]' + \frac{1}{4} \frac{([q(t)p(t)]')^2}{(p(t))^{3/2}(q(t))^{5/2}} \right| dt < \infty.$$

Then

(a) *if* $\int_x^b |p(t)q(t)|^{-1/2}dt = \infty$ *for all* x,

then τ *has no boundary values at* b;

(b) *if for sufficiently large* x,

$$\int_x^b |p(t)q(t)|^{-1/2}dt < \infty,$$

then τ *has two boundary values at* b.

Proof. Using Corollary 2.21, we may pass with no essential change in the situation from the interval $[a, b)$ to any interval $[x, b)$, where $x > a$. Thus, we may assume without loss of generality that $p(t) > 0$ and $q(t) > 0$ for $t \in [a, \infty)$. This being the case, put

$$s(t) = \int_a^t q(t)^{1/2}p(t)^{-1/2}dt,$$

so that $s'(t) = q(t)^{1/2}p(t)^{-1/2}$. Write a solution f of the equation $\tau f = 0$ as $f(t) = g(s(t))$. Then

$$\begin{aligned}
&f'(t) = g'(s(t))q(t)^{1/2}p(t)^{-1/2},\\
&p(t)f'(t) = g'(s(t))(q(t)p(t))^{1/2},\\
&(p(t)f'(t))' = g''(s(t))q(t)+((q(t)p(t))^{1/2})'g'(s(t)).
\end{aligned}$$

Thus, g satisfies the equation

[*] $$g''(s)+B(s)g'(s)+g(s) = 0,$$

where

$$\begin{aligned}
B(s(t)) = b(t) &= (q(t))^{-1}[(p(t)q(t))^{1/2}]'\\
&= \tfrac{1}{2}(p(t))^{-1/2}(q(t))^{-3/2}(p(t)q(t))'.
\end{aligned}$$

The interval $[a, b)$ is transformed by the map $t \to s(t)$ into the interval $[0, c)$, where

$$c = \int_a^b (q(t))^{1/2}(p(t))^{-1/2}dt.$$

Because

$$B'(s) = (s'(t))^{-1}b'(t) = \tfrac{1}{2}[q(t)^{-1/2}p(t)^{1/2}]\left[\frac{(p(t)q(t))'}{p(t)^{1/2}q(t)^{3/2}}\right]',$$

it follows that

$$\int_0^c |2B'(s)+(B(s))^2|ds = \int_a^b |2(s'(t))^{-1}b'(t)$$
$$+\tfrac{1}{4}p(t)^{-1}q(t)^{-3}((p(t)q(t))')^2|s'(t)dt$$
$$= \int_a^b |2b'(t) + \tfrac{1}{4}[q(t)^{1/2}p(t)^{-1/2}][p(t)^{-1}q(t)^{-3}]\{(p(t)q(t))'\}^2|dt$$
$$= \int_a^b |2b'(t) + \tfrac{1}{4}[p(t)^{-3/2}q(t)^{-5/2}]\{(p(t)q(t))'\}^2|dt < \infty.$$

Thus, by hypothesis, equation [$*$] satisfies the hypothesis of the previous corollary. It follows that the equation $\tau\sigma = 0$ has two solutions of the forms

$$f_1(t) = (p(a)q(a))^{-1/4}e^{is(t)}\exp(-\tfrac{1}{2}\int_0^{s(t)} B(s)ds)(1+o(1))$$
$$= (p(a)q(a))^{-1/4}e^{is(t)}\exp(-\tfrac{1}{2}\int_a^t b(t)q(t)^{1/2}p(t)^{-1/2}dt)(1+o(1))$$
$$= (p(a)q(a))^{-1/4}e^{is(t)}\exp(-\tfrac{1}{2}\int_a^t (p(t)q(t))^{-1/2}$$
$$((p(t)q(t))^{1/2})'dt)(1+o(1))$$
$$= (p(a)q(a))^{-1/4}e^{is(t)}\exp\left(-\tfrac{1}{2}\log (p(x)q(x))^{1/2}\Big|_{x=a}^{x=t}\right)(1+o(1))$$
$$= e^{is(t)}(p(t)q(t))^{-1/4}(1+o(1))$$

and

$$f_2(t) = e^{-is(t)}(p(t)q(t))^{-1/4}(1+o(1))$$

in the neighborhood of $t = b$. If

$$\int_a^b (p(t)q(t))^{1/2}dt = \infty,$$

then neither of these solutions lies in $L_2[a, b)$. By Corollary 2.14, the deficiency indices of τ are equal, and by Corollary 2.23 and Theorem 11, they cannot be (0, 0) or (2, 2). They are therefore (1, 1). Since there are two boundary values at a, Lemma XII.4.21 shows that there cannot be any boundary values at b.

If $\int_a^b (p(t)q(t))^{-1/2} < \infty$, then both f_1 and f_2 are in $L_2[a, b)$. Furthermore, they are linearly independent, for if $f_1 = cf_2$, then $e^{2is(\cdot)}$, and hence $s(\cdot)$, would be constant in a neighborhood of b, and consequently

$$s'(t) = q(t)^{1/2}p(t)^{-1/2} = 0,$$

contrary to assumption. By Theorem 11 and 2.19 and by Lemma XII.4.22 it follows that the operator has two boundary values at b. Q.E.D.

21 Corollary. *Under the hypotheses of the theorem we have:*

(a) *If $\int_x^b \{p(t)q(t)\}^{-1/2}dt < \infty$ for some sufficiently large x, the spectrum of every self adjoint extension T of τ consists entirely of isolated points.*

(b) *If $\int_x^b \{p(t)q(t)\}^{-1/2}dt = \infty$ for every $x > a$, and q is monotone increasing, the spectrum of every self adjoint extension T of $T_0(\tau)$ is entirely continuous, and covers the whole real axis.*

Proof. (a) The notation of the preceding theorem and of its proof will be used. Again we can assume without loss of generality that p and q are positive; in case (a) the deficiency indices are (2, 2) by Theorem 11. The assertion then follows from Corollary 12.

(b) In the course of the proof of the preceding theorem it was shown that in a neighborhood of b the equation $\tau f = 0$ has solutions of the forms

$$f_1(t) = e^{is(t)}\{p(t)q(t)\}^{-1/4}(1+o(1))$$

and

$$f_2(t) = e^{-is(t)}\{p(t)q(t)\}^{-1/4}(1+o(1)),$$

where

$$s(t) = \int_a^t q(t)^{1/2}p(t)^{-1/2}dt.$$

Thus $\tau f = 0$ has solutions of the forms

$$\frac{1}{2i}(f_1(t) - f_2(t)) = \{\sin s(t)\}\{p(t)q(t)\}^{-1/4}(1+o(1)),$$

and

$$\{\cos s(t)\}\{p(t)q(t)\}^{-1/4}(1+o(1)).$$

Consequently, any non-zero real solution of $\tau f = 0$ is of the form

$$[*] \qquad k_1 \sin\big(s(t)+k\big)\big(p(t)q(t)\big)^{-1/4}\big(1+o(1)\big), \qquad k_1 \neq 0.$$

Suppose now that

$$\int_a^b \big(p(t)q(t)\big)^{-1/2}dt = \infty.$$

We shall prove that no function of the form in [*] can lie in $L_2[a, b)$. Suppose the contrary is true. Then for some constant k

$$\int_a^b \sin^2\big(s(t)+k\big)\big(p(t)q(t)\big)^{-1/2}dt < \infty.$$

Making the same change of variables as in the proof of the preceding theorem,

$$\int_0^c \sin^2\big(s(t)+k\big)\big(p(t(s))q(t(s))\big)^{-1/2}q(t(s))^{-1/2}p(t(s))^{1/2}ds$$
$$= \int_0^c \sin^2(s+k)\big(q(s)\big)^{-1}ds < \infty,$$

where the function $t(s)$ is defined by $t(s(t)) = t$. Because $t(\cdot)$ and $q(\cdot)$ are monotone increasing we have

$$\begin{aligned}\int_{\pi/2}^c \cos^2(s+k)q(t(s))^{-1}ds &= \int_{\pi/2}^c \sin^2\Big(s+k-\frac{\pi}{2}\Big)\big(q(t(s))\big)^{-1}ds \\ &\leq \int_{\pi/2}^c \sin^2\Big(s+k-\frac{\pi}{2}\Big)\Big[q\Big(t\Big(s-\frac{\pi}{2}\Big)\Big)\Big]^{-1}ds \\ &\leq \int_0^c \sin^2(s+k)[q(t(s))]^{-1}ds < \infty.\end{aligned}$$

Therefore

$$\int_a^b [p(t)q(t)]^{-1/2}dt = \int_0^c [q(t(s))]^{-1}ds$$
$$= \int_0^{\pi/2} [q(t(s))]^{-1}ds + \int_{\pi/2}^c [\sin^2(s+k)+\cos^2(s+k)][q(t(s))]^{-1}ds < \infty,$$

contrary to assumption.

It follows that $\tau f = 0$ has *no* solutions which lie in $L_2[a, b)$. Thus, 0 cannot be in the point spectrum of a symmetric extension T of $T_0(\tau)$. Now, by Corollary 8, $0 \in \sigma_e(\tau)$. Hence, by Corollary 3 and Theorem 5, $0 \in \sigma(T)$. It follows easily from Theorem XII.2.6 and

Corollary XII.2.7 that $E((0); T) = 0$, and hence that $T\mathfrak{D}(T)$ is dense. Thus λ is in the continuous spectrum of T. Since, for each real λ, $q(t)+\lambda$ satisfies the same hypotheses as $q(t)$, every real λ is in the continuous spectrum of T. Q.E.D.

The special case $p = 1$ of Theorem 20 yields the following corollary.

22 COROLLARY. *Let there be given the second order differential operator*

$$\tau = -\left(\frac{d^2}{dt^2}\right) - q(t)$$

on an interval of the form $[a, b)$, *where* $a < b \leq \infty$. *Assume that*

(a) $q(t)$ *is positive for* t *sufficiently near* b, *and*

(b) *for* x *sufficiently near* b,

$$\int_x^b \left| \left[\frac{(q(t))'}{(q(t))^{3/2}}\right]' + \frac{1}{4}\frac{[(q(t))']^2}{(q(t))^{5/2}} \right| dt < \infty$$

We conclude that

(a) *if for all* x,

$$\int_x^b |q(t)|^{-1/2} dt = \infty,$$

then τ *has no boundary values at* b;

(b) *if for* x *sufficiently near* b,

$$\int_x^b |q(t)|^{-1/2} dt < \infty,$$

then τ *has two boundary values at* b.

The theorems developed so far give information as to the existence of boundary values of a differential operator τ in an interval of the form $[a, b)$, where b is finite or infinite. The following set of theorems will consider the cases in which the interval is of the form $(a, b]$, where a is known to be finite. Without loss of generality it can be assumed that $a = 0$.

23 THEOREM. *Let*

$$\tau = -\left(\frac{d^2}{dt^2}\right) + q(t)$$

be a real self adjoint second order formal differential operator defined on an interval $I = (0, b]$. *Then*

(a) *if* $\liminf_{t\to 0} t^2 q(t) > 3/4$, τ *has no boundary values at zero;*

(b) *if* $\limsup_{t\to 0} |t^2 q(t)| < 3/4$, τ *has two boundary values at zero.*

PROOF. (a) Using Corollary 2.21, we can pass without any essential change in the situation to the consideration of any interval $(0, b_1]$, $0 < b_1 < b$. Thus, we may and shall assume without loss of generality that $q(t) \geqq (3/4)t^{-2}$ for $t \in I$. Let f be the unique solution of $\tau\sigma = 0$ satisfying the boundary conditions $f(b) = 0$, $f'(b) = -2$, and let f_1 be the unique solution of the equation

$$[*] \qquad \sigma'' - (3/4)t^{-2}\sigma = 0$$

satisfying the boundary conditions $f_1(b) = 0$, $f_1'(b) = -1$. The general solution of [*] is of the form

$$\sigma(t) = at^{-1/2} + bt^{3/2}.$$

Therefore

$$f_1(t) = \tfrac{1}{2}(b^{3/2}t^{-1/2} - b^{-1/2}t^{3/2}),$$

and hence f_1 is not square-integrable in $(0, b]$. We shall prove that f_1 is positive and $f(t) > f_1(t)$ for all t in the interval $(0, b)$ so that f is not square-integrable in $(0, b)$. The statement will then follow from Theorem 11 and the fact that, since by Corollary 2.14 the deficiency indices of τ are equal, there are by Lemma XII.4.21 an even number of boundary values at zero. The function f_1 is positive in $(0, b)$; indeed, suppose it had a second zero at c. By a partial integration, we find

$$0 = \int_c^b [-f_1''(t) + (3/4)t^{-2}f_1(t)]f_1(t)\,dt$$

$$= -[f_1'(t)f_1(t)]_c^b + \int_c^b \{[f_1'(t)]^2 + \tfrac{3}{4}t^{-2}[f_1(t)]^2\}\,dt > 0,$$

a contradiction.

The boundary conditions imposed on f and on f_1 imply that

$$[\dagger] \qquad f_1'(t) > f'(t), \qquad f(t) > f_1(t),$$

in an interval of the form (c_0, b), where $0 \leqq c_0 < b$. The proof will be completed by showing that $c_0 = 0$. Let c_0 be the largest

number for which [†] fails in $[c_0, b)$. If $c_0 \neq 0$, then either $f_1(c_0) = f(c_0)$ or $f_1'(c_0) = f'(c_0)$. Now,

$$\begin{aligned} f_1'(c_0) &= -1-\int_{c_0}^{b} f_1''(s)ds = -1-\int_{c_0}^{b} (3/4)s^{-2}f_1(s)ds \\ &> -1-\int_{c_0}^{b} (3/4)s^{-2}f(s)ds \geq -1-\int_{c_0}^{b} q(s)f(s)ds \\ &= -1-\int_{c_0}^{b} f''(s)ds = 1+f'(c_0) > f'(c_0), \end{aligned}$$

and hence $f_1'(c_0) > f'(c_0)$. We therefore necessarily have $f(c_0) = f_1(c_0)$. But this is impossible, because

$$f(c_0) = -\int_{c_0}^{b} f'(t)dt > -\int_{c_0}^{b} f_1'(t)dt = f_1(c_0).$$

We conclude that $c_0 = 0$ and that $f(t) > f_1(t)$ throughout the interval $(0, b)$. Because f_1 is positive, and not square-integrable, f cannot be square-integrable.

(b) By Theorem 11, it will suffice to show that every solution of the equation $\tau\sigma = 0$ is square-integrable. Let f be a real solution of this equation. Let f_2 be a solution of the equation

$$[**] \qquad \sigma'' - kt^{-2}\sigma = 0 \qquad (0 < k < \tfrac{3}{4}).$$

Let f_2 be subjected to the boundary conditions

$$f_2(b) = |f(b)|+1; \; f_2'(b) = -|f'(b)|-1.$$

Every solution of equation [**] is of the form $at^{e_1}+bt^{e_2}$, where

$$e_1 = \tfrac{1}{2} - (\tfrac{1}{4}+k)^{1/2} > -\tfrac{1}{2}$$

and

$$e_2 = \tfrac{1}{2}+(\tfrac{1}{4}+k)^{1/2} > 0.$$

Therefore f_2 is square-integrable. It is evident that f_2 is positive, and convex upwards.

It can again be assumed without loss of generality that for some constant k such that $0 < k < 3/4$,

$$0 \leq |q(t)| < kt^{-2}, \qquad 0 < t \leq b.$$

From the boundary conditions it follows that $f_2(t) > |f(t)|$ and $f_2'(t) < -|f'(t)|$ for t in a neighborhood of b. Let c be the smallest real

number in $(0, b)$ such that these two inequalities hold throughout the interval $(c, b]$. Then,

$$
\begin{aligned}
f_2'(c) &= -|f'(b)| - 1 - \int_c^b f_2''(t)dt = -|f'(b)| - 1 - \int_c^b kt^{-2}f_2(t)dt \\
&< -|f'(b)| - 1 - \int_c^b |q(t)f(t)|dt \\
&= -|f'(b)| - 1 - \int_c^b f''(t)dt = -|f'(b)| - 1 - |f'(b) - f'(c)| \\
&< -|f'(c)|.
\end{aligned}
$$

Hence $f_2(c) = |f(c)|$. But this is impossible, because

$$
\begin{aligned}
f_2(c) = |f(b)| + 1 - \int_c^b f_2'(t)dt &> |f(b)|+1+\int_c^b |f'(t)|dt \\
&\geq |f(b)|+1+\left|\int_c^b f'(t)dt\right| \\
&= |f(b)|+1+|f(b) - f(c)| \\
&> |f(c)|.
\end{aligned}
$$

Q.E.D.

The preceding theorem covers most cases in which q is positive and also certain cases in which $q(t) \to -\infty$ as $t \to 0$, but not "too rapidly". The next result covers most cases in which $q(t) \to -\infty$ rather rapidly.

24 THEOREM. *Let $\tau = -(d/dt)^2+q(t)$ be a real self adjoint formal differential operator of second order defined on an interval $I = (0, b]$. Then, if $q(t)$ is monotone increasing, τ has two boundary values at zero.*

The proof will be preceded by three lemmas.

25 LEMMA. *In the closed interval $[a, b]$, let the functions f_1 and f_2 satisfy the equations $f_1'' = -q_1 f_1$ and $f_2'' = -q_2 f_2$. Assume that*

(a) $f_1(t) \geq 0$ *for* $a < t \leqq b$,

(b) $q_1(t) \geq q_2(t) \geq 0$, $a \leq t \leq b$,

(c) $f_1(a) = f_2(a)$,

(d) $f_1'(a) = f_2'(a)$.

Then $f_1(t) \leq f_2(t)$ for $a \leqq t \leqq b$.

PROOF. Suppose first that $q_1(t) > q_2(t) \geq 0$ for $a \leqq t \leqq b$.

(a) We have $f_1(t) \leqq f_2(t)$ for t sufficiently near (but not equal to) a. Indeed,

$$f_1''(a) = -q_1(a)f_1(a) < -q_2(a)f_2(a) = f_2''(a);$$

that is, for t sufficiently near a,

$$\frac{f_1'(t)-f_1'(a)}{t-a} \leqq \frac{f_2'(t)-f_2'(a)}{t-a},$$

and hence $f_1'(t) \leqq f_2'(t)$. Since $f_1(a) = f_2(a)$, it clearly follows that $f_1(t) \leqq f_2(t)$.

(b) Let c be the point farthest from a such that for $a \leqq t \leqq c$ we have $f_1(t) \leqq f_2(t)$. Thus for $t > c$ and t sufficiently near to c, $f_1(t) \geqq f_2(t)$; that is, the graph of f_1 intersects the graph of f_2 at c. Hence $f_1'(c) \geqq f_2'(c)$. By a partial integration we find

$$\begin{aligned} 0 &= \int_a^c [f_2''(t)+q_2(t)f_2(t)]f_1(t)dt \\ &= \int_a^c [f_2(t)f_1''(t)+q_2(t)f_2(t)f_1(t)]dt \\ &\quad + [f_2'(t)f_1(t) - f_1'(t)f_2(t)]_a^c \\ &= \int_a^c f_1(t)f_2(t)[q_2(t)-q_1(t)]dt \\ &\quad + f_1(c)[f_2'(c)-f_1'(c)] < 0, \end{aligned}$$

because $f_1(t)f_2(t) > 0$ and $q_2(t) - q_1(t) < 0$ in the first term, and $f_2'(c) - f_1'(c) \leqq 0$ and $f_1(c) \geqq 0$ in the second term. This contradiction establishes the theorem in the special case when $q_1(t) > q_2(t) \geqq 0$ for $a \leqq t \leqq b$. The general case when $q_1(t) \geqq q_2(t) > 0$ for $a \leqq t < b$ now follows from this by an evident limiting argument the details of which we leave to the reader. Q.E.D.

26 COROLLARY. *Let q be continuous, negative and monotone decreasing in a finite interval $[a, b)$. Then every real solution f of the equation $f'' = qf$ is uniformly bounded.*

PROOF. We need only consider solutions not identically zero. Two cases arise: that in which f has a finite number of zeros in $[a, b)$, and that in which f has an infinite number of zeros in $[a, b)$. In the first case, we may assume without loss of generality that f has no zeros in $[a, b)$, and hence may assume without loss of generality that f

is positive in $[a, b)$. Since q is negative, f is convex downward in $[a, b)$. Thus, if g_1 is the linear function defined by $g(t) = f(a)+f'(a)(t-a)$ it follows that $g \geqq f$, from which the boundedness of f is evident.

Suppose now that f has an infinite number of zeros in $[a, b)$. If c is an accumulation point of the zeros and $a \leqq c < b$ then $f(c) = f'(c) = 0$ which implies that $f(t) = 0$ for all t. Thus, only the case in which f has an infinite increasing sequence $s_1, s_2, \ldots$ of zeros in $[a, b)$ need be considered. If $f(t) > 0$ between s_i and s_{i+1}, then f is convex downwards between s_i and s_{i+1}, so that f has a single maximum at $t = m_i$ between s_i and s_{i+1}. Moreover, since f is not identically zero, $f'(s_{i+1}) \neq 0$. Since $f(t) > 0$ for $s_i < t < s_{i+1}$, $f'(s_{i+1})$ is negative. Thus f is negative between s_{i+1} and s_{i+2}, positive between s_{i+2} and s_{i+3}, etc. Clearly, f has a single minimum $f(m_{i+1})$ between s_{i+1} and s_{i+2}, a single maximum $f(m_{i+2})$ between s_{i+2} and s_{i+3}, etc. We will show that $|f(m_i)| \geq |f(m_{i+1})| \geqq |f(m_{i+2})| \geqq \ldots$, which will clearly establish the desired result. On the interval $[s_{i+1}, m_{i+1}]$, consider the two functions $-f(t)$ and $f_1(t) = +f(2s_{i+1} - t)$. We have $(-f)'' = q(-f)$, $f_1'' = q_1 f_1$, where $q_1(t) = q(2s_{i+1} - t) \geq q(t)$, since q is monotone decreasing. By the preceding lemma, $-f(t) < f_1(t)$ in $[s_{i+1}, m_{i+1}]$. In particular $-f(m_{i+1}) = |f(m_{i+1})| \leqq f_1(m_{i+1})$. Since $0 \leq -f(t) \leq f_1(t) = f(2s_{i+1} - t)$ for $t \in [s_{i+1}, m_{i+1}]$, none of the points $2s_{i+1} - t$ can lie in the interval $[s_{i-1}, s_i]$ (where $f(t)$ is negative). Thus, $2s_{i+1} - m_{i+1} \in [s_i, s_{i+1}]$, so that $f_1(m_{i+1}) = f(2s_{i+1} - m_{i+1}) < f(m_i)$. Q.E.D.

27 Corollary. *Let q be continuous, negative, and monotone increasing on a finite interval $(a, b]$. Then every real solution f of the equation $f'' = qf$ is uniformly bounded.*

Proof. To prove the corollary it suffices to make the change of variable $t \to -t$ in the preceding corollary. Q.E.D.

Proof of Theorem 24. If the function q of Theorem 24 is not negative for t sufficiently close to zero, then it is bounded, and Theorem 23 applies to give the desired result. If q is negative for t sufficiently close to zero, then the preceding corollary applies to give the desired result. Q.E.D.

We now wish to prove a result, similar to the theorems given above but not nearly so far-reaching, on the deficiency indices of a differential operator of order n.

28 THEOREM. *Let there be given two formally self adjoint formal differential operators τ and τ', the latter being of no greater order than the former. Assume that*

(a) $\mathfrak{D}(T_1(\tau')) \supseteq \mathfrak{D}(T_1(\tau))$.

(b) *Let A be any bounded subset in $\mathfrak{D}(T_1(\tau))$. If A is considered as a subset of $\mathfrak{H}$, then the restriction of $T_1(\tau')$ to A is a continuous mapping of A into $\mathfrak{H}$.*

Then, assuming that $\tau+\tau'$ has a non-zero leading coefficient.

(A) *the Hilbert spaces $\mathfrak{D}(T_1(\tau+\tau'))$ and $\mathfrak{D}(T_1(\tau))$ have the same elements and equivalent topologies;*

(B) *the differential operators τ and τ' have the same deficiency indices.*

PROOF. First we shall prove (A). Let f be in the domain of $T_1(\tau)$. Then by assumption (a), f is in the domain of $T_1(\tau')$, that is, both τf and $\tau' f$ are square-integrable. Hence, so is $(\tau+\tau')f$, and thus $\mathfrak{D}(T_1(\tau)) \subseteq \mathfrak{D}(T_1(\tau+\tau'))$.

The remainder of the proof is broken up into a succession of steps.

(a') The topology of the Hilbert space $\mathfrak{D}(T_1(\tau))$ is the same as its relative topology as a subspace of the Hilbert space $\mathfrak{D}(T_1(\tau+\tau'))$.

Indeed, let $\{f_n\}$ be a sequence in $\mathfrak{D}(T_1(\tau))$. Suppose that $\{f_n\}$ converges to zero in the topology of $\mathfrak{D}(T_1(\tau))$. Then, by assumption (b), $\{f_n\}$ converges to zero in the topology of $\mathfrak{D}(T_1(\tau+\tau'))$. Conversely, let $\{f_n\}$ converge to zero in the topology of $\mathfrak{D}(T_1(\tau+\tau'))$, that is, let

[*] $$|f_n| + |T_1(\tau+\tau')f_n| \to 0.$$

If $\{f_n\}$ is not bounded in $\mathfrak{D}(T_1(\tau))$, there is a subsequence $\{f_{n_i}\}$ such that $h_{n_i} = f_{n_i}/|T_1(\tau)f_{n_i}|$ converges to zero in $\mathfrak{H}$ and is bounded in $\mathfrak{D}(T_1(\tau))$. By hypothesis (b) it follows that $T_1(\tau')h_{n_i}$ converges to zero in $\mathfrak{H}$. But [*] implies that $|T_1(\tau+\tau')h_{n_i}| \to 0$, and consequently

$$1 = |T_1(\tau)h_{n_i}| \leq |T_1(\tau+\tau')h_{n_i}| + |T_1(\tau')h_{n_i}| \to 0,$$

a contradiction. Hence $\{f_n\}$ is bounded in $\mathfrak{D}(T)$ and, by [*], it converges to zero in $\mathfrak{H}$. It follows from hypothesis (b) that $T_1(\tau')f_n \to 0$. Therefore by [*],

$$|T_1(\tau)f_n| \leqq |T_1(\tau+\tau')f_n| + |T_1(\tau')f_n| \to 0,$$

which is what was to be shown.

(b′) $\mathfrak{D}(\overline{T_0(\tau)}) = \mathfrak{D}(\overline{T_0(\tau+\tau')})$.

Let $z \in \mathfrak{D}(\overline{T_0(\tau)})$, and (cf. Lemma XII.4.5(c)) let $z_n \in \mathfrak{D}(T_0(\tau))$, $z_n \to z$ in the topology of $\mathfrak{D}(T_1(\tau))$. By Definition 2.8 $\mathfrak{D}(T_0(\tau)) = \mathfrak{D}(T_0(\tau+\tau'))$. Thus, by (a'), $z_n \to z$ in the topology of $\mathfrak{D}(T_1(\tau+\tau')) \supseteq \mathfrak{D}(T_1(\tau))$. so that $z \in \mathfrak{D}(\overline{T_0(\tau+\tau')})$. Conversely, let $z \in \mathfrak{D}(\overline{T_0(\tau+\tau')})$, and let $z_n \in \mathfrak{D}(T_0(\tau+\tau'))$, and $z_n \to z$ in the topology of $\mathfrak{D}(T_1(\tau+\tau'))$. Then $\lim_{m,n\to\infty}(z_m - z_n) = 0$ in the topology of $\mathfrak{D}(T_1(\tau+\tau'))$, so that by (a), $\{z_n\}$ is a Cauchy sequence in the (complete) Hilbert space $\mathfrak{D}(T_1(\tau))$. Hence it converges to some element z_∞ in $\mathfrak{D}(T_1(\tau))$, and it is clear that $z_\infty \in \mathfrak{D}(\overline{T_0(\tau)})$. On the other hand, it follows from (a) that $z_\infty = z$. Thus, (b′) is proved.

(c′) Let $\mathfrak{D}_+$ and $\mathfrak{D}_-$ be the deficiency spaces of $T_0(\tau)$, and $\mathfrak{D}'_+$, $\mathfrak{D}'_-$ be the deficiency spaces of $T_0(\tau+\tau')$. Then

$$\dim \mathfrak{D}_+ \geqq \dim \mathfrak{D}'_+; \qquad \dim \mathfrak{D}_- \geqq \dim \mathfrak{D}'_-,$$

$\dim \mathfrak{X}$ denoting the dimension of the (finite dimensional) subspace $\mathfrak{X}$ of Hilbert space.

Suppose for the sake of definiteness that the first of these inequalities is false. For simplicity in notation, put $\mathfrak{D}_0 = \mathfrak{D}(\overline{T_0(\tau)}) \cap \mathfrak{D}(\overline{T_0(\tau+\tau')})$. Then, by Theorem XII.4.19 and by our supposition,

$$\begin{aligned}\dim \{(T_1(\tau+\tau') - \lambda I)(\mathfrak{D}_0 + \mathfrak{D}_+)\}^\perp &> \dim ((T_1(\tau+\tau') - \lambda I)\mathfrak{D}_0)^\perp \\ &\quad - \dim \{(T_1(\tau+\tau') - \lambda I)\mathfrak{D}_+\} \geqq \dim \mathfrak{D}'_+ - \dim \mathfrak{D}_+ > 0,\end{aligned}$$

for any λ such that $\mathscr{I}\lambda < 0$. Consequently it follows that for no λ such that $\mathscr{I}\lambda < 0$ is $(T_1(\tau+\tau') - \lambda I)(\mathfrak{D}_0 + \mathfrak{D}_+)$ dense in L_2. We shall obtain a contradiction by showing that $(T_1(\tau+\tau') + niI)(\mathfrak{D}_0 + \mathfrak{D}_+)$ is dense in L_2 for sufficiently large n. Note to begin with that

$$\begin{aligned}|(T_1(\tau) + \mu iI)(d_0 + d_+)|^2 &> \mu^2|d_0 + d_+|^2 + (T_1(\tau)(d_0 + d_+), \mu i(d_0 + d_+)) \\ &\qquad + (\mu i(d_0 + d_+), T_1(\tau)(d_0 + d_+)) \\ &= \mu^2|d_0 + d_+|^2 + (id_+, \mu i(d_0 + d_+)) \\ &\qquad + (\mu id_+, T_1(\tau)(d_0 + d_+)) \\ &= \mu^2|d_0 + d_+|^2 + \mu(d_+, d_0 + d_+) - \mu(d_+, d_0) \\ &\qquad + \mu|d_+|^2 \\ &\geq \mu^2|d_0 + d_+|^2, \qquad d_0 \in \mathfrak{D}_0, \qquad d_+ \in \mathfrak{D}_+,\end{aligned}$$

for μ positive. Thus

$$[\dagger] \qquad |(T_1(\tau)+\mu iI)x| \geqq \mu|x|, \qquad x \in \mathfrak{D}_0+\mathfrak{D}_+, \qquad \mu > 0.$$

Let S be the restriction of $T_1(\tau)$ to $\mathfrak{D}_0+\mathfrak{D}_+$. By Lemma XII.4.11, S is a closed operator. It is clear from [†] that

$$[\dagger\dagger] \qquad |(S+\mu iI)x| \geq \mu|x|, \qquad x \in \mathfrak{D}(S), \qquad \mu > 0.$$

Moreover, for each $\mu > 0$, the range of $S+\mu iI$ is closed. Indeed, if $z = \lim_{n\to\infty}(S+\mu iI)x_n$, then $\lim_{m,n\to\infty}|(S+\mu iI)(x_m-x_n)| = 0$, so that by [††], $\{x_n\}$ is a Cauchy sequence. If x is its limit, it is clear since S is closed that $x \in \mathfrak{D}(S)$, and $(S+\mu iI)x = z$.

Let $y \in ((S+iI)\mathfrak{D}(S))^{\perp}$. Then $y \in ((T_0(\tau)+iI)\mathfrak{D}_0)^{\perp}$; hence, by Definition XII.4.9, $y \in \mathfrak{D}_+$. However, since $(S+iI)\mathfrak{D}(S) \supseteq (S+iI)\mathfrak{D}_+ = \mathfrak{D}_+$, we have $y \in \mathfrak{D}_+^{\perp}$. Consequently, $y = 0$. This shows that $(S+iI)\mathfrak{D}(S)$ is dense, and since it is closed, it must be all of Hilbert space. Thus, from [††] is follows that $-i$ is in the resolvent set of S. Let μ_0 be the largest real number such that the whole interval $[-i, -\mu_0 i)$ of the negative imaginary axis is in the resolvent set of S. Since, by Lemma XII.1.3, the resolvent set is open, μ_0 is not in the resolvent set. Suppose $\mu_0 < \infty$, and let μ_n be a sequence of real numbers approaching μ_0 from below. By [††], $|R(-\mu_n i; S)| < \mu_n^{-1}$. It follows from Lemma XII.1.3 that $\{R(-\mu_n i; S)\}$ is a Cauchy sequence in the uniform operator topology. Let R be the limit of this Cauchy sequence. Then since

$$\lim_{n\to\infty}(S+\mu_0 iI)R(-\mu_n i; S)x = \lim_{n\to\infty}(x+(\mu_0-\mu_n)R(-\mu_n i; S)x) = x,$$

and since S is closed, it follows that $Rx \in \mathfrak{D}(S)$, and $(S+\mu_0 iI)Rx = x$. This shows that $(S+\mu_0 iI)\mathfrak{D}(S)$ is all of Hilbert space, so that $\mu_0 i$ is in the resolvent set of S, contrary to assumption. Hence we conclude that for each $n > 1$, $(S+niI)$ has an inverse R_n which by [††] has norm at most n^{-1}. Since $-ni \in \rho(S)$, R_n maps L_2 onto $\mathfrak{D}_0+\mathfrak{D}_+ \subseteq \mathfrak{D}(T_1(\tau)) \subseteq \mathfrak{D}(T_1(\tau'))$ by assumption (a).

We shall now prove that the everywhere defined operator $T_1(\tau_1')R_n$ is bounded, and that for n sufficiently large, $|T_1(\tau')R_n| < 1$. Indeed, if this is not the case, then there exists an $\varepsilon > 0$ and a sequence f_n of elements such that $|f_n| = 1$ and $|T_1(\tau')R_n| > \varepsilon$. Put

$g_n = R_n f_n$. Then since $|nR_n| \leq 1$, $|g_n| \to 0$. Moreover, $T_1(\tau)g_n = f_n - nR_n f_n$, and since $|nR_n| \leq 1$, $|T_1(\tau)g_n|$ is bounded. It follows from hypothesis (b) that $|T_1(\tau')g_n| \to 0$. This contradiction proves our assertion.

Consequently, there exists an $n \geqq 1$ such that $|T_1(\tau')R_n| < 1$. Let x be an arbitrary element of Hilbert space Then

$$(T_1(\tau) + T_1(\tau') + niI)\sum_{k=0}^{m}(-1)^k R_n(T_1(\tau')R_n)^k x = \\ (I + (-1)^m(T_1(\tau')R_n)^{m+1})x.$$

Thus, since R_n maps L_2 into $\mathfrak{D}_0 + \mathfrak{D}_+$, $(S + T_1(\tau') + niI)\mathfrak{D}(S)$ is dense in Hilbert space. This proves (c').

Now we conclude our proof as follows. By Lemma XII.4.10, the Hilbert spaces $\mathfrak{D}(T_1(\tau))$ and $\mathfrak{D}(T_1(\tau+\tau'))$ have the following orthogonal direct sum decompositions:

$$[\dagger\dagger\dagger] \quad \mathfrak{D}(T_1(\tau)) = \mathfrak{D}_0 \oplus \mathfrak{D}_+ \oplus \mathfrak{D}_-; \quad \mathfrak{D}(T_1(\tau+\tau')) = \mathfrak{D}_0 \oplus \mathfrak{D}'_+ \oplus \mathfrak{D}'_-.$$

Let P be the orthogonal projection of $\mathfrak{D}(T_1(\tau+\tau'))$ onto $\mathfrak{D}'_+ \oplus \mathfrak{D}'_-$. Since by (c')

$$\dim \mathfrak{D}_+ \oplus \mathfrak{D}_- \geqq \dim \mathfrak{D}'_+ \oplus \mathfrak{D}'_-,$$

either we must have $P(\mathfrak{D}_+ \oplus \mathfrak{D}_-) = \mathfrak{D}'_+ \oplus \mathfrak{D}'_-$, or there must exist a non-zero element y in $\mathfrak{D}_+ \oplus \mathfrak{D}_-$ such that $Py = 0$. But then we would have $y \in \mathfrak{D}_0$, which is impossible by the first equation of [†††]. Thus $P(\mathfrak{D}_+ \oplus \mathfrak{D}_-) \supseteq \mathfrak{D}'_+ \oplus \mathfrak{D}'_-$. It follows that $\mathfrak{D}_0 \oplus \mathfrak{D}_+ \oplus \mathfrak{D}_- \supseteqq \mathfrak{D}'_+ \oplus \mathfrak{D}'_-$, and hence that $\mathfrak{D}_0 \oplus \mathfrak{D}_+ \oplus \mathfrak{D}_- \supseteqq \mathfrak{D}_0 \oplus \mathfrak{D}'_+ \oplus \mathfrak{D}'_-$. That is, $\mathfrak{D}(T_1(\tau)) \supseteqq \mathfrak{D}(T_1(\tau+\tau'))$. Thus (A) is proved.

This last argument also shows that

$$\dim \mathfrak{D}_+ \oplus \mathfrak{D}_- > \dim \mathfrak{D}'_+ \oplus \mathfrak{D}'_-$$

is impossible. Hence

$$\dim \mathfrak{D}_+ + \dim \mathfrak{D}_- = \dim \mathfrak{D}'_+ + \dim \mathfrak{D}'_-.$$

Since $\dim \mathfrak{D}_\pm \geqq \dim \mathfrak{D}'_\pm$ by (c'), it follows that $\dim \mathfrak{D}_\pm = \dim \mathfrak{D}'_\pm$, proving (B). Q.E.D.

29 Corollary. *Under the hypotheses and in the notation of the preceding theorem, every boundary value for τ is a boundary value for $\tau+\tau_1$ and conversely.*

The proof is immediate from the preceding lemma and Definition 2.17 of a boundary value.

30 Corollary. *Let τ be a formal differential operator and let q be a bounded function. Then*

(a) $\mathfrak{D}(T_1(\tau)) = \mathfrak{D}(T_1(\tau+q))$;

(b) *τ and $\tau+q$ have the same deficiency indices; and*

(c) *every boundary value for τ is a boundary value for $\tau+q$.*

31 Lemma. *Let $f(t)$ be a C^∞ function defined on a finite or infinite interval $[a, b)$. Let t_n be an increasing sequence of elements of $[a, b)$ with t_n approaching b. Let $\mu_i(n) = \max_{a\leq s\leq t_n}|f^{(i)}(s)|$. Then if $\lim_{n\to\infty}(\mu_0(n)/\mu_1(n)) = 0$, it follows that $\lim_{n\to\infty}(\mu_j(n)/\mu_{j+1}(n)) \to 0$ for all j.*

Proof. If the function is identically zero, then the statement is trivial. If f is not identically zero, then clearly $\mu_1(n) \to \infty$. Given $\varepsilon > 0$, choose n so large that

$$[*] \qquad \mu_0(n) < \frac{\varepsilon}{16}\mu_1(n).$$

Let s_0 be a point in $[a, t_n]$ such that $\mu_1(n) = |f'(s_0)|$. We shall prove that there exists a point s_1 in $[s_0-(\varepsilon/4), s_0]$ such that

$$[**] \qquad |f'(s_0)-f'(s_1)| > \tfrac{1}{2}|f'(s_0)|.$$

We can assume without loss of generality that $f'(s_0)$ is positive. If [**] does not hold, then we have

$$\mathscr{R}f'(s) > \tfrac{1}{2}f'(s_0), \qquad s_0-\frac{\varepsilon}{4} \leq s \leq s_0,$$

and integrating,

$$\left|\mathscr{R}f(s_0)-\mathscr{R}f\left(s_0-\frac{\varepsilon}{4}\right)\right| > \frac{\varepsilon}{8}f'(s_0).$$

Hence, either $|\mathscr{R}f(s_0)|$ or $|\mathscr{R}f(s_0-\varepsilon/4)|$ is greater than or equal to

$(\varepsilon/16)f'(s_0)$, which contradicts [*]. From the validity of [**] we infer that either

$$|\mathscr{I}f'(s_0)-\mathscr{I}f'(s_1)| > \tfrac{1}{4}|f'(s_0)|$$

or

$$|\mathscr{R}f'(s_0)-\mathscr{R}f'(s_1)| > \tfrac{1}{4}|f'(s_0)|,$$

where $|s_1 - s_0| \leq \varepsilon/4$. Then by the mean value theorem there exists a point s_2 in $[s_1, s_0]$ such that

$$\frac{\varepsilon}{4}|f''(s_2)| \geqq |s_0 - s_1|\,|f''(s_2)| > \tfrac{1}{4}|f'(s_0)|,$$

that is,

$$|f'(s_0)| < \varepsilon\,|f''(s_2)|,$$

and hence $\mu_1(n) < \varepsilon\mu_2(n)$. Q.E.D.

32 LEMMA. *Let* $\tau = \sum_{k=0}^{n} a_k(t)(d/dt)^k$ *be a formal differential operator of order* n *defined on an interval* $I = [a, \infty)$. *Suppose that* $a_n(t) = 1$, *and that all the coefficients* a_i *are bounded on* I. *Then, if* $\tau f = 0$ *and* $f \in L_2(a, \infty)$, $f, f', \ldots, f^{(n)}$ *are all uniformly bounded on* I.

PROOF. We shall prove that f is bounded. It follows from the preceding lemma that there exists a constant k such that for all t in $[a, \infty)$,

$$[*] \qquad k \max_{a\leq s\leq t_m} |f(s)| \geqq \max_{a\leq s\leq t_m} |f'(s)|.$$

Indeed, if this were not the case, then to every integer m we could associate a point t_m in $[a, \infty)$ such that

$$m \max_{a\leq s\leq t_m} |f(s)| < \max_{a\leq s\leq t_m} |f'(s)|.$$

The sequence $\{t_m\}$ would thus satisfy the hypothesis of the preceding lemma. Hence, using the notation of the preceding lemma.

$$\lim_{m\to+\infty} \mu_j(m)/\mu_{j+1}(m) = 0, \qquad 0 \leqq j < n,$$

from which it follows also that for $j < n$,

$$\lim_{m\to\infty}\frac{\mu_j(m)}{\mu_n(m)} = \lim_{m\to\infty}\frac{\mu_j(m)}{\mu_{j+1}(m)}\cdot\frac{\mu_{j+1}(m)}{\mu_{j+2}(m)}\cdots\frac{\mu_{n-1}(m)}{\mu_n(m)} = 0.$$

Let $\{s_m\}$ be a sequence such that

[†] $$|f^{(n)}(s_m)| = \max_{a\leq s\leq t_m}|f^{(n)}(s)|, \qquad 0 \leq s_m \leq t_m.$$

We have, then, *a fortiori*, for $j < n$,

[††] $$\lim_{m\to\infty}\frac{|f^{(j)}(s_m)|}{|f^{(n)}(s_m)|} = 0.$$

Let M be a common bound for the coefficients a_j, $1 \leq j < n$. Choose m so large that for $1 \leq j < n$,

$$|f^{(j)}(s_m)| < \frac{\varepsilon}{Mn}|f^{(n)}(s_m)|.$$

Then

$$\begin{aligned} 0 = \tau f &\geq |f^{(n)}(s_m)| - |\sum_{k=0}^{n-1} a_k(s_m)f^{(k)}(s_m)| \\ &\geq |f^{(n)}(s_m)| - \varepsilon|f^{(n)}(s_m)| \\ &= (1-\varepsilon)|f^{(n)}(s_m)|. \end{aligned}$$

Because ε is arbitrary, it follows that $f^{(n)}(s_m) = 0$ for large m. In view of [†] and [††], this is clearly impossible except when f vanishes identically. We conclude that inequality [*] holds. In the same way we can show that there exists a constant k such that

[**] $$k\max_{a\leq s\leq t}|f^{(j)}(s)| \geq \max_{a\leq s\leq t}|f^{(j+1)}(s)|, \qquad t \geq a, \qquad 1 \leq j < n.$$

Suppose now that f is not bounded. Then, given any integer N, we can find a point t in $[a, \infty)$ such that $|f(t)| > N$. Multiplying f by an appropriate constant of modulus 1 we may assume that $f(t) > N$.

By [*] we have

$$\max_{a\leq s\leq t}|\mathscr{R}f'(s)| \leq \max_{a\leq s\leq t}|f'(s)| \leq kf(t).$$

By the mean value theorem,

$$|t-s|^{-1}|\mathscr{R}f(s)-\mathscr{R}f(t)| \leq |\mathscr{R}f'(s_0)| \leq kf(t),$$

where $s \leqq s_0 \leqq t$. In particular for $t-(1/2k) \leqq s < t$,

$$|\mathscr{R}f(s) - f(t)| \leqq \frac{f(t)}{2},$$

and therefore $|\mathscr{R}f(s)| > f(t)/2 > N/2$. Consequently,

$$\int_a^\infty |f(s)|^2 ds \geqq \int_{t-(1/2k)}^t |f(s)|^2 ds \geqq \frac{N^2}{8k}.$$

Because N is arbitrarily large, this contradicts the assumption that f is square-integrable. Thus f is bounded, so that by [$*$] and [$**$], $f^{(j)}$ is bounded, $0 \leqq j \leqq n$. Q.E.D.

38 Lemma. *Let f be a function in A^n defined on the real axis and vanishing outside a compact subset of the real axis. Then, if* $0 \leqq k \leqq n$,

$$\left(\int_{-\infty}^\infty |f(t)|^2 dt\right)^{1-(k/n)} \left(\int_{-\infty}^\infty |f^{(n)}(t)|^2\right)^{k/n} \geqq \left(\int_{-\infty}^\infty |f^{(k)}(t)|^2 dt\right).$$

Proof. Assume $f^{(n)}$ to be square-integrable since if it is not, the inequality is obvious. Consider the Fourier transform

$$F(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(s) e^{ist} ds.$$

We have

$$(-it)^k F(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f^{(k)}(s) e^{its} ds, \qquad 0 \leqq k \leqq n.$$

By Hölder's inequality

$$\int_{-\infty}^\infty |t^k F(t)|^2 dt = \int_{-\infty}^\infty |F(t)|^{2(1-k/n)} |t^{2n} F(t)^2|^{k/n}$$
$$\leqq \left\{\int_{-\infty}^\infty |F(t)|^2 dt\right\}^{1-k/n} \left\{\int_{-\infty}^\infty |t^n F(t)|^2 dt\right\}^{k/n}.$$

The lemma now follows immediately from Plancherel's theorem (cf. XI.3.21). Q.E.D.

34 LEMMA. *Let $\tau = \sum_{k=0}^{n} a_k(t)(d/dt)^k$ be a formal differential operator of order n defined on an interval $I = [a, \infty)$. Suppose that $a_n(t) = 1$, and that all the coefficients a_i are bounded in I. Then, if $\tau f = 0$ and f is in $L_2(I)$, all the $n-1$ functions $f', f'', \ldots, f^{(n)}$ belong to $L_2(I)$.*

PROOF. Assume for simplicity that $a = 0$. Choose an infinitely differentiable function h on $(-\infty, \infty)$ which is identically equal to one for $t > 1$ and which vanishes identically for $t < 0$. Let $f_m(t) = h(m-t)h(t)f(t)$. Then f_m is infinitely differentiable, and vanishes together with its derivatives for $t < 0$ and for $t > m$. Furthermore, f_m coincides with f for $1 < t < m-1$, and hence the function $\tau(f_m - f)$ is identically zero in t except on the intervals $m-1 \leqq t \leqq m$ and $0 \leqq t \leqq 1$. Applying Leibniz' rule we see that τf_m is a linear combination of derivatives of $h(m-t)$, of $h(t)$, and of f, and of the coefficients of τ, all of which are bounded by Lemma 32 and by the hypothesis. Let M be a common bound for all these functions and for the functions f_m.

Then

$$[\dagger] \qquad |\tau f_m|_2^2 = |\tau(f_m - f)|_2^2 = \left(\int_0^1 + \int_{m-1}^{m}\right)|\tau(f_m)(t)|^2 dt < 2M^2,$$

uniformly in n. We shall prove that the assumption that some derivative of order not greater than n is not square-integrable contradicts this.

Since $f \in L_2$, we have clearly

$$C = \sup_{0 \leqq m < \infty} \int_{\infty}^{\infty} |f_m(t)|^2 dt \leqq \sup_{0 \leqq m < \infty} \int_{1}^{m-1} |f(t)|^2 dt + 2M^2$$
$$< |f|_2^2 + 2M^2 < \infty.$$

By the preceding lemma we have

$$[*] \qquad |f_m^{(k)}|_2 = O(|f_m^{(n)}|_2^{k/n})$$

and

$$[**] \qquad |f_m^{(k)}|_2 / |f_m^{(n)}|_2 \leqq C^{1/2 - k/2n} |f_m^{(n)}|_2^{k/n - 1}, \qquad 0 \leqq k \leqq n.$$

Suppose now that for some $k_0 \leqq n$, $f^{(k_0)}$ is not square-integrable. Then

clearly $\lim_{m\to\infty}|f_m^{(k_0)}|_2 = \infty$, and by [*], $\lim_{m\to\infty}|f_m^{(n)}|_2 = \infty$. Consequently, by [**], $|f_m^{(k)}|_2 = o(|f_m^{(n)}|)$ for $0 \leq k < n$. Furthermore,

$$|\tau f_m - f_m^{(n)}|_2 = |\sum_{k=0}^{n-1} a_k f_m^{(k)}|_2 \leq M \sum_{k=0}^{n-1} |f_m^{(k)}|_2,$$

and hence

$$||\tau f_m|_2 - |f_m^{(n)}|_2| \leq |\tau f_m - f_m^{(n)}|_2 = o(|f_m^{(n)}|_2).$$

Thus $\lim_{m\to\infty}|\tau f_m|_2 = \infty$, which contradicts [†]. Q.E.D.

35 THEOREM. *Let*

$$\tau = \sum_{k=0}^{n} a_k(t)\left(\frac{d}{dt}\right)^k$$

be a formally self adjoint formal differential operator of order n defined on an interval $I = [a, \infty)$. Assume that

(a) $|a_n(t)|$ *is bounded away from zero;*

(b) $|a_k(t)|$ *is bounded,* $0 \leq k \leq n$.

Then τ has no boundary values at infinity.

PROOF. Without loss of generality, assume that $a = 0$. Let τ' be the formally self adjoint formal differential operator $i^n(d/dt)^n$. The proof will consist in establishing that the Hilbert spaces $\mathfrak{D}(T_1(\tau'))$ and $\mathfrak{D}(T_1(\tau))$ are equivalent and that τ' has no boundary values at infinity. Since, by definition, $\mathfrak{D}(T_0(\tau))$ coincides with $\mathfrak{D}(T_0(\tau'))$, it will then follow that the boundary values of τ and τ' coincide, and from this it will be easy to prove the theorem.

We proceed in several steps.

(a) τ' has no boundary values at infinity. Indeed, let $\omega_1, \ldots, \omega_n$ be the nth roots of $(-i)^{n-1}$. Then the functions $\exp(\omega_i t)$ form a basis for the solutions of $\tau'\sigma = i\sigma$. Among these, the square-integrable solutions are those for which the real part of ω_i is negative. Similarly, let $\omega'_1, \ldots, \omega'_n$ be the nth roots of $-(-i)^{n-1}$. Then the square-integrable solutions of $\tau'\sigma = -i\sigma$ are the functions $\exp(\omega'_i t)$ for which ω'_i has a negative real part. If n is odd, the nth roots of $-t$ are the negative nth roots of t. Thus, in this case, the sum of the deficiency indices of τ' is the number of nth roots of $(-i)^{n-1}$ which are not pure imaginary. Since if n is odd, $i^n = \pm i$ and $(-i)^{n-1} = \pm 1$, this number

is n. If n is even, we see by a similar elementary computation that the sum of the deficiency indices of τ' is n. Our assertion now follows immediately from Corollary 2.23 and Lemma XII.4.21.

(b) $\mathfrak{D}(T_1(\tau)) \supseteq \mathfrak{D}(T_1(\tau'))$.

Suppose f is in $\mathfrak{D}(T_1(\tau'))$. This means by definition that both f and $f^{(n)}$ are square-integrable. Now form the functions $f_m^{(k)}$, $0 \leqq k \leqq n$, as in the first paragraph of the proof of Lemma 34. Then $|f_m^{(n)}|_2$ is bounded in m, because $f^{(n)}$ is square-integrable, and because, by Lemma 32, $f^{(k)}(t)$ is bounded on the interval $[a, \infty)$ for $0 \leqq k \leqq n-1$. Therefore by Lemma 33, $|f_m^{(k)}|$ is bounded in k and in m, and hence $|f^{(k)}|_2$ is finite for $0 \leqq k \leqq n$. Let M be an upper bound for the functions $|a_k(\cdot)|$. Then

$$|\tau f|_2 \leqq M \sum_{k=0}^{n} |f^{(k)}|_2 < \infty$$

and therefore f is in $\mathfrak{D}(T_1(\tau))$. Hence $\mathfrak{D}(T_1(\tau')) \subseteq \mathfrak{D}(T_1(\tau))$.

(c) The identity mapping of the Hilbert space $\mathfrak{D}(T_1(\tau'))$, into $\mathfrak{D}(T_1(\tau'))$ considered as a subspace of $\mathfrak{D}(T_1(\tau))$, is closed, and hence continuous.

Let $\{f_n\}$ be a sequence in $\mathfrak{D}(T_1(\tau'))$ which converges to f in the topology of $\mathfrak{D}(T_1(\tau'))$ and to g in the topology of $\mathfrak{D}(T_1(\tau))$. Let J be any compact subinterval of I. Then, by Corollary 2.16(b), the restriction of $\{f_n\}$ to J converges in $H_n(J)$ to both f and g. Therefore, since f and g are continuous functions, they coincide on J. Because J is arbitrary, they coincide everywhere on I.

(d) Let $\{g_n\}$ be a sequence in $\mathfrak{D}(T_0(\tau'))$ which converges to zero in the norm of $\mathfrak{D}(T_1(\tau))$. Then $\{g_m\}$ converges to zero in the norm of $\mathfrak{D}(T_1(\tau'))$. Indeed, we have $|g_m|_2 \to 0$ and $|\tau g_m|_2 \to 0$, and

$$[*] \qquad \left| \left|\sum_{k=0}^{n-1} a_k g_m^{(k)}\right|_2 - |a_n g_m^{(n)}|_2 \right| \leqq \left| \sum_{k=0}^{n-1} a_k g_m^{(k)} + a_n g_m^{(n)} \right|_2 = |\tau g_m|_2 \to 0.$$

Then $|a_n g_m^{(n)}|_2$ is bounded, for otherwise (passing to a subsequence for which $|a_n g_m^{(n)}|_2 \to \infty$) it would follow from Lemma 33 that

$$\begin{aligned} |g_m^{(k)}|_2 &= O(|g_m^{(n)}|_2^{k/n}) \\ &\qquad o(|g_m^{(n)}|_2), \qquad 0 \leqq k < n. \end{aligned}$$

Hence, because the coefficients a_k are bounded,

$$|\sum_{k=0}^{n-1} a_k g_m^{(k)}|_2 = o(|g_m^{(n)}|_2) = o(|a_n g_m^{(n)}|_2).$$

But then it is clear that

$$\begin{aligned} |\tau g_m|_2 &\geq |a_n g_m^{(n)}|_2 - |\sum_{k=0}^{n-1} a_k g_m^{(k)}|_2 \\ &= |a_n g_m^{(n)}|_2 (1-o(1)) \to \infty, \end{aligned}$$

contradicting [*]. Let $M = \sup_m |g_m^{(n)}|_2$. From Lemma 33 we obtain

$$|g_m|_2^{1-k/n} M^{k/n} \geq |g_m^{(k)}|_2.$$

Because $|g_m|_2 \to 0$, it follows that $|g_m^{(k)}|_2 \to 0$ for $0 \leq k < n$ and therefore also that

$$|\sum_{k=0}^{n-1} a_k g_m^{(k)}|_2 \to 0.$$

By [*] this implies that $|a_n g^{(n)}|_2 \to 0$, and since $|a_n(\cdot)|^{-1}$ is bounded by hypothesis, $|g_m^{(n)}|_2 \to 0$. Thus $\{g_m\}$ converges to zero in the norm of $\mathfrak{D}(T_1(\tau'))$.

(e) The closure of $\mathfrak{D}(T_0(\tau'))$ in the norm of $\mathfrak{D}(T_1(\tau'))$ coincides with the closure of $\mathfrak{D}(T_0(\tau'))$ in the norm of $\mathfrak{D}(T_1(\tau))$.

Let $\mathfrak{D}_1$ and $\mathfrak{D}_2$ be the closures of $\mathfrak{D}(T_0(\tau'))$ in the norms of $\mathfrak{D}(T_1(\tau'))$ and $\mathfrak{D}(T_1(\tau))$ respectively. By step (c) we have that $\mathfrak{D}_2 \supseteq \mathfrak{D}_1$. Let $g \in \mathfrak{D}_2$, and let $\{g_m\}$ be a Cauchy sequence in $\mathfrak{D}(T_0(\tau'))$ which converges to g in the norm of $\mathfrak{D}(T_1(\tau))$. To show that g is in $\mathfrak{D}_1$ it suffices by (c) to show that $\{g_n\}$ is a Cauchy sequence in the norm of $\mathfrak{D}(T_1(\tau'))$. Suppose that this is not so. Then the generalized sequence $\{g_{mn}\} = \{g_m - g_n\}$ does not converge to zero in $\mathfrak{D}(T_1(\tau'))$; therefore a subsequence of it, which we shall call $\{f_i\}$, does not converge to zero in $\mathfrak{D}(T_1(\tau'))$, whereas it does converge to zero in $\mathfrak{D}(T_1(\tau))$. This, however, contradicts the result of step (d).

(f) $\mathfrak{D}(T_1(\tau')) \supseteqq \mathfrak{D}(T_1(\tau))$.

Clearly $\mathfrak{D}(T_0(\tau)) = \mathfrak{D}(T_0(\tau'))$, so that by step (e) $\mathfrak{D}(\overline{T_0(\tau)}) = \mathfrak{D}(\overline{T_0(\tau')})$. Let $\mathfrak{D}_1 = \mathfrak{D}(\overline{T_0(\tau)}) = \mathfrak{D}(\overline{T_0(\tau')})$. Then (cf. XII.4.10) $\mathfrak{D}(T_1(\tau)) = \mathfrak{D}_1 \oplus \mathfrak{D}_+ \oplus \mathfrak{D}_-$, where $\mathfrak{D}_+$ and $\mathfrak{D}_-$ are the deficiency

spaces of τ. It suffices to prove that $\mathfrak{D}_+ \oplus \mathfrak{D}_- \subseteq \mathfrak{D}(T_1(\tau'))$. Let f be an element of $\mathfrak{D}_+$. Then $[a_n(t)]^{-1}((\tau-i)f)(t) = 0$ for t in I. The operator $a_n^{-1}(\tau-i)$ satisfies the hypothesis of the preceding lemma. We conclude that $f^{(n)}$ is square-integrable, that is, f is in $\mathfrak{D}(T_1(\tau'))$. Thus, $\mathfrak{D}_+ \subseteq \mathfrak{D}(T_1(\tau'))$. Similarly, $\mathfrak{D}_- \subset \mathfrak{D}(T_1(\tau'))$. It follows from (c) and II.2.2 that the two topologies on $\mathfrak{D}(T_1(\tau)) = \mathfrak{D}(T_1(\tau'))$ are equivalent. Q.E.D.

In concluding our analysis of the methods available for the calculation of the deficiency indices of a formal differential operator, we note one case which, though more special than many of the cases studied above, is nevertheless of great practical importance. This is the case in which the coefficients of the formal differential operator are analytic in I and have poles at the free end points of I. In this case, exact information on the asymptotic nature of the solutions of $\tau f = \lambda f$ is available.

Suppose first that the end point under consideration is finite so that without loss of generality we can suppose it to be at zero, Then, dividing through if necessary by the leading coefficient a_n of τ, we can write the equation $(\tau-\lambda)f = 0$ in the form

$$[*] \qquad \sum_{k=0}^{n} \frac{\alpha_k(z)}{z^{\nu(n-k)}} f^{(k)}(z) = 0,$$

where $\alpha_n = 1$, α_k is analytic in the neighborhood of zero for $0 \leq \alpha_k \leq n$, and where we suppose that ν is minimal; i.e., that the differential equation [*] does not have the form $\sum_{k=0}^{n} \beta_k(t)t^{-\mu(n-k)}f^{(k)}(z) = 0$ where β_k is analytic in the neighborhood of zero for $0 \leq k \leq n$ and $\mu < \nu$. In this case, ν is called the *order* of the singularity of equation [*] at zero. If $\nu = 0$, there is no singularity at all, and zero is called a *regular* point of the differential equation. If $\nu = 1$, the singularity of equation [*] at zero is called a *regular singularity*; if $\nu > 1$, an *irregular singularity*. If [*] has a regular singularity at $z = 0$, the equation

$$\begin{aligned}\mu(\mu-1)\ldots(\mu-n+1)&+\alpha_{n-1}(0)\mu(\mu-1)\ldots(\mu-n+2)\\ &+\alpha_{n-2}(0)\mu(\mu-1)\ldots(\mu-n+3)\\ &+\ldots+\alpha_1(0)\mu+\alpha_0(0) = 0\end{aligned}$$

is called the ***indicial equation of* [*] *at zero.*** If the indicial equation has distinct roots $e_1, \ldots, e_n$, no two of which differ by an integer, then the set of solutions of [*] has a basis of the form $\sigma_j(z) = z^{e_j}\varphi_j(z)$, where φ_j is analytic and non-zero in the neighborhood of $z = 0$. Thus, in this case, the number of solutions of $\tau f = \lambda f$ which are square-integrable in the neighborhood of $z = 0$ is exactly the number of roots of the indicial equation which have real parts greater than $-(1/2)$. In case the indicial equation has two roots which differ by an integer or has multiple roots, a corresponding result may be stated; but here the basis for solutions of [*] may have a more complex form, involving logarithmic terms. We shall not give the details of this result here, but instead refer the reader to the excellent exposition of this point in Poole [1] and Coddington and Levinson [1]; to which sources the reader is also referred for proofs. It should be emphasized, however, that in all these cases the problem of determining the number of solutions of [*] which are square-integrable in the neighborhood of zero can be reduced to a finite algebraic problem.

In case $\nu > 1$, so that [*] has an irregular singularity at zero, then the equation

$$(-\nu\mu)^n + \alpha_{n-1}(0)(-\nu\mu)^{n-1} + \ldots + \alpha_0(0) = 0$$

is called the ***characteristic equation of* [*] *at zero.*** If the characteristic equation has distinct roots $\mu_1, \ldots, \mu_n$ then the set of solutions of [*] has a basis of the form

$$\sigma_j(z) = \{\exp(\mu_j z^{1-\nu} + k_j^{(2)} z^{2-\nu} + \ldots + k_j^{(\nu-1)} z^{-1})\} z^{e_j}(1 + c_j^{(1)} z + \ldots);$$

where, however, the infinite series constituting the final factor of this expression is not convergent but is a divergent asymptotic series. The coefficients $k_j^{(2)}, \ldots, k_j^{(\nu-1)}$, e_j, $c_j^{(1)}$, $c_j^{(2)}, \ldots$ can be determined by formally substituting the asymptotic expression for σ_j in equation [*] and comparing coefficients. In case the characteristic equation has multiple roots, a corresponding result may be stated; but here the asymptotic expressions for solutions of [*] may have a more complex form, involving series in fractional powers of z and logarithmic terms. We shall not give the details of this result, but shall

instead refer the reader to the appropriate chapter of Coddington and Levinson [1]. What is decisive for our purposes is that the existence of an asymptotic series for the solutions of [*] reduces the problem of finding the number of solutions of [*] which are square-integrable in the neighborhood of zero to a finite algebraic problem.

By making the change of variable $z \to (1/z)$, similar results can be obtained for singularities at infinity. Suppose that we are dealing with an equation of the form

$$[**] \qquad \sum_{k=0}^{n} \alpha_k(z) z^{\nu(n-k)} f^{(k)}(z) = 0,$$

where $\alpha_n(z) = 1$ and the coefficients α_k are analytic in the neighborhood of $z = \infty$ for $0 \leq k \leq n$, and where the integer ν is minimal in the sense that [**] cannot be written in similar form with a smaller index ν. Then, if $\nu = -1$, [**] is said to have a *regular singularity at infinity*; if $\nu > -1$, [**] is said to have an *irregular singularity of order* $\nu+2$ *at infinity*. If [**] has a regular singularity at infinity, the equation

$$\mu(\mu+1)\ldots(\mu+n-1) - \alpha_{n-1}(\infty)\mu(\mu+1)\ldots(\mu+n-2) + \ldots + (-1)^{n-1}\alpha_1(\infty)\mu + (-1)^n \alpha_0(\infty) = 0$$

is called the *indicial equation of* [**] *at infinity*. If the roots $e_1, \ldots, e_n$ of this equation are all distinct and no two differ by an integer, then the set of solutions of [**] has a basis of the form $\sigma_j(z) = z^{-e_j}\varphi_j(z)$, where φ_j is analytic and non-zero in the neighborhood of $z = \infty$.

If [**] has an irregular singularity of order $\nu+2$ at infinity, the equation

$$((\nu+1)\mu)^n + \alpha_{n-1}(\infty)((\nu+1)\mu)^{n-1} + \ldots + \alpha_0(\infty) = 0$$

is called the *characteristic equation of* [**] *at infinity*. If the roots $\mu_1, \ldots, \mu_n$ of the characteristic equation are distinct, then [**] has a set of solutions of the form

$$\sigma_j(z) = \{\exp(\mu_j z^{\nu+1} + k_j^{(\nu)} z^{\nu} + \ldots + k_j^{(1)} z)\} z^{-e_j}(1 + c_j^{(1)} z^{-1} + c_j^{(2)} z^{-2} + \ldots),$$

where, however, the infinite series constituting the final factor of this expression is not convergent but is a divergent asymptotic series. The coefficients $k_j^{(\nu)}, \ldots, k_j^{(1)}, c_j, c_j^{(1)}, c_j^{(2)}, \ldots$ can be determined by formally substituting the asymptotic expression for σ_j in [**] and comparing coefficients of z^{-n}.

Thus, in all cases in which we deal with a formal differential operator on an interval I having coefficients analytic in I and with poles at the free end points of I, the theory of regular and irregular singularities enables us to reduce the problem of determining the deficiency indices of τ to a finite algebraic problem.

We will return to the theory of regular and irregular singular points below in connection with the study of several specific examples of formally symmetric formal differential operators in Section 8. In that section, we will have an opportunity to examine a number of facets of the theory in somewhat greater detail

7. Qualitative Theory of the Spectrum

It will be seen in this section how a variety of qualitative results concerning the spectrum of a formally self adjoint formal differential operator τ may readily be obtained from the analytical properties of the coefficients defining τ. For example, if these coefficients are periodic the spectrum coincides with the continuous spectrum and consists of a set of intervals (cf. Theorem 64). It will also be seen how the classical separation theorems of Sturm are related to spectral theory. The methods used are, for the most part, elementary, and are analogous to those used in the investigation of the deficiency indices of τ given in the preceding section.

1 THEOREM. *Let T be a closed operator in a B-space $\mathfrak{X}$ and suppose that for each λ in the point spectrum of T, the null-manifold $\{x|(T-\lambda I)x = 0\}$ is finite dimensional. Then $\lambda_0 \in \sigma_e(T)$ if and only if there exists a bounded sequence $\{f_n\}$ of elements of $\mathfrak{D}(T)$ such that $\lim_{n\to\infty}(T-\lambda_0 I)f_n$ exists but $\{f_n\}$ has no strongly convergent subsequence.*

PROOF. Passing without any essential change in the situation from the consideration of T to the consideration of $T-\lambda_0 I$, we may (and shall) suppose without loss of generality that $\lambda_0 = 0$. Suppose

then that $0 \notin \sigma_e(T)$. Since $\mathfrak{N} = \{x | Tx = 0\}$ is finite dimensional, we can find a basis $\varphi_1, \ldots, \varphi_k$ of $\mathfrak{N}$. By Corollary IV.3.2, every finite dimensional subspace of a B-space is closed. Thus, by the Hahn-Banach theorem (II.3.13) there exists a set $x_1^*, \ldots, x_k^*$ of continuous linear functionals on the B-space such that $x_i^*(\varphi_j) = 0$ for $0 < i \neq j < k$, $x_i^*(\varphi_i) = 1$. Let $\mathfrak{D} = \{x \in \mathfrak{D}(T) | x_j^*(x) = 0, \ i = 1, \ldots, k\}$. Then, by Lemma 6.2, the restriction T_1 of T to $\mathfrak{D}$, which is evidently closed, has a closed range $\mathfrak{R}$. The inverse T_1^{-1} of T_1 is clearly closed, and is a one-to-one mapping of $\mathfrak{R}$ into $\mathfrak{X}$. Thus by the closed graph theorem (II.2.4), T_1^{-1} is bounded. It follows that if $\{f_n\}$ is a sequence of elements of $\mathfrak{D}$ such that $\{Tf_n\}$ converges, then $\{f_n\}$ converges. Let $\{g_n\}$ be a bounded sequence of elements of $\mathfrak{D}(T)$ such that $\{Tg_n\}$ converges. Find a subsequence $\{g_{n_i}\} = \{h_i\}$ such that $x_j^*(h_i)$ converges for each $j, 1 \leq j \leq k$. Then $\tilde{h}_i = h_i - \sum_{j=1}^k x_j^*(h_i)\varphi_j$ is in $\mathfrak{D}$, and $T\tilde{h}_i = Th_i$. Thus $\{\tilde{h}_i\}$ converges, so that $\{h_i\} = \{\tilde{h}_i + \sum_{j=1}^k x_j^*(h_i)\varphi_j\}$ converges. Consequently, $\{g_n\}$ has a convergent subsequence, which proves the first part of our theorem.

Conversely, suppose that if $\{f_n\}$ is bounded, and $\{Tf_n\}$ converges, then $\{f_n\}$ has a convergent subsequence. We wish to show that $T\mathfrak{D}(T)$ is closed; so let $g \in \overline{T\mathfrak{D}(T)}$, let $g_n = Th_n$, and let $g_n \to g$. Let d_n be the distance from h_n to the closed manifold $\mathfrak{N}$. Let y_n be a sequence of elements of $\mathfrak{N}$ such that $|h_n - y_n| - d_n \to 0$. Let $h_n - y_n = k_n$. Then, if d_n is bounded, k_n is bounded, and $Tk_n = Th_n$ converges. Hence k_n has a convergent subsequence k_{n_i}, which converges to an element k. Since $Tk_{n_i} \to g$ and T is closed, $k \in \mathfrak{D}(T)$ and $Tk = g$. Consequently, to prove that $T\mathfrak{D}(T)$ is closed it suffices to show that d_n is bounded. If this is false, then, by passing to a subsequence, we can suppose that $d_n \to \infty$, so that $|k_n| \to \infty$. Putting $\tilde{k}_n = k_n/|k_n|$, we have $|\tilde{k}_n| = 1$, while the distance $\tilde{d}_n$ from $\tilde{k}_n$ to $\mathfrak{N}$ is $d_n/|k_n| \to 1$. But, since $T\tilde{k}_n \to 0$, $\tilde{k}_n$ has a convergent subsequence $\tilde{k}_{n_i}$; and clearly $\tilde{k}_{n_i}$ converges to an element of $\mathfrak{N}$, contradicting the fact that $\tilde{d}_n \to 1$. This proves the converse part of our theorem. Q.E.D.

2 Corollary. *Let τ be a formal differential operator. Then $\lambda_0 \in \sigma_e(\tau)$ if and only if there exists a bounded sequence $\{f_n\}$ of functions in $\mathfrak{D}(T_0(\tau))$ such that $\{(\tau - \lambda_0)f_n\}$ converges, but $\{f_n\}$ has no strongly convergent subsequence.*

Proof. Suppose that a bounded sequence $\{f_n\}$ of elements of $\mathfrak{D}(T_0(\tau))$ exists such that $\{(\tau-\lambda_0)f_n\}$ converges but the sequence $\{f_n\}$ has no convergent subsequence. Then, since $T_0(\tau) \subseteq T_1(\tau)$, it follows immediately from the preceding lemma that $\lambda_0 \in \sigma_e(T_1(\tau))$, so that by Definition 6.1, $\lambda_0 \in \sigma_e(\tau)$.

Conversely, let $\lambda_0 \in \sigma_e(\tau)$. Let $\mathfrak{D}_1$ be the closure in the Hilbert space $\mathfrak{D}(T_1(\tau))$ of $\mathfrak{D}(T_0(\tau))$, and let $\overline{T_0(\tau)}$ be the restriction of $T_1(\tau)$ to $\mathfrak{D}_1$. Let $\mathfrak{D}$ be the orthocomplement in $\mathfrak{D}(T_1(\tau))$ of $\mathfrak{D}_1$. Then $\mathfrak{D}(T_1(\tau)) = \mathfrak{D}(\overline{T_0(\tau)}) \oplus \mathfrak{D}$. Moreover, if we have $f \in \mathfrak{D}$, then $(f, g) + (T_1(\tau)f, T_0(\tau)g) = 0$ for g in $\mathfrak{D}(T_0(\tau))$. Hence $f + T_0(\tau)^*T_1(\tau)f = 0$, so that by Theorem 2.10 and Corollary 1.4, f is a C^∞ solution of the differential equation $f+\tau^*\tau f = 0$. Since this equation has finite order, it follows that $\mathfrak{D}$ is finite dimensional. Thus, it follows from Lemma 6.2 that λ_0 is in the essential spectrum of the closed operator $\overline{T_0(\tau)}$. Hence, by the preceding theorem, there exists a bounded sequence $\{\tilde{f}_n\}$ of elements of $\mathfrak{D}(\overline{T_0(\tau)})$ such that $\{(\overline{T_0(\tau)}-\lambda I)\tilde{f}_n\}$ converges but $\{\tilde{f}_n\}$ has no convergent subsequence. By definition of $\mathfrak{D}(\overline{T_0(\tau)})$, there exist elements f_n in $\mathfrak{D}(T_0(\tau))$ such that

$$|f_n-\tilde{f}_n| \to 0 \qquad \text{and} \qquad |T_0(\tau)f_n-\overline{T_0(\tau)}\tilde{f}_n| \to 0.$$

Then $\{f_n\}$ is bounded and contains no convergent subsequence, while $\{(T_0(\tau)-\lambda I)f_n\} = \{(\tau-\lambda)f_n\}$ converges. This proves the converse part of our theorem. Q.E.D.

8 **Corollary.** *Let τ be a formal differential operator, and B_i, $i = 1, \ldots, k$ be a set of boundary values for τ. Let T be the operator derived from τ by imposition of the set $B_i(f) = 0$, $i = 1, \ldots, k$, of boundary conditions. Then $\sigma_e(T) = \sigma_e(\tau)$.*

Proof. We have shown in the course of the preceding proof that $\mathfrak{D}(T_1(\tau)) = \mathfrak{D}(\overline{T_0(\tau)}) \oplus \mathfrak{D}$, where $\mathfrak{D}$ is finite dimensional. Hence $\mathfrak{D}(T) = \mathfrak{D}(\overline{T_0(\tau)}) \oplus \mathfrak{D}_1$, and $\mathfrak{D}(T_1(\tau)) = \mathfrak{D}(T) \oplus \mathfrak{D}_2$, where $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are finite dimensional. Hence, by Lemma 6.2, $\sigma_e(T) = \sigma_e(T_1(\tau)) = \sigma_e(\tau)$. Q.E.D.

The next result is the analog for the spectrum of Corollary 2.26 on deficiency indices.

4 THEOREM. *Let τ be a formal differential operator defined on an interval I with end points a, b, and let I be the union of two subintervals I_1 and I_2. Let the restriction of τ to I_1 (to I_2) be denoted by τ_1 (by τ_2). Then*

$$\sigma_e(\tau) = \sigma_e(\tau_1) \cup \sigma_e(\tau_2).$$

PROOF. Let $\lambda_0 \in \sigma_e(\tau_1)$. Then by Corollary 2 there exists a bounded sequence $\{f_n\}$ of elements of $\mathfrak{D}(T_0(\tau_1))$ such that $((\tau-\lambda_0)f_n\}$ converges but $\{f_n\}$ has no convergent subsequence. Since $\mathfrak{D}(T_0(\tau_1)) \subseteq \mathfrak{D}(T_0(\tau))$, it follows immediately from Lemma 2 that $\lambda_0 \in \sigma_e(\tau)$. Thus $\sigma_e(\tau_1) \subseteq \sigma_e(\tau)$. Similarly, $\sigma_e(\tau_2) \subseteq \sigma_e(\tau)$. Thus we have

$$\sigma_e(\tau_1) \cup \sigma_e(\tau_2) \subseteq \sigma_e(\tau).$$

Conversely, let $\lambda_0 \in \sigma_e(\tau)$ and suppose that $\lambda_0 \notin \sigma_e(\tau_1) = \sigma_e(T_1(\tau_1))$ and $\lambda_0 \notin \sigma_e(\tau_2) = \sigma_e(T_1(\tau_2))$. Let $\{f_n\}$ be a bounded sequence in $\mathfrak{D}(T_0(\tau))$ such that $\{(\tau-\lambda_0)f_n\}$ converges but such that $\{f_n\}$ contains no convergent subsequence. Then since the restriction $\rho_1 f_n$ of f_n to I_1 belongs to $\mathfrak{D}(T_1(\tau))$, it follows that $\{\rho_1 f_n\}$ has a convergent subsequence $\{\rho_1 f_{n_i}\}$. In the same way, the sequence of restrictions of $\{f_{n_i}\}$ to I_2 has a convergent subsequence. Hence the sequence $\{f_n\}$ has a subsequence which converges in the topology of $L_2(I)$, contrary to assumption. Q.E.D.

5 THEOREM. *Let τ be a formal differential operator defined on an interval I, and let C be the smallest closed convex set containing all the values $(\tau f, f)$ with f in $\mathfrak{D}(T_0(\tau))$ and $|f| = 1$. Then $\sigma_e(\tau) \subseteq C$.*

PROOF. We have only to show that if H is a closed half-plane of the complex plane which contains C, then $\sigma_e(\tau) \subseteq H$ (cf. V.2.12). By passing without any essential change in the situation from the consideration of τ to the consideration of $\alpha\tau+\beta$, (where $\alpha \neq 0$) we may, and shall, suppose without loss of generality that H is the right half-plane, and that λ is real. Thus we have $\mathscr{R}(\tau f, f) \geqq 0$ for f in $\mathfrak{D}(T_0(\tau))$ and wish to show that no $\lambda < 0$ belongs to $\sigma_e(\tau)$. Suppose that this is false, so that there exists a bounded sequence $\{f_n\}$ of elements of $\mathfrak{D}(T_0(\tau))$ such that $\{(\tau-\lambda)f_n\}$ converges but $\{f_n\}$ has no convergent subsequence. Then

$$\lim_{i,j\to\infty} (\tau-\lambda)(f_i-f_j) = 0.$$

so that

$$\lim_{i,j\to\infty} \mathscr{R}((\tau-\lambda)(f_i-f_j), (f_i-f_j))$$
$$-\lim \mathscr{R}(\tau(f_i-f_j), (f_i-f_j))-\lambda|f_i-f_j|^2 = 0.$$

Since $\mathscr{R}(\tau(f_i-f_j), (f_i-f_j)) \geqq 0$ and $-\lambda|f_i-f_j|^2 \geq 0$, it follows that

$$\lim_{i,j\to\infty} -\lambda|f_i-f_j|^2 = 0.$$

Thus $\{f_n\}$ is a Cauchy sequence, and hence convergent. This contradiction proves our theorem. Q.E.D.

6 Definition. A formal differential operator τ is said to be *formally positive* if $(\tau f, f) \geqq 0$ for every $f \in \mathfrak{D}(T_0(\tau))$.

An example of such an operator is given by

$$\tau - \sum_{k=0}^{n} (-1)^k \left(\frac{d}{dt}\right)^k p_k(t) \left(\frac{d}{dt}\right)^k,$$

where all the coefficients p_k are non-negative, since in this case we have

$$(\tau f, f) = \int_I \tau f(t)\overline{f(t)}dt = \sum_{k=0}^{n} \int p_k(t)|f^{(k)}(t)|^2 dt.$$

7 Corollary. *If a formal differential operator is formally positive, its essential spectrum is a subset of the positive real axis.*

Proof. This follows immediately from the preceding definitlon and Theorem 5. Q.E.D.

8 Theorem. *Suppose that all the values of the coefficients p_k of the formal differential operator*

$$\tau = \sum_{k=0}^{n} (-1)^k \left(\frac{d}{dt}\right)^k p_k(t) \left(\frac{d}{dt}\right)^k, \qquad t \in [a, b)$$

lie in the right half-plane and that $\mathscr{R}p_0(t) \to \infty$ as $t \to b$. Then the essential spectrum of τ is void.

Proof. We have $\mathscr{R}p_k(t) \geq 0$, $0 \leqq k \leq n$, and $\mathscr{R}p_0(t) \to \infty$ as $t \to b$. If τ' is the restriction of τ to an interval $[c, b)$ then by Theorem 4 $\sigma_e(\tau) = \sigma_e(\tau')$. If any real number λ_0 is given, we can choose c such that $\mathscr{R}p_0(t) \geq \lambda_0$ for $t \in [c, b)$. Then we have

$$\mathscr{R}(\tau' f, f) = \sum_{k=0}^{n} \int_c^b [\mathscr{R} p_k(t)] |f^{(k)}(t)|^2 dt \geq \lambda_0 |f|^2, \qquad f \in \mathfrak{D}(T_0(\tau')).$$

It follows from Theorem 5 that $\sigma_e(\tau')$ and hence $\sigma_e(\tau)$ lies entirely in the half-plane $\mathscr{R}z \geq \lambda_0$. Since λ_0 is arbitrary, $\sigma_e(\tau)$ is void. Q.E.D.

9 THEOREM. *Suppose that τ is a formal differential operator of order $2n$ defined on an interval $I = [a, b)$, and that τ has the form*

$$\tau = (-1)^n \left(\frac{d}{dt}\right)^{2n} + \sum_{j=1}^{2n-1} p_j(t) \left(\frac{d}{dt}\right)^j + p_0(t),$$

where the coefficients $p_1, \ldots, p_{2n-1}$ are uniformly bounded and $\mathscr{R}p_0(t) \to \infty$ as $t \to b$. Then $\sigma_e(\tau)$ is void.

PROOF. Let c be real. Consider the differential operator

$$\tau_c = (-1)^n \left(\frac{d}{dt}\right)^{2n} + \sum_{j=1}^{2n-1} p_j(t) \left(\frac{d}{dt}\right)^j + c.$$

We will first show that for c sufficiently large, $\mathscr{R}(\tau_c f, f) \geq 0$ for $f \in \mathfrak{D}(T_0(\tau))$. If this is not the case, there exists a sequence $\{g_m\}$ of elements of $\mathfrak{D}(T_0(\tau))$ such that

$$\mathscr{R}(\tau_0 g_m, g_m) \leq -m|g_m|_2^2.$$

Hence there exists a sequence $\{f_m\}$ of elements of $\mathfrak{D}(T_0(\tau))$ such that $|f_m|_2 = 1$, but $\mathscr{R}(\tau_0 f_m, f_m) \to -\infty$. If $|f_m^{(2n)}|_2$ is bounded, it follows from Lemma 6.33 that $|f_m^{(k)}|_2$ is bounded for all $0 < k < 2n$, and hence that $(\tau_0 f_m, f_m)$ is bounded, contrary to assumption. Hence we may assume without loss of generality that $|f_m^{(2n)}|_2$ approaches ∞. Then, by Lemma 6.33 again,

$$|f_m^{(k)}|_2 = o(|f_m^{(2n)}|_2), \qquad 0 < k \leq 2n,$$

so that

$$\begin{aligned}(\tau_0 f_m, f_m) &= (-1)^n (f_m^{(2n)}, f_m)(1 + o(1)) \\ &= |f_m^{(n)}|_2^2 (1 + o(1)),\end{aligned}$$

contradicting the fact that $\mathscr{R}(\tau_0 f_m, f_m) \to -\infty$.

Thus $\mathscr{R}(\tau_c f, f) \geq 0$ for $f \in \mathfrak{D}(T_0(\tau))$ if c is sufficiently large. Let λ_0 be real and positive. Passing by means of Theorem 4 to a sufficient-

ly small subinterval $[a_0, b)$ of $[a, b)$, we may assume without loss of generality that $\mathscr{R}(p_0(t) - c) \geqq \lambda_0$. Then

$$\mathscr{R}(\tau f, f) \geqq (\tau_c f, f) + \lambda_0(f, f)$$

for $f \in \mathfrak{D}(T_0(\tau))$. It follows from Theorem 5 that $\sigma_e(\tau)$ lies entirely in the half-plane $\mathscr{R}z \geqq \lambda_0$. Since λ_0 is arbitrary, $\sigma_e(\tau)$ is void. Q.E.D.

10 Theorem. *Suppose that τ is a formal differential operator of order $2n$ defined on an interval $I = [a, b)$, and that τ has the form*

$$\tau = (-1)^n \left(\frac{d}{dt}\right)^{2n} + \sum_{k=1}^{n-1} (-1)^k \left(\frac{d}{dt}\right)^k p_k(t) \left(\frac{d}{dt}\right)^k + p_0(t).$$

Suppose that $\mathscr{R}p_k$ is bounded below for $1 \leqq k \leqq n-1$, and that $\mathscr{R}p_0(t) \to \infty$ as $t \to b$. Then $\sigma_e(\tau)$ is void.

Proof. Putting

$$\tau_c = (-1)^n \left(\frac{d}{dt}\right)^{2n} + \sum_{k=1}^{n-1} (-1)^k \left(\frac{d}{dt}\right)^k p_k(t) \left(\frac{d}{dt}\right)^k + c,$$

we will show that for c sufficiently large $\mathscr{R}(\tau_c f, f) \geqq 0$ for f in $\mathfrak{D}(T_0(\tau))$. This will allow us to complete the proof of the present theorem exactly as in the last paragraph of the preceding theorem. If our assertion is false, then there exists a sequence $\{g_m\}$ of elements of $\mathfrak{D}(T_0(\tau))$ such that $\mathscr{R}(\tau_0 g_m, g_m) \leqq -m|g_m|_2^2$. Hence, there exists a sequence $\{f_m\}$ of elements of $\mathfrak{D}(T_0(\tau))$ such that $\mathscr{R}(\tau_0 f_m, f_m) \to -\infty$ and such that $|f_m|_2 = 1$. If $|f_m^{(n)}|_2$ were unbounded, it would follow by Lemma 6.33 that $|f_m^{(k)}|_2 = O(|f_m^{(n)}|_2)$ for $0 < k < n$, so that, if B is a lower bound for $\mathscr{R}p_1, \mathscr{R}p_2, \ldots, \mathscr{R}p_{n-1}$, we would have

$$\mathscr{R}(\tau_0 f_m, f_m) = \int_I |f_m^{(n)}(t)|^2 dt + \sum_{j=1}^{n-1} \int_I \mathscr{R}p_j(t)|f_m^{(j)}(t)|^2 dt$$

$$\geqq |f_m^{(n)}|_2^2 - B \sum_{j=1}^{n-1} |f_m^{(j)}|_2^2 \geqq |f_m^{(n)}|_2^2(1+O(1)),$$

contradicting the fact that $\mathscr{R}(\tau_0 f_m, f_m) \to -\infty$. If $|f_m^{(n)}|_2$ were bounded it would follow from Lemma 6.33 that $|f_m^{(k)}|_2$ was bounded for each k with $0 \leqq k < n$, and from the first inequality displayed above that $\mathscr{R}(\tau_0 f_m, f_m)$ was bounded, again contradicting the fact that $\mathscr{R}(\tau_0 f_m, f_m) \to -\infty$.

Thus $\mathscr{R}(\tau_0 f, f) \geqq 0$ for all f in $\mathfrak{D}(T_0(\tau))$ if c is sufficiently large, and, as remarked above, this proves the theorem. Q.E.D.

11 THEOREM. *Suppose that τ is a formal differential operator of order n defined on an interval $I = [a, b)$, and that k is an integer not greater than n. Suppose that there exists a finite constant M such that*

$$|f^{(k)}|_2^2 < M\{|\tau f|_2^2 + |f|_2^2\}, \qquad f \in \mathfrak{D}(T_0(\tau)).$$

Let τ_1 be a (regular or irregular) formal differential operator of the form

$$\tau_1 = \sum_{j=0}^{k} a_j(t)\left(\frac{d}{dt}\right)^j,$$

where $\lim_{t\to b} a_j(t) = 0$, $j = 0, \ldots, k$. *Then, if $\tau + \tau_1$ has a non-vanishing leading coefficient,*

$$\sigma_e(\tau) = \sigma_e(\tau + \tau_1).$$

PROOF. Let $\lambda_0 \in \sigma_e(\tau)$ and assume that $\lambda_0 \notin \sigma_e(\tau + \tau_1)$. Then, by Corollary 2, there exists a bounded sequence of functions f_m in $\mathfrak{D}(T_0(\tau))$ such that $\{(\tau - \lambda_0)f_m\}$ converges, and such that $\{f_m\}$ has no strongly convergent subsequence. Since every bounded sequence of functions in Hilbert space contains a weakly convergent subsequence, we may assume without loss of generality that $\{f_m\}$ is weakly convergent. Then by Lemma 2.16, $\{f_m\}$ converges weakly in the topology of $H^{(n)}(J)$ for each compact subinterval $J = [a, b_0]$ of I. It follows from Lemma 2.16(a) that $\{f_m^{(p)}(t)\}$ converges for each t and $0 \leq p \leq n-1$, and by the uniform boundedness principle that $\{f_m^{(p)}(t)\}$ is uniformly bounded for $0 \leq p \leq n-1$ and for t in any compact subinterval J of I. Thus, by the Lebesgue dominated convergence theorem

$$[\dagger] \qquad \lim_{m, m_1 \to \infty} \int_J |f_m^{(p)}(t) - f_{m_1}^{(p)}(t)|^2 dt = 0, \; 0 \leq p \leq n-1.$$

Let

$$\tau = \sum_{j=0}^{n} b_j(t)\left(\frac{d}{dt}\right)^j.$$

Then, since $b_n(t) \neq 0$ and $\{(T - \lambda_0)f_n\}$ converges strongly, it follows from [†] that

$$\lim_{m, m_1 \to \infty} \int^J |f_m^{(n)}(t) - f_{m_1}^{(n)}(t)|^2 dt = 0.$$

Thus

$$\limsup_{m, m_1 \to \infty} \int_J |\tau_1(f_m(t) - f_{m_1}(t))|^2 dt = 0.$$

By Lemma 6.33, $|f^j|_2$ is bounded by some constant N for $j = 0, \ldots, k$. If J is chosen so large that $|a_j(t)| \leq (kN)^{-1}\varepsilon$ for $x \in I - J$, we have

$$\limsup_{m, n \to \infty} \int_{I-J} |\tau_1(f_m(t) - f_n(t))|^2 dt < \varepsilon^2.$$

This proves that

$$\limsup_{m, n \to \infty} |\tau_1(f_m - f_n)|_2 < \varepsilon^2,$$

and, since ε is arbitrary, that

$$\lim_{m, n \to \infty} |\tau_1(f_m - f_n)|_2 = 0.$$

Consequently, $\{\tau_1 f_m\}$ converges. Thus $\lim_{m \to \infty}(\tau + \tau_1 - \lambda_0) f_m$ exists. Since $\lambda_0 \notin \sigma_e(\tau + \tau_1)$, $\{f_m\}$ has a strongly convergent subsequence. But we have seen above that this is impossible. Hence, we have proved that $\sigma_e(\tau) \supseteq \sigma_e(\tau + \tau_1)$.

The remainder of the theorem will follow by symmetry once it is shown that there exists a constant M_1 such that

$$|f^{(k)}|_2^2 < M_1\{|(\tau + \tau_1)f|_2^2 + |f|_2^2\}, \qquad f \in \mathfrak{D}(T_0(\tau + \tau_1)) = \mathfrak{D}(T_0(\tau)).$$

To prove this, it is evidently sufficient to show that there exists a constant M_2 such that

$$|\tau f|_2^2 + |f|_2^2 < M_2\{|(\tau + \tau_1)f|_2^2 + |f|_2^2\}, \qquad f \in \mathfrak{D}(T_0(\tau)).$$

If this is not the case, there evidently exists a sequence $\{f_m\}$ of elements of $\mathfrak{D}(T_0(\tau))$ such that $|(\tau + \tau_1)f_m|_2 \to 0$ and $|f_m|_2 \to 0$, while $|\tau f_m|_2$ does not converge to zero, so that $|\tau_1 f_m|_2$ does not converge to zero. By hypothesis and Lemma 6.38, $|f_m^{(j)}|_2$ is uniformly bounded in m for $j = 0, \ldots, k$. Thus, since the coefficients of τ_1 approach zero as $t \to b$, we have

$$\lim_{c\to b}\int_c^b |(\tau_1 f_m)(t)|^2 dt = 0$$

uniformly in m. On the other hand, we have

$$\lim_{m\to\infty}\int_a^c |(\tau_1 f_m)(t)|^2 = 0$$

for each $a < c < b$, by Corollary 2.15. Thus $|\tau_1 f_m|_2 \to 0$. This contradiction proves the theorem. Q.E.D.

12 COROLLARY. *Let τ be a formal differential operator of the form*

$$\tau = \sum_{j=0}^{n} p_j(t)\left(\frac{d}{dt}\right)^j$$

defined on the interval $I = [a, b)$. Suppose that all the coefficients $p_0, \ldots, p_{n-1}$ are uniformly bounded on I, and that $|p_n(t)|$ is bounded below on I. Let τ_1 be a (regular or irregular) formal differential operator of the form

$$\tau_1 = \sum_{j=0}^{n} a_j(t)\left(\frac{d}{dt}\right)^j,$$

where $\lim_{t\to b} a_j(t) = 0$, $j = 0, \ldots, n$. Then, if $\tau+\tau_1$ has a non-vanishing leading coefficient,

$$\sigma_e(\tau) = \sigma_e(\tau+\tau_1).$$

PROOF. This will follow from the preceding theorem as soon as we establish the existence of a constant M such that

$$|f^{(n)}|_2^2 \leq M\{|\tau f|_2^2 + |f|_2^2\}, \qquad f \in \mathfrak{D}(T_0(\tau)).$$

If no such M exists, it is clear that we can find a sequence $\{f_m\}$ of elements of $\mathfrak{D}(T_0(\tau))$ such that $|f_m^{(n)}|_2 = 1$, $|\tau f_m|_2 \to 0$, $|f_m|_2 \to 0$. Since the coefficients $p_0, \ldots, p_{n-1}$ are bounded, it follows immediately from Lemma 6.33 that

$$|\tau f_m - p_n(\cdot) f_m^{(n)}(\cdot)|_2 \to 0$$

Since p_n is bounded below and $|f_m^{(n)}|_2 = 1$, $|p_n(\cdot) f_m^{(n)}(\cdot)|_2$ is bounded below. Thus $|\tau f_m|_2$ does not approach zero, a contradiction. Q.E.D.

13 Corollary. *Let τ be a formal differential operator of the form*

$$\tau = \sum_{j=0}^{n} p_j(t)\left(\frac{d}{dt}\right)^j,$$

defined on an interval $I = [a, \infty)$. Suppose that $\lim_{t\to\infty} p_j(t) = q_j$ exists for $j = 0, \ldots, n$, and that $q_n \neq 0$. Then

$$\sigma_e(\tau) = \{\lambda | \lambda = \sum_{0}^{n} q_j(it)^j, -\infty < t < +\infty\}.$$

Proof. Let τ_1 be the formal differential operator

$$\tau_1 = \sum_{j=0}^{n} q_j\left(\frac{d}{dt}\right)^j,$$

defined on the interval $(-\infty, +\infty)$, and let τ_2 be its restriction to $[a, \infty)$. By the preceding theorem we have only to show that

$$\sigma_e(\tau_2) = \{\lambda | \lambda = \sum_{0}^{n} q_j(it)^j, -\infty < t < +\infty\} = V.$$

To do this, we will first show that $\sigma_e(\tau_2) = \sigma_e(\tau_1)$, and then show that $\sigma_e(\tau_1) = V$. Since $\mathfrak{D}(T_0(\tau_2)) \subseteq \mathfrak{D}(T_0(\tau_1))$, it follows immediately from Corollary 2 that $\sigma_e(\tau_2) \subseteq \sigma_e(\tau_1)$. On the other hand, let $\lambda \in \sigma_e(\tau_1)$. Then by Corollary 2 there exists a bounded sequence of functions f_n in $\mathfrak{D}(T_0(\tau))$ such that $\{(\tau_1 - \lambda)f_n\}$ converges, but $\{f_n\}$ has no strongly convergent subsequence. Since $\{f_n\}$ is not a Cauchy sequence, there is an $\varepsilon > 0$ and a subsequence $\{f_{n_j}\}$ of $\{f_n\}$ such that $|f_{n_{i+1}} - f_{n_i}| \geq \varepsilon$, $i \geq 1$. Put $g_i = f_{n_{i+1}} - f_{n_i}$. Then $\{(\tau_1 - \lambda)g_n\}$ converges to zero, but $\{g_n\}$ does not converge to zero. Let the translation operator T_p be defined by $(T_pf)(t) = f(t-p)$. It is clear that if $f, g \in \mathfrak{D}(T_0(\tau_1))$, T_pf and g are orthogonal for sufficiently large p, so that

$$|T_pf - g| = (|f|^2 + |g|^2)^{1/2}$$

for sufficiently large p. For $n \geq 1$, let p_n be chosen so large that $h_n = T_{p_n}g_n$ vanishes outside $[a, \infty)$, and so large that h_n is orthogonal to $h_1, \ldots, h_{n-1}$. Then $h_n \in \mathfrak{D}(T_0(\tau_2))$,

$$|(\tau_2 - \lambda)h_n| = |(\tau_1 - \lambda)g_n| \to 0,$$

while

$$|h_n - h_m| - (|g_n|^2 + |g_m|^2)^{1/2} > \varepsilon, \qquad n, m \geq 1.$$

Thus $\{(\tau_2 - \lambda)h_m\}$ converges to zero, while $\{h_m\}$ contains no convergent subsequences, so that, by Corollary 2, $\lambda \in \sigma_e(\tau_2)$. Thus $\sigma_e(\tau_2) = \sigma_e(\tau_1)$.

Let ρ be the formally symmetric formal differential operator

$$\rho = \frac{1}{i}\frac{d}{dt}$$

defined on the interval $(-\infty, +\infty)$, and let T be the unique self adjoint extension of ρ (cf. remarks following Definition 2.20, Case 3) Then, as established in Section 5 (cf. remarks after Corollary 5.30) the spectrum of T is a continuous spectrum covering the entire real axis. Let T_1 be the polynomial function $\sum_{j=0}^{n} q_j(iT)^j$ of T. The operator T_1 is clearly an extension of $T_0(\tau_1)$. By Corollary XII.2.8 and Theorem XII.2.6, T_1 is a closed operator. By XII.2.6(d)

$$T_1^* = \sum_{j=0}^{n} \overline{q_j}(-i)^j T^j.$$

Thus $T_1^* \supseteq T_0(\tau_1^*)$, so that

$$T_1 = T_1^{**} \subseteq T_0(\tau_1^*)^* = T_1(\tau_1)$$

by Theorem 2.10. By Corollary 3, $\sigma_e(\tau_1) = \sigma_e(T_1)$. By Theorem XII.2.9(b), $\sigma(T_1)$ is included in the range V of the polynomial $\sum_{j=0}^{n} q_j(it)^j$. Thus $\sigma_e(T_1) \subseteq V$. On the other hand, let

$$\lambda = \sum_{j=0}^{n} q_j(it_0)^j.$$

Let $\{I_n\}$ be a sequence of disjoint open intervals of the real axis, such that $I_n \subseteq \{t \mid |t - t_0| < 1/n\}$. Since $\sigma(T)$ is the entire real axis, $E(I_n)$ is non-zero for each n by Theorem XII.2.9(b). Hence, there exists a sequence $\{f_n\}$ in $\mathfrak{D}(T)$ such that $E(I_n)f_n = f_n$, $n \geq 1$, and $|f_n| = 1$. By XII.2.6(c), $T_1 f_n - \lambda f_n$ approaches zero as $n \to \infty$. On the other hand, since f_n and f_m are orthogonal for $n \neq m$,

$$|f_n - f_m|^2 = |f_n|^2 + |f_m|^2 = 2, \qquad n \neq m,$$

so that $\{f_n\}$ has no convergent subsequence. Thus $\lambda \in \sigma_e(T_1) = \sigma_e(\tau)$. Q.E.D.

14 Corollary. *Let τ be a formally self adjoint formal differential operator of the form*

$$\tau = \sum_{j=0}^{n} p_j(t)\left(\frac{d}{dt}\right)^j,$$

defined on an interval $[a, \infty)$. Suppose that $\lim_{t\to\infty} p_j(t) = q_j$ exists for $j = 0, \ldots, n$ and that $q_n \neq 0$. Then

(a) *if n is odd, $\sigma_e(\tau)$ is the entire real axis;*

(b) *if n is even, and $(-1)^{n/2} q_n \geq 0$, $\sigma_e(\tau)$ is the positive half-axis bounded below by*

$$\lambda_0 = \min_{-\infty < t < +\infty} \sum_{j=0}^{n} q_j (it)^j.$$

Proof. Since τ is formally self adjoint, $(\tau f, f) = (f, \tau f)$ for f in $\mathfrak{D}(T_0(\tau))$, so that $(\tau f, f)$ is real. Hence, by Theorem 5, $\sigma_e(\tau)$ is contained in the real axis. By the previous theorem the polynomial $P(t) = \sum_{j=0}^{n} q_j (it)^j$ is real. If n is odd, this polynomial is of odd order. Hence, it converges to $+\infty$ as t approaches $+\infty$, and to $-\infty$ as t approaches $-\infty$, and consequently takes on all real values. Statement (a) follows immediately from this by the preceding theorem.

If n is even and $i^n q_n = (-1)^{n/2} q_n \geq 0$, then $P(x)$ converges to $+\infty$ as t approaches $\pm\infty$. Thus $P(x)$ takes on all values from $+\infty$ to its minimum. This proves (b). Q.E.D.

15 Corollary. *Let τ be a formal differential operator of the form*

$$\tau = \sum_{j=0}^{n} (-1)^j \left(\frac{d}{dt}\right)^j p_j(t) \left(\frac{d}{dt}\right)^j,$$

defined on an interval $I = [a, b)$. Suppose that $\mathscr{R}p_j$ is bounded below, $j = 0, \ldots, n$, and that $\mathscr{R}p_n$ is bounded below by a positive constant. Let τ_1 be a regular or irregular formal differential operator of the form

$$\tau_1 = \sum_{j=0}^{n} a_j(t)\left(\frac{d}{dt}\right)^j,$$

where $\lim_{t\to b} a_j(t) = 0$, $j = 0, \ldots, n$. Then τ and $\tau + \tau_1$ have the same essential spectrum.

Proof. This will follow immediately from Theorem 11 as soon as it is established that there exists a positive constant M such that

$$|f^{(n)}|_2^2 \le M\{|\tau f|_2^2 + |f|_2^2\}, \qquad f \in \mathfrak{D}(T_0(\tau)).$$

If this is false, we can find a sequence $\{f_m\}$ of elements of $\mathfrak{D}(T_0(\tau))$ such that $|\tau f_m|_2 \to 0$, $|f_m|_2 \to 0$, while $|f_m^{(n)}|_2 = 1$. By Lemma 6.88, $|f_m^{(k)}|_2 \to 0$ for $0 \le k < n$. Thus, since $\mathscr{R}p_0, \ldots, \mathscr{R}p_{n-1}$ are bounded below,

$$\liminf_{m\to\infty} \mathscr{R}\int_I \left\{\sum_{j=0}^{n-1} p_j(t)|f_m^{(j)}(t)|^2\right\}dt \ge 0,$$

so that, if M is a positive lower bound for $\mathscr{R}p_n$, we have

$$\begin{aligned} 0 = \liminf_{m\to\infty} \mathscr{R}(\tau f_m, f_m) &\ge \liminf_{m\to\infty} \mathscr{R}\int_I \left\{\sum_{j=0}^{n-1} p_j(t)|f_m^{(j)}(t)|^2\right\}dt \\ &\ge M \liminf_{m\to\infty} |f_m^{(n)}|_2^2 = M. \end{aligned}$$

This contradiction establishes the result. Q.E.D.

The theorems given above are only a small sampling of the results which can be given. Additional results are given in the exercises at the end of the chapter.

The theorems proved above give a rather complete picture of the nature of the spectrum of real self adjoint second order operators, all the parts of which are brought together for convenient reference in the following two theorems.

16 Theorem. *Let τ be a real second order operator of the form*

$$\left(\frac{d}{dt}\right)^2 + q(t),$$

defined on an interval $I = [a, \infty)$.

(a) *If $q(t) \to \infty$ as $t \to \infty$, then $\sigma_e(\tau)$ is void.*

(b) *If $q(t) \to c$ as $t \to \infty$, then $\sigma_e(\tau) = \{\lambda | \lambda \ge c\}$.*

(c) *If $q(t) \to -\infty$, if*

$$\int_{a_0}^{\infty} \left| \left(\frac{q(t)'}{|q(t)|^{3/2}}\right)' - \frac{1}{4}\frac{(q(t)')^2}{|q(t)|^{5/2}} \right| dt < \infty,$$

for a_0 sufficiently large, and if

$$\int_{a_0}^{\infty} |q(t)|^{-1/2} dt < \infty$$

for a_0 sufficiently large, then $\sigma_e(\tau)$ is void.

(d) *If $q(t) \to -\infty$, if q is monotone decreasing for sufficiently large t, if*

$$\int_{a_0}^{\infty} \left| \left(\frac{q(t)'}{|q(t)|^{3/2}} \right)' - \frac{1}{4} \frac{(q(t)')^2}{|q(t)|^{5/2}} \right| dt < \infty$$

for a_0 sufficiently large, and

$$\int_{a_0}^{\infty} |q(t)|^{-1/2} dt = \infty$$

for all a_0, then $\sigma_e(\tau)$ is the entire real axis.

PROOF. Parts (a) and (b) follow immediately from Theorem 9 and Corollary 14 respectively. Part (c) follows from Corollary 6.22 and from Corollary 6.11. Part (d) follows from Corollary 6.21(b). Q.E.D.

It should be noted that by use of Theorem 4 the preceding theorem may also be made to cover the case in which I is the interval $(-\infty, +\infty)$. We leave the detailed formulation of the appropriate results to the reader.

A result similar to the preceding theorem may be stated for the case in which I is an interval $(a, b]$ with a finite. For the sake of simplicity, we suppose without loss of generality that $a = 0$.

17 THEOREM. *Let τ be a real second order operator of the form*

$$-\left(\frac{d}{dt}\right)^2 + q(t),$$

defined on an interval $I = (0, b]$.

(a) *If $q(t) \to \infty$ as $t \to 0$, then $\sigma_e(\tau)$ is void.*

(b) *If $\limsup_{t\to 0}|t^2 q(t)| < 3/4$, then $\sigma_e(\tau)$ is void.*

(c) *If $q(t) \to -\infty$ as $t \to 0$, if*

$$\int_0^{b_0} \left| \left(\frac{q'(t)}{|q(t)|^{3/2}}\right)' - \frac{1}{4}\frac{(q(t)')^2}{|q(t)|^{5/2}} \right| dt < \infty$$

for sufficiently small b_0, *and if*

$$\int_0^{b_0} |q(t)|^{-1/2} dt < \infty$$

for sufficiently small b_0, *then* $\sigma_e(\tau)$ *is void.*

(d) *If* $q(t) \to -\infty$ *as* $t \to 0$, $q(t)$ *is monotone decreasing for sufficiently small* t,

$$\int_0^{b_0} \left| \left(\frac{q'(t)}{|q(t)|^{3/2}}\right)' - \frac{1}{4}\frac{q(t)'^2}{|q(t)|^{5/2}} \right| dt < \infty$$

for sufficiently small b_0, *and if*

$$\int_0^{b_0} |q(t)|^{-1/2} dt = \infty$$

for all $b_0 > 0$, *then* $\sigma_e(\tau)$ *is the entire real axis.*

PROOF. Part (a) follows immediately from Theorem 9. Part (b) follows from Corollary 6.12, and from Theorem 6.23(b). Part (c) follows from Theorem 6.23(c) and from Corollary 6.12. Part (d) follows from Corollary 6.21(b). Q.E.D.

A number of additional criteria of this sort for the second order operator

$$\left(\frac{d}{dt}\right)^2 + q(t)$$

follow from the corresponding theorems of Section 6. Several of these are given as exercises at the end of the chapter.

Theorem 8 may evidently be generalized as follows.

18 THEOREM. *Suppose that all the values of the coefficients* p_k, $k > 1$, *of the formal differential operator*

$$\tau = \sum_{k=0}^{n} (-1)^k \left(\frac{d}{dt}\right)^k p_k(t) \left(\frac{d}{dt}\right)^k, \qquad t \in [a, b),$$

lie in the right half-plane, and that $\liminf_{t\to b} \mathscr{R}p_0(t) \geq \lambda_0$. *Then the essential spectrum* $\sigma_e(\tau)$ *lies in the half-plane* $\mathscr{R}z \geqq \lambda_0$.

Proof. If τ' is the restriction of τ to an interval $[c, b)$, then, by Theorem 4, $\sigma_e(\tau) = \sigma_e(\tau')$. If any real number $\lambda < \lambda_0$ is given, we can choose c such that $\mathscr{R}p_0(t) \geq \lambda$ for $x \in [c, b)$. Then we have

$$\mathscr{R}(\tau' f, f) = \sum_{k=0}^{n} \int_c^b (\mathscr{R}p_k(t))|f^{(k)}(t)|^2 dt \geq \lambda |f|^2, \qquad f \in \mathfrak{D}(T_0(\tau')).$$

It follows from Theorem 5 that $\sigma_e(\tau)$ lies entirely in the half-plane $\mathscr{R}z \geqq \lambda$. Since λ is any number smaller than λ_0, $\sigma_e(\tau)$ lies entirely in the half-plane $\mathscr{R}z \geqq \lambda_0$. Q.E.D.

19 Corollary. *Let* τ *be a real self adjoint second order formal differential operator of the form*

$$\tau = -\left(\frac{d}{dt}\right) p(t) \left(\frac{d}{dt}\right) + q(t)$$

defined on an interval $[a, b)$. *Suppose that* $p(t) > 0$ *for* $t \in [a, b)$. *Then, if* $\liminf_{t\to b} q(t) = \lambda_0$, *every* $\lambda \in \sigma_e(\tau)$ *satisfies* $\lambda \geqq \lambda_0$.

Somewhat stronger results can be stated for formally symmetric formal differential operators which are bounded below in the sense of Definition XII.5.1, which is reproduced below for the convenience of the reader.

Definition XII.5.1. A symmetric operator T is *bounded above* (*bounded below*) if there is a real number c such that $(Tx, x) \leqq c(x, x)$ ($(Tx, x) \geq c(x, x)$) for all x in $\mathfrak{D}(T)$. The number c is called a *bound* for T, and the smallest (largest) such c is called the *upper* (*lower*) bound for T. If $(Tx, x) \geq 0$ for all x in $\mathfrak{D}(T)$, then T is called *non-negative.*

20 Definition. Let τ be a formally symmetric formal differential operator. Then τ is said to be *bounded below* if and only if $T_0(\tau)$ is bounded below.

21 Lemma. *A self adjoint operator* T *is bounded below in the sense of Definition* XII.5.1 *if and only if* $\sigma(T)$ *is bounded below as a*

subset of the real axis, i.e., if and only if there exists a finite μ such that $\mu \leq \lambda$ for each $\lambda \in \sigma(T)$.

Proof. Suppose that such a μ exists. Then, by Theorem XII.2.6,

$$(Tx, x) = \int_{\mu}^{\infty} \lambda |E(d\lambda)x|^2 \geq \mu |x|^2, \qquad x \in \mathfrak{D}(T),$$

so that if $|x|$ is bounded, (Tx, x) is bounded below.

Conversely, suppose that for each n, $e_n = (-\infty, -n) \cap \sigma(T)$ is non-void. By Theorem XII.2.9(b), $E(e_n) \neq 0$ for any n. Using this fact, choose an x_n such that $E(e_n)x_n = x_n$, $x_n \in \mathfrak{D}(T)$, $|x_n| = 1$. Then, by Theorem XII.2.6,

$$(Tx_n, x_n) = \int_{-\infty}^{-n} \lambda |E(d\lambda)x_n|^2 < -n|x_n|^2, \qquad n \to \infty.$$

Thus T is not bounded below. Q.E.D.

22 Lemma. *Let T_1 and T_2 be two symmetric operators in Hilbert space. Suppose that $T_1 \subsetneqq T_2$, and that $\mathfrak{D}(T_2) = \mathfrak{D}(T_1) + \mathfrak{N}$, where $\mathfrak{N}$ is finite dimensional. Then T_1 is bounded below if and only if T_2 is bounded below.*

Proof. It is evident from Definition XII.5.1 that if T_2 is bounded below, T_1 must also be bounded below. Conversely, let T_1 be bounded below. Suppose that the lemma is false, so that T_2 is not bounded below.

We shall show by induction that for each $K > 0$ and each integer n there exists an n-dimensional subspace $\mathfrak{S}_n(K)$ of $\mathfrak{D}(T_2)$ such that

$$(T_2 x, x) < -K|x|^2, \qquad x \in \mathfrak{S}_n(K).$$

For $n = 1$, this is evident from Definition XII.5.1 and the hypothesis that T_2 is not bounded below. Suppose that $\mathfrak{S}_n(K)$ exists for a given n and for each K, but that K_0 is a positive constant for which $\mathfrak{S}_{n+1}(K_0)$ does not exist. Let E be the orthogonal projection of Hilbert space $\mathfrak{H}$ on $\mathfrak{S}_n(2K_0)$. Let T be the mapping of $\mathfrak{S}_n(2K_0)$ into $\mathfrak{H}$ defined by the equation $Tx = T_2 x$, $x \in \mathfrak{S}_n(2K_0)$, and let $|T|$ be its norm. Let $m = \max(|T|, |T|/K_0, K_0)$.

It cannot be the case that

$$(T_2 y, y) \geq -m|y|^2, \qquad y \in \mathfrak{D}(T_2) \cap (\mathfrak{S}_n(2K_0))^{\perp}.$$

For if this were the case, then we would have

$$\begin{aligned}(T_2 x, x) &= (T_2(Ex+(I-E)x), Ex+(I-E)x)\\ &= (T_2Ex, Ex)+(T_2(I-E)x, (I-E)x)\\ &\quad+(T_2Ex, (I-E)x)+((I-E)x, T_2Ex)\\ &\geq -|T|\,|Ex|^2-2|T|\,|Ex|\,|(I-E)x|-m|(I-E)x|^2\\ &\geq -2m(|Ex|^2+|(I-E)x|^2) = -2m|x|^2,\end{aligned}$$

contradicting the assumption that T_2 is not bounded below. Consequently, there exists a unit vector y, $y \in \mathfrak{D}(T_2) \cap (\mathfrak{S}_n(2K_0))^{\perp}$, such that $(T_2 y, y) < -m$. Let $\mathfrak{S}$ be the space spanned by $\mathfrak{S}_n(2K_0)$ and y. Then every x in $\mathfrak{S}$ may be written uniquely as $x = u+\alpha y$ with u in $\mathfrak{S}_n(2K_0)$. We have

$$\begin{aligned}(T_2 x, x) &= (T_2(u+\alpha y), (u+\alpha y))\\ &= (T_2u, u)+(T_2u, \alpha y)+(\alpha y, T_2u)+(T_2\alpha y, \alpha y)\\ &\leq -2K_0|u|^2+2|T||u||\alpha|-m|\alpha|^2\\ &\leq -K_0(|u|^2+|\alpha|^2)-K_0|u|^2+2|T|\,|u|\,|\alpha|\\ &\qquad\qquad -(|T|/K_0)|\alpha|^2\\ &= -K_0|x|^2-K_0(|u|-(|T|/K_0)|\alpha|)^2 \leq -K_0|x|^2\end{aligned}$$

for each x in $\mathfrak{S}$. We may consequently take $\mathfrak{S}_{n+1}(K_0) = \mathfrak{S}$, thus proving the existence of $\mathfrak{S}_{n+1}(K_0)$, and hence the existence of $\mathfrak{S}_n(K)$ for all $n \geq 1$, $K > 0$.

It is clear from Definition XII.5.1 that there exists a constant K such that

$$(T_1 x, x) \geq -K(x, x), \qquad x \in \mathfrak{D}(T_1),$$

from which it is evident (cf. XII.4.6) that we have, more generally,

$$(\overline{T}_1 x, x) \geq -K(x, x), \qquad x \in \mathfrak{D}(\overline{T}_1).$$

The graph $\overline{\Gamma}_1$ of $\overline{T}_1$ is a subspace of the graph $\overline{\Gamma}_2$ of T_2, which is in turn a subspace of the direct sum $\mathfrak{H} \oplus \mathfrak{H}$ of two replicas of Hilbert space. Since T_1 is closed, Γ_1 is a closed subspace of $\overline{\Gamma}_2$. By hypothesis, $\Gamma_2 = \overline{\Gamma}_1+\mathfrak{M}$, where $\mathfrak{M}$ is finite dimensional. Let P be the orthogonal

projection of Γ_2 on Γ_1. Then $\Gamma_2 = \Gamma_1 \oplus (I - P)\Gamma_2$. On the other hand, $\mathfrak{M}_1 = (I - P)\Gamma_2 \subseteq (I - P)\mathfrak{M}$ is finite dimensional. We have consequently $\Gamma_2 = \Gamma_1 \oplus \mathfrak{M}_1$, where $\mathfrak{M}_1$ is finite dimensional; let the dimension of $\mathfrak{M}_1$ be n_1. Let P_1 be the orthogonal projection of Γ_2 on $\mathfrak{M}_1$. For x in $\mathfrak{D}(T_2)$, define $P_2 x = P_1[x, T_2 x]$. Then P_2 is a (discontinuous) linear mapping of $\mathfrak{D}(T_2)$ into the n-dimensional space $\mathfrak{M}_1$. There consequently exists an element $x \neq 0$ in the $(n+1)$-dimensional space $\mathfrak{S}_{n+1}(2K)$ constructed above such that $P_2 x = 0$. But, if $P_2 x = 0$, it is clear that x is in $\mathfrak{D}(T_1)$. Thus, we have on the one hand $x \in \mathfrak{D}(T_1)$, so that $(T_1 x, x) \geq K(x, x)$, and on the other hand $x \in \mathfrak{S}_{n+1}(2K)$, so that $(T_1 x, x) = (T_2 x, x) \leq -2K(x, x)$. This contradiction proves the present lemma. Q.E.D.

23 Lemma. *If T is a closed symmetric operator in Hilbert space, and T is bounded below, then*

(a) *the essential spectrum of T is a subset of the real axis which is bounded below;*

(b) *the deficiency indices of T are equal.*

Proof. To prove (a), note that if T is bounded below, there exists a constant K such that

$$(Tx, x) \geq -K(x, x), \qquad x \in \mathfrak{D}(T).$$

The proof of Theorem 5 now shows that $\sigma_e(T)$ is a subset of the half-axis $\infty > t \geq -K$. Since T is bounded below, it follows from Theorem XII.5.2 and Corollary XII.4.13(a) that the deficiency indices of T are equal. This proves (b). Q.E.D.

24 Corollary. *If τ is a formally symmetric formal differential operator, and τ is bounded below, then*

(a) *the essential spectrum of τ is a subset of the real axis which is bounded below;*

(b) *the deficiency indices of $T_0(\tau)$ are equal;*

(c) *all symmetric extensions of $T_0(\tau)$ are bounded below.*

Proof. Statements (a) and (b) follow immediately from the preceding lemma, by virtue of Definitions 20 and 6. Statement (c) follows from Lemma 22 by virtue of the finiteness of the deficiency indices of $T_0(\tau)$ (cf. 2.18), and from Lemma XII.4.11. Q.E.D.

25 Definition. (a) If T is a closed symmetric operator in Hilbert space which is bounded below and whose essential spectrum $\sigma_e(T)$ does not intersect the interval $(-\infty, \lambda)$ of the real axis, we say that *T is finite below λ.*

(b) If τ is a formally symmetric formal differential operator which is bounded below, and whose essential spectrum $\sigma_e(\tau)$ does not intersect the interval $(-\infty, \lambda)$ of the real axis, we say that *τ is finite below λ.*

26 Corollary. *Let τ be a formally symmetric formal differential operator, and let T be any closed symmetric extension (in particular, any self adjoint extension) of $T_0(\tau)$. Then τ is finite below λ if and only if T is finite below λ.*

Proof. This follows immediately from Lemma 22 and Corollary 6.3. Q.E.D.

27 Corollary. *A self adjoint operator T is finite below λ if and only if for each $\varepsilon > 0$ the set $(-\infty, \lambda-\varepsilon) \cap \sigma(T)$ is finite.*

Proof. Suppose that $(-\infty, \lambda-\varepsilon) \cap \sigma(T)$ is finite. Then $\sigma(T)$ is clearly a subset of the real axis which is bounded below, so that T is bounded below by Lemma 21. By Theorem 6.5, $\sigma_e(T) \cap (-\infty, \lambda-\varepsilon)$ is void. Thus T is finite below λ.

Conversely, suppose that T is finite below λ. Then by Theorem 6.5 and Lemma 21, $(-\infty, \lambda-\varepsilon) \cap \sigma(T)$ is finite for each $\varepsilon > 0$. Q.E.D.

The next theorem is closely related to Theorem 4.

28 Theorem. *Let τ be a formal differential operator defined on an interval I with end points a, b, and let I be the union of two (not necessarily disjoint) subintervals I_1 and I_2. Let the restriction of τ to I_1 (to I_2) be denoted by τ_1 (by τ_2). Then τ is bounded below if and only if τ_1 and τ_2 are both bounded below.*

Proof. First suppose that τ is bounded below. Let $\{f_n\}$ be a sequence of elements in $\mathfrak{D}(T_0(\tau_1))$ which is bounded in L_2-norm. Then f_n is in $\mathfrak{D}(T_0(\tau))$, and since $T_0(\tau)$ is bounded below, the sequence of quantities $(T_0(\tau_1)f_n, f_n) = (T_0(\tau)f_n, f_n)$ is bounded below. That is, $(T_0(\tau_1)f, f)$ is bounded below as f varies over any set in $\mathfrak{D}(T_0(\tau_1))$ for

which (f, f) is bounded. From this, it is evident that $T_0(\tau_1)$, and hence τ_1, is bounded below. Similarly, τ_2 is bounded below.

Conversely, suppose that τ is of order n and that τ_1 and τ_2 are both bounded below. Suppose for definiteness that I_1 contains a neighborhood of the left end point a of I, so that, unless $I_1 = I$ (in which case $\tau = \tau_1$ and τ is evidently bounded below), I_2 contains a neighborhood of the right end point b of I.

Unless $I_2 = I$, in which case we again have nothing to prove, both the right end point c_1 of I_1 and the left end point c_2 of I_2 are interior to I. Let $\hat{I}_1 = I_1 \cup c_1$, $\hat{I}_2 = I_2 \cup c_2$, and let $\hat{\tau}_1$ and $\hat{\tau}_2$ be the restriction of τ to $\hat{I}_1$ and $\hat{I}_2$ respectively. Then obviously $\mathfrak{D}(T_0(\hat{\tau}_1)) = \mathfrak{D}(T_0(\tau_1))$, $i = 1, 2$. We may, consequently, suppose without loss of generality that $c_1 \in I_1$, $c_2 \in I_2$, i.e., that I_1 contains its right end point, and I_2 contains its left end point. Let c be a point interior to I which is common to I_1 and I_2. Then $(a, c] \subseteq I_1$, and $[c, b) \subseteq I_2$. Let $g_0, g_1, g_2, \ldots, g_{n-1}$ be a set of n functions in $\mathfrak{D}(T_0(\tau))$ such that

$$
\begin{aligned}
g_i^{(j)}(c) &= 1, \qquad 0 \leq i = j \leq n-1, \\
&= 0, \qquad 0 \leq i \neq j \leq n-1,
\end{aligned}
$$

and let $\mathfrak{N}$ be the n-dimensional subspace of $\mathfrak{D}(T_0(\tau))$ spanned by these functions. Let $\mathfrak{D}$ be the subspace of $\mathfrak{D}(T_0(\tau))$ determined by the condition

$$
\mathfrak{D} = \{f \in \mathfrak{D}(T_0(\tau)) \mid f^{(j)}(c) = 0,\ 0 \leq j \leq n-1\}.
$$

Then $\mathfrak{D}(T_0(\tau)) = \mathfrak{D} + \mathfrak{N}$. Hence, if T denotes the restriction of $T_0(\tau)$ to $\mathfrak{D}$, it follows from Lemma 22 that to prove $T_0(\tau)$ bounded below, it suffices to show that T is bounded below.

Since τ_1 and τ_2 are bounded below, there exists a constant K such that

$$
\begin{aligned}
(T_0(\tau_1)f, f) &\geq -K(f, f), \qquad f \in \mathfrak{D}(T_0(\tau_1)), \\
(T_0(\tau_2)f, f) &\geq -K(f, f), \qquad f \in \mathfrak{D}(T_0(\tau_2)).
\end{aligned}
$$

It then follows immediately (cf. Definition XII.4.7) that

$$
\begin{aligned}
(\overline{T_0(\tau_1)}f, f) &\geq -K(f, f), \qquad f \in \mathfrak{D}(\overline{T_0(\tau_1)}), \\
(\overline{T_0(\tau_2)}f, f) &\geq -K(f, f), \qquad f \in \mathfrak{D}(\overline{T_0(\tau_2)}).
\end{aligned}
$$

Let $h \in \mathfrak{D}$. Put

$$h_1(t) = h(t), \quad t \leq c; \qquad h(t) = 0, \quad t > c,$$
$$h_2(t) = h(t), \quad t \geqq c; \qquad h(t) = 0, \quad t < c.$$

Then it is evident that $h_1 \in \mathfrak{D}(T_1(\tau_1))$, $h_2 \in \mathfrak{D}(T_1(\tau))$. Since h_1 vanishes in a neighborhood of the left end point of I_1, and vanishes together with its first $n-1$ derivatives at the right end point of I_1, it follows from Theorem 2.19 and Corollary 2.23 that if B is any boundary value for $T_0(\tau_1)$, $B(h_1) = 0$. It then follows from Definition XII.4.7 and Lemma XII.4.26 that $h_1 \in \mathfrak{D}(\overline{T_0(\tau_1)})$. Hence

$$(T_1(\tau_1)h_1, h_1) > -K(h_1, h_1).$$

Similarly,

$$(T_1(\tau_2)h_2, h_2) > -K(h_2, h_2).$$

Adding, we find

$$(Th, h) = (T_1(\tau)h, h) \geq -K(h, h), \qquad h \in \mathfrak{D}(T),$$

so that T is bounded below. Q.E.D.

29 LEMMA. *Let τ be a formally symmetric formal differential operator on a finite closed interval I. Then, if τ is of odd order, it is not bounded below. If τ is of even order $2n$, and the leading coefficient $a_{2n}(t)$ of τ (which is necessarily real and non-vanishing) satisfies $(-1)^n a_{2n}(t) > 0$, τ is finite below any finite λ, but $-\tau$ is not finite below.*

PROOF. Let $\tau = a_m(t)(d/dt)^m + \ldots$, and m be odd. Then $\tau = \tau^* = -\overline{a_m(t)}(d/dt)^m + \ldots$, so that a_m is pure imaginary. (Similarly, if m is even, a_m is real.) Let φ be a function in $C^\infty(I)$ which vanishes near the end points of I. Then $\tau(\varphi(t)e^{ikt})$ has the asymptotic behavior of $(ik)^m a_m(t)e^{ikt}\varphi(t)$ as k approaches $\pm\infty$. Consequently, by making k approach plus (or minus) infinity, we can make $(\tau\varphi(\cdot)e^{ik\cdot}, \varphi(\cdot)e^{ik\cdot})$ approach $\pm\infty$ (or $\mp\infty$). Since the set of functions $\{\varphi(\cdot)e^{ik\cdot}\}$ is bounded this shows that τ is bounded neither above nor below. In the same way we may show, in case $m = 2n$ is even and $(-1)^n a_{2n}(t) > 0$, that $-\tau$ is not finite below.

Since $\sigma_e(\tau) = \phi$ by Theorems 4.1 and 4.2, in order to complete our proof, it suffices to show that in case $m - 2n$ is even and $(-1)^n a_{2n}(t) \geqq 0$, τ is bounded below. To do this, it is evidently sufficient by Theorem 28 to show that there exists an $\varepsilon > 0$ so small that if J is a subinterval of I of length at most ε, then the restriction $\hat{\tau}$ of τ to J is bounded below.

Let b be a lower bound for $(-1)^n a_{2n}(t)$, and let B be an upper bound for $|a_k^{(j)}(t)|$, $0 \leqq j, k \leqq 2n$. Then, if $f \in \mathfrak{D}(T_0(\hat{\tau}))$,

$$
\begin{aligned}
(T_0(\hat{\tau})f, f) = (-1)^n \int_J a_{2n}(t) |f^{(n)}(t)|^2 \, dt \\
+ \sum_{j=1}^{2n-1} \int_J a_j(t) f^{(j)}(t) \overline{f(t)} \, dt \\
\geq b \int_J |f^{(n)}(t)|^2 \, dt + \sum_{j=1}^{2n-1} \int_J a_j(t) f^{(j)}(t) \overline{f(t)} \, dt.
\end{aligned}
$$

If the sum on the right is repeatedly integrated by parts, it may be written as the sum of a certain number N of integrals with integrands of the form $\pm a_j^{(k)}(t) f^{(l)}(t) f^{(p)}(t)$, $0 \leq k \leqq 2n$, $0 < l \leqq n$, $0 < p < n$, where N depends only on n. Thus we find

[*] $$
\begin{aligned}
(T_0(\hat{\tau})f, f) \geqq b \int_J |f^{(n)}(t)|^2 \, dt \\
- NB \max_{\substack{0 \leq l \leq n \\ 0 \leq p < n}} \int_J |f^{(l)}(t)| \, |f^{(p)}(t)| \, dt.
\end{aligned}
$$

Now, if g vanishes in a neighborhood of the left end point a of J, and J is of length at most ε, then for t in the interval J

$$
\begin{aligned}
|g(t)| = \left| \int_a^t g'(t) \, dt \right| \leqq \varepsilon \int_J |g'(t)| \, dt \\
\leqq \varepsilon^{3/2} \left\{ \int_J |g'(t)|^2 \, dt \right\}^{1/2},
\end{aligned}
$$

so that

$$
\int_J |g(t)|^2 \, dt \leq \varepsilon^4 \int_J |g'(t)|^2 \, dt.
$$

From this it follows immediately by induction that if $\varepsilon < 1$,

$$\int_J |g^{(p)}(t)|^2\,dt \leq \varepsilon^4 \int_J |g^{(n)}(t)|^2\,dt, \qquad 0 \leqq p < n.$$

It then follows from Schwarz's inequality and [*] that

$$(T_0(\hat{\tau})f, f) > b\int_J |f^{(n)}(t)|^2 - NB\varepsilon^2 \int_J |f^{(n)}(t)|^2\,dt.$$

If ε is so small that $b - NB\varepsilon^2 \geqq 0$, it follows that $\hat{\tau}$ is bounded below. Q.E.D.

30 Corollary. *A formally positive formally symmetric formal differential operator τ is finite below zero.*

Proof. It is obvious from Definition 20 that τ is bounded below. Thus the present corollary follows from Corollary 7 and Definition 25(b). Q.E.D.

31 Corollary. *Suppose in addition to the hypotheses of Theorem 8 that the coefficients p_k are real. Then τ is finite below any finite λ.*

Proof. We use the notations of the proof of Theorem 8. By Lemma 29 and Theorem 28 it is sufficient to show that τ' is finite below λ_0 in order to conclude that τ is finite below λ_0. But it was shown in the proof of Theorem 8 that c may be chosen so that $\tau' - \lambda_0$ is formally positive. By the preceding corollary, $\tau' - \lambda_0$ is finite below zero, from which it is evident by Definition 25 that τ' is finite below λ_0. Q.E.D.

32 Corollary. *Suppose in addition to the hypotheses of Theorem 9 that τ is formally symmetric. Then τ is finite below any finite λ.*

Proof. We use the notations of the proof of Theorem 9. This proof shows that $\tau = \tau_c + p_0$ is bounded below (since τ_c is formally positive for sufficiently large c, and p_0 is bounded below). Since $\sigma_e(\tau) = \phi$ by Theorem 9, the present corollary follows immediately from Definition 25. Q.E.D.

33 Corollary. *Suppose in addition to the hypotheses of Theorem 10 that the coefficients p_k are real. Then τ is finite below any finite λ.*

Proof. This may be proved in the same way as Corollary 32.

A number of additional results, related to those expressed by Theorem 11, . . ., Corollary 19 in much the same way as Corollaries

31, 32, and 33 are related to Theorems 8, 9 and 10, can be established. Some results of this sort are to be found in the exercises at the end of the present chapter.

The following "comparison theorem" is often useful in determining the essential spectrum of a formal differential operator.

34 THEOREM. *Let τ be a formally self adjoint formal differential operator of order n on an interval I, and suppose that τ is bounded below. Let τ_1 be a regular or irregular differential operator of order less than or equal to n defined on the same interval I as τ, and let τ_1 be formally self adjoint and formally positive. Then, if τ is finite below λ, and the leading coefficient of $\tau+\tau_1$ never vanishes, $\tau+\tau_1$ is finite below λ.*

PROOF. It is clear that we may suppose without loss of generality that $\lambda = 0$. By Corollary 24(b), Corollary XII.4.13, and Corollary 26, $T_0(\tau)$ has a self adjoint extension T which is also finite below 0. Let $\varepsilon > 0$. Then, by Corollary 27, $(-\infty, -\varepsilon) \cap \sigma(T)$ is finite. Let $\lambda_1, \ldots, \lambda_p$ be an enumeration of these points, let $E(\cdot)$ be the spectral resolution of T, and let $E_i = E(\lambda_i)$. By Theorem XII.2.6, every function f in $E_i L_2$ satisfies $Tf = \lambda_i f$. Since by Theorem 2.10 and Lemma XII.4.1(*b*), $Tf = \lambda_i f$ implies $T_1(\tau)f = \lambda_i f$, it follows that $E_i(L_2(I))$ is finite dimensional for each $i = 1, \ldots, p$. We may, consequently, find a finite orthonormal basis $\varphi_1, \ldots, \varphi_n$ for $E_1(L_2(I)) + \ldots + E_p(L_2(I))$. If $f \in L_2(I)$ satisfies $(f, \varphi_i) = 0$, $i = 1, \ldots, n$, then $(I - \sum_{i=1}^{n} E_i)f = f$, so that $E([-\varepsilon, \infty))f = f$. Consequently, if $f \in \mathfrak{D}(T)$ satisfies $(f, \varphi_i) = 0$, $i = 1, \ldots, n$, it follows by Theorem XII.2.6 that

$$[*] \qquad (Tf, f) = \int_{-\varepsilon}^{\infty} \lambda |E(d\lambda)f|^2 \geq -\varepsilon |f|^2.$$

For each integer $k \leq n$, we may write $\mathfrak{D}(T_0(\tau))$ as the sum of $\mathfrak{D}_k = \{f \in \mathfrak{D}(T_0(\tau)) | (f, \varphi_i) = 0, i = 1, \ldots, k\}$ and a finite dimensional subspace $\mathfrak{N}_k$ such that there exists a bounded projection P_k of $\mathfrak{D}(T_0(\tau))$ on $\mathfrak{N}_k$ such that $(I - P_k)\mathfrak{D}(T_0(\tau)) = \mathfrak{D}_k$. This is easily proved by induction on k; using induction, we have only to show that $\mathfrak{D}_k$ may be written as the sum of $\mathfrak{D}_{k+1}$ and a finite dimensional subspace $\hat{\mathfrak{N}}_k$ such that there exists a bounded projection $\hat{P}_k$ of $\mathfrak{D}_k$ on $\hat{\mathfrak{N}}_k$ such that $(I - \hat{P}_k)\mathfrak{D}_k = \mathfrak{D}_{k+1}$. For, in this case, we have only to put $\mathfrak{N}_{k+1} =$

$\mathfrak{N}_k + \hat{\mathfrak{N}}_k$, and to define P_{k+1} inductively by $P_{k+1} = P_k + \hat{P}_k(I - P_k)$. If $(f, \varphi_{k+1}) = 0$ for all $f \in \mathfrak{D}_k$, $\mathfrak{D}_k = \mathfrak{D}_{k+1}$, $\hat{\mathfrak{N}}_k = 0$. If, on the other hand, there exists a $g \in \mathfrak{D}_k$ such that $(g, \varphi_{k+1}) = 1$, then $f - (f, \varphi_{k+1})g \in \mathfrak{D}_{k+1}$ for each $f \in \mathfrak{D}_k$, so that $\mathfrak{D}_k$ is the sum of $\mathfrak{D}_{k+1}$ and the one-dimensional space $\hat{\mathfrak{N}}_k$ spanned by g. Here we put $\hat{P}_k f = (f, \varphi_{k+1})g$. Thus we are able to conclude that

$$\mathfrak{D}(T_0(\tau)) = \{f \in \mathfrak{D}(T_0(\tau)) \mid (f, \varphi_i) = 0, i = 1, \ldots, n\} + \mathfrak{N} = \mathfrak{D}_n + \mathfrak{N}_n,$$

where $\mathfrak{N}$ is a finite dimensional subspace of $\mathfrak{D}(T_0(\tau))$, and where there exists a bounded projection $P = P_n$ of $\mathfrak{D}(T_0(\tau)) = \mathfrak{D}(T_0(\tau+\tau_1))$ on $\mathfrak{N}_n$ such that $(I - P)\mathfrak{D}(T_0(\tau+\tau_1)) = \mathfrak{D}_n$.

Let $\mu > 0$; we shall show, using Corollary 2, that $-(\mu+\varepsilon) \notin \sigma_e(\tau+\tau_1)$. If this is false, then by Corollary 2 there exists a bounded sequence $\{f_m\}$ of $\mathfrak{D}(T_0(\tau+\tau_1))$ such that $\{(\tau+\tau_1+\mu+\varepsilon)f_m\}$ converges, but such that $\{f_m\}$ has no convergent subsequence. Let $g_m = Pf_m$. Then $\{g_m\}$ is a bounded sequence in the finite dimensional space $\mathfrak{N}_n$, so that (passing without loss of generality to a subsequence) we may suppose that $\{g_m\}$ converges. The mapping $T_0(\tau+\tau_1)$ is defined and linear on the *finite dimensional* space $\mathfrak{N}_n$, hence continuous on this space. Consequently, $\{(\tau+\tau_1+\mu+\varepsilon)g_m\}$ converges. Consequently, putting $\hat{f}_m = f_m - g_m$, $\{\hat{f}_m\}$ is a bounded sequence of elements of $\mathfrak{D}_n$ such that $\{(\tau+\tau_1+\mu+\varepsilon)\hat{f}_m\}$ converges, but such that $\{\hat{f}_m\}$ contains no convergent subsequence. On the other hand, we have

$$\limsup_{m,q\to\infty} |\hat{f}_m - \hat{f}_q|^2 \le \mu^{-1} \limsup_{m,q\to\infty} ((\tau+\tau_1+\mu+\varepsilon)(\hat{f}_m - \hat{f}_q), \hat{f}_m - \hat{f}_q)$$
$$= 0$$

by [*], since $|\hat{f}_m - \hat{f}_q|$ is bounded and $\{(\tau+\tau_1+\mu+\varepsilon)\hat{f}_m\}$ converges. Thus $\{\hat{f}_m\}$ is a Cauchy sequence, and hence converges. This contradiction proves the present theorem. Q.E.D.

In the case of the real formally self adjoint formal differential operator

$$\tau = -\left(\frac{d}{dt}\right) p(t) \left(\frac{d}{dt}\right) + q(t),$$

there exists an intimate connection between the spectral theory of the various self adjoint extensions of $T_0(\tau)$, and the more classical

"oscillation theory" of Sturm. We now propose to establish this connection.

In developing our analysis we will find use for the basic facts of the Sturmian theory of zeros of solutions of second order differential equations, which we present in the next few lemmas.

35 LEMMA. *Let*

$$\tau_1 = -\left(\frac{d}{dt}\right) p_1(t) \left(\frac{d}{dt}\right) + q_1(t)$$

and

$$\tau_2 = -\left(\frac{d}{dt}\right) p_2(t) \left(\frac{d}{dt}\right) + q_2(t)$$

be two real, formally self adjoint formal differential operators defined on a closed interval I. Suppose that $p_1(t) \geq p_2(t) > 0$ and $q_1(t) \geq q_2(t)$ for $t \in I$. Let f_1 be a non-zero real solution of $\tau_1 f_1 = 0$, and let f_2 be a non-zero real solution of $\tau_2 f_2 = 0$. Let $f_1(c) = 0$, $f_1(d) = 0$. Then, unless f_2 is a constant multiple of f_1 in $[c, d]$, and $p_1(t) \equiv p_2(t)$, $q_1(t) \equiv q_2(t)$ for $t \in [c, d]$, there exists a point b in the open interval (c, d) such that $f_2(b) = 0$.

PROOF. Suppose that this is not the case. Then the function $f_1(t)/f_2(t)$ is continuous in $[c, d]$. Indeed, even if f_2 vanishes at one of the points c, d, say at c, we must have $f_2'(c) \neq 0$, (since $f_2(c) = f_2'(c) = 0$ implies $f_2 \equiv 0$, contrary to assumption), and then

$$\lim_{t \to c} f_1(t)/f_2(t) = f_1'(c)/f_2'(c).$$

The function g defined by

$$g(t) = f_1(t) f_2^{-1}(t)(p_1(t) f_1'(t) f_2(t) - p_2(t) f_1(t) f_2'(t))$$

is consequently continuous in $[c, d]$. Moreover, it vanishes at the end points of $[c, d]$, even if $f_2(c) = 0$ or $f_2(d) = 0$. Now

$$g = p_1 f_1' f_1 - (p_2 f_2') f_1^2 f_2^{-1},$$

so that

$$\begin{aligned} g' &= q_1 f_1^2 - q_2 f_1^2 + p_1 (f_1')^2 - 2 p_2 f_2' f_1' f_1 f_2^{-1} + p_2 f_2' f_1^2 f_2' f_2^{-2} \\ &= (q_1 - q_2) f_1^2 + p_1 (f_1')^2 \\ &\quad + p_2 f_2^{-2}((f_2')^2 f_1^2 - 2 f_1 f_1' f_2 f_2' + (f_1')^2 f_2^2) - p_2 (f_1')^2 \\ &= (q_1 - q_2) f_1^2 + (p_1 - p_2)(f_1')^2 + p_2 f_2^{-1}(f_1' f_2 - f_2' f_1)^2. \end{aligned}$$

Thus

$$0 = g(d) - g(c) = \int_c^d \{(q_1-q_2)f_1^2 + (p_1 - p_2)(f_1')^2 + p_2 f_2^{-1}(f_1' f_2 - f_2' f_1)^2\}.$$

Since all the terms in the integral on the right are non-negative, we must have $f_1' f_2 - f_2' f_1$ identically zero in $[c, d]$. Thus

$$(f_1 f_2^{-1})' = f_2^{-2}(f_1' f_2 - f_2' f_1)$$

is identically zero in $[c, d]$, so that $f_1 f_2^{-1}$ is constant. Moreover, since f_1 and f_1' have only a finite number of zeros in $[c, d]$, we must have $p_1(t) = p_2(t)$, $q_1(t) = q_2(t)$ for $t \in [c, d]$. Q.E.D.

36 Corollary. *Let*

$$\tau = -\left(\frac{d}{dt}\right) p(t) \left(\frac{d}{dt}\right) + q(t)$$

be a real, formally self adjoint formal differential operator defined on an open, half open, or closed interval I. Then

(a) *between any two consecutive zeros of any solution f_1 of $\tau f_1 = 0$ which is not identically zero, there exists a zero of any linearly independent solution f_2;*

(b) *if any solution of $\tau f_1 = 0$ which is not identically zero has an infinite number of zeros in I, then every solution of $\tau f = 0$ which is not identically zero has an infinite number of zeros in I.*

Proof. Statement (a) is the special case $\tau_1 = \tau_2$ of the preceding theorem. Statement (b) is an evident consequence of statement (a). Q.E.D.

The theorem is illustrated by the pair of functions $\sin \lambda t$ and $\sin \mu t$ for $\mu > \lambda$; the corollary by the pair of functions $\sin t$ and $\cos t$.

Another illustration of the power of Sturm's comparison theorems is given by the following theorem of Kneser, which, as we will see below, yields interesting information on the spectrum of the self adjoint extensions of $T_0(\tau)$ in the case where it applies.

37 Corollary. *Let*

$$\tau = -\left(\frac{d}{dt}\right)^2 + q(t)$$

be a real, formally self adjoint formal differential operator defined on an interval $[a, \infty)$. *Then*

(a) *if* $\limsup_{t\to\infty} t^2 q(t) < -(1/4)$, *every solution of* $\tau f = 0$ *has an infinite number of zeros on* $[a, \infty)$;

(b) *if* $\liminf_{t\to\infty} t^2 q(t) > -(1/4)$, *no solution, not identically zero, of* $\tau f = 0$ *has more than a finite number of zeros on* $[a, \infty)$.

Proof. According to the theorem above, it suffices to show that for $\alpha < -(1/4)$, every solution of the equation $f'' + \alpha t^{-2} f = 0$ has an infinite number of zeros, and that for $\alpha > -(1/4)$, no solution, not identically zero, does. Now, two solutions of $f'' + \alpha t^{-2} f = 0$ are t^{e_1} and t^{e_2}, where $e_1, e_2 = 1/2 \pm \sqrt{\alpha + (1/4)}$. If $\alpha \geqq -(1/4)$, e_1 and e_2 are real, and t^{e_1}, t^{e_2} have clearly no zeros. If $\alpha < -(1/4)$, so that $i\mu = \sqrt{\alpha + (1/4)}$ is pure imaginary, then

$$t^{e_1} + t^{e_2} = t^{1/2}(t^{i\mu} + t^{-i\mu}) = 2t^{1/2}\cos(\mu \log t)$$

clearly has infinitely many zeros. Q.E.D.

38 Corollary. *Let*

$$\tau = -\left(\frac{d}{dt}\right)^2 + q(t)$$

be a real, formally self adjoint formal differential operator defined on an interval $(0, a]$. *Then*

(a) *if* $\limsup_{t\to 0} t^2 q(t) < -(1/4)$, *every solution* $\tau f = 0$ *has an infinite number of zeros on* $(0, a]$;

(b) *if* $\liminf_{t\to 0} t^2 q(t) > -(1/4)$, *no solution, not identically zero, of* $\tau f = 0$ *has more than a finite number of zeros on* $(0, a]$.

Proof. The reader who examines the proof of the preceding corollary will have no difficulty in modifying it to obtain a proof of the present corollary. Q.E.D.

We now give a first result relating oscillation theory with spectral theory.

39 Lemma. *Let*

$$\tau = -\left(\frac{d}{dt}\right)p(t)\left(\frac{d}{dt}\right) + q(t)$$

be a real formally self adjoint formal differential operator defined on an interval $I = (a, b)$. Let $p(x) > 0$ for $x \in I$ and let λ_0 be a real number. Then

(a) *if τ is finite below λ_0, then for each $\varepsilon > 0$, no real solution of the equation $\tau\sigma = (\lambda_0 - \varepsilon)\sigma$ has infinitely many zeros in I;*

(b) *if, for all $\varepsilon > 0$, every real solution of the equation $\tau\sigma = (\lambda_0 - \varepsilon)\sigma$ has only finitely many zeros in I, then τ is finite below λ_0.*

Proof. Let σ be real and satisfy the equation $\tau\sigma = (\lambda_0 - \varepsilon)\sigma$, and let σ have infinitely many zeros in I. Then σ has either an increasing or a decreasing infinite sequence of distinct zeros. Suppose, for the sake of definiteness, that σ has an increasing infinite sequence of zeros, which we enumerate as $z_1, z_2, z_3, \ldots$. Let $g_n(t) = k_n\sigma(t)$ for $z_n \leq t \leq z_{n+1}$, where the constant k_n is chosen so that

$$\int_{z_n}^{z_{n+1}} g_n(t)^2\, dt = 1,$$

and let g_n be zero elsewhere.

Next we need the following general auxiliary principle. Let g be a continuously differentiable function in a finite closed interval J, and let g vanish at the end points of J. Let $\varepsilon > 0$ be given. Then there exists a function $h \in C^\infty(J)$ such that

$$\max_{t \in J} |h(t) - g(t)| < \varepsilon,$$

$$|h' - g'|_2 = \left\{\int_J |h'(t) - g'(t)|^2\, dt\right\}^{1/2} < \varepsilon,$$

and such that h vanishes in a neighborhood of the end-points of J. This may be proved as follows. First of all, we may evidently suppose for the sake of definiteness and without loss of generality that $J = [-1, +1]$. Extend the definition of g by putting $g(t) = 0$ for $|t| > 1$. Then, as $\delta \to 0$, it is clear that

$$\max_{t \in J} |g(t) - g((1+\delta)t)| \to 0,$$

$$\int \left| g'(t) - \frac{d}{dt} g((1+\delta)t) \right|^2 dt \to 0.$$

Hence, by choosing δ sufficiently small and putting $\hat{g}(t) = g((1+\delta)t)$, we find a continuous function $\hat{g}$ with a derivative continuous except

at the two points $\pm(1+\delta)^{-1}$, such that

$$\max_{t\in J} |g(t)-\tilde{g}(t)| < \varepsilon/2,$$
$$|g'-\tilde{g}'|_2 < \varepsilon/2,$$

and such that $\tilde{g}$ vanishes outside an interval $[-1+\eta, 1-\eta]$, where η is some number greater than zero. Next let φ be an infinitely differentiable real valued and non-negative function vanishing outside the interval $J=(-1, +1)$ such that

$$\int_J \varphi(t)dt = 1.$$

Then, as $\alpha \to \infty$, it is clear that

$$\int_{-\infty}^{+\infty} \alpha\varphi(\alpha s)\tilde{g}(t-s)ds \to \tilde{g}(t) \text{ uniformly;}$$
$$\int_{-\infty}^{+\infty} \alpha\varphi(\alpha s)\tilde{g}(t-s)ds \text{ vanishes outside } [-1+\eta-\alpha^{-1}, 1-\eta+\alpha^{-1}];$$
$$\int_{-\infty}^{+\infty} \alpha\varphi(\alpha s)\tilde{g}'(t-s)ds \to \tilde{g}'(t) \text{ at each point of continuity of } \tilde{g}';$$
$$\left|\int_{-\infty}^{+\infty} \alpha\varphi(\alpha s)\tilde{g}'(t-s)ds\right| \leq \max_{t\in J} |\tilde{g}'(t)|.$$

From the last three statements and the Lebesgue dominated convergence theorem we conclude that as $\alpha \to \infty$

$$\int_J \left|\int_{-\infty}^{+\infty} \alpha\varphi(\alpha s)\tilde{g}'(t-s)ds - \tilde{g}'(t)\right|^2 dt \to 0.$$

Since $\tilde{g}(t-s)$ vanishes at the points of discontinuity of its derivative, we find on integrating by parts that

$$\begin{aligned}
\int_{-\infty}^{+\infty} \alpha\varphi(\alpha s)\tilde{g}'(t-s)ds &= \int_{-\infty}^{+\infty} \left\{\frac{d}{ds}\alpha\varphi(\alpha s)\right\}\tilde{g}(t-s)ds \\
&= -\int_{-\infty}^{+\infty} \left\{\frac{d}{ds}\alpha\varphi(\alpha(t-s))\right\}\tilde{g}(t)ds \\
&= \int_{-\infty}^{+\infty} \frac{d}{dt}\alpha\varphi(\alpha(t-s))\tilde{g}(s)ds \\
&= \frac{d}{dt}\int_{-\infty}^{+\infty} \alpha\varphi(\alpha(t-s))\tilde{g}(s)ds \\
&= \frac{d}{dt}\int_{-\infty}^{+\infty} \alpha\varphi(\alpha s)\tilde{g}(t-s)ds;
\end{aligned}$$

here, the fact that $\varphi \in C^{\infty}(J)$ has also been used. Thus, if we choose α to be sufficiently large and put $h(t) = \int_{-\infty}^{+\infty}\alpha\varphi(\alpha s)g(t-s)ds$, we have

$$\max_{t\in J}|h(t)-g(t)| < \varepsilon/2+\varepsilon/2 = \varepsilon,$$
$$|h'-g'|_2 < \varepsilon/2+\varepsilon/2 = \varepsilon,$$

h vanishes outside an interval $[-1+\eta/2, 1+\eta/2]$, where $\eta > 0$. Moreover, since $h(t) = \int_{-\infty}^{+\infty}\alpha\varphi(\alpha(t-s))g(s)ds$, it is clear that $h \in C^{\infty}(I)$, which completes the proof of the general auxiliary principle.

Now, using the principle which has just been established, let $\{h_{nm}\}$ be a sequence of infinitely differentiable functions such that $h_{nm}(t)$ vanishes except when $z_n < t < z_{n+1}$, such that

$$\lim_{m\to\infty}|h'_{nm}-g'_n|_2 = 0,$$

and such that $h_{nm}(t) \to g_n(t)$ uniformly as $m \to \infty$. Then

$$\begin{aligned}(\tau h_{nm}, h_{nm}) &= \int_{z_n}^{z_{n+1}}\{p(t)|h'_{nm}(t)|^2+q(t)|h_{nm}(t)|^2\}dt \\ &\to \int_{z_n}^{z_{n+1}}\{p(t)|g_n(t)|^2+q(t)|g_n(t)|^2\}dt \\ &= \int_{z_n}^{z_{n+1}}(\tau g_n)(t)\overline{g_n(t)}dt \\ &\le (\lambda_0-\varepsilon)|g_n|_2^2 = (\lambda_0-\varepsilon).\end{aligned}$$

Hence, by choosing a sufficiently large $m = m_n$ and putting $f_n = h_{nm_n}/|h_{nm_n}|_2$, we can construct a sequence of functions $f_n \in \mathfrak{D}(T_0(\tau))$ such that

$$[*] \qquad \begin{aligned}&(f_n, f_m) = 0, && 1 \le n \ne m,\\ &|f_n| = 1, && 1 \le n,\\ &(\tau f_n, f_n) < \lambda_0-\varepsilon/2.\end{aligned}$$

Now suppose that τ is finite below λ_0. By Corollary 24(b), Corollary XII.4.13, and Corollary 26, $T_0(\tau)$ has a self adjoint extension T which is also finite below λ_0. Then, by Corollary 27, $(-\infty, \lambda_0-\varepsilon/4) \cap \sigma(\tau)$ is finite. Let $\lambda_1, \ldots, \lambda_p$ be an enumeration of these points, let $E(\cdot)$ be the spectral resolution of T, and let $E_i = E(\lambda_i)$. By Theorem XII.2.6, every function f in E_iL_2 satisfies $Tf = \lambda_i f$. Since by Theorem 2.10 and Lemma XII.4.1, $Tf = \lambda_i f$ implies

$T_1(\tau)f = \lambda_i f$, it follows that $E_i L_2$ is finite dimensional for each $i = 1, \ldots, p$. We may consequently find a finite orthonormal basis $\varphi_1, \ldots, \varphi_m$ for $E_1 L_2 \oplus \ldots \oplus E_p L_2$. If $f \in L_2$ satisfies $(f, \varphi_i) = 0$, $i = 1, \ldots, m$, then $(I - \sum_{i=1}^{n} E_i)f = f$, so that $E([\lambda_0 - \varepsilon/4, \infty))f = f$. Consequently, if $f \in \mathfrak{D}(T)$ satisfies $(f, \varphi_i) = 0$, $i = 1, \ldots, m$, it follows from Theorem XII.2.6 that

$$[**] \qquad (Tf, f) = \int_{\lambda_0 - \varepsilon/4}^{\infty} \lambda |E(d\lambda)f|^2 \geq (\lambda_0 - \varepsilon/4)|f|^2.$$

On the other hand, since infinitely many functions f_n have been constructed above, we can clearly find a non-trivial linear combination $f = \sum_{i=1}^{n} \alpha_i f_i$ of the functions f_n such that $(f, \varphi_i) = 0$, $i = 1, \ldots, m$. Then we have [**]; but, by [*],

$$\begin{aligned}(Tf, f) &= \sum_{i=1}^{n} |\alpha_i|^2 (Tf_i, f_i) \leq (\lambda_0 - \varepsilon/2) \sum_{i=1}^{n} |\alpha_i|^2 \\ &= (\lambda_0 - \varepsilon/2)|f|^2.\end{aligned}$$

This contradiction proves that τ is not finite below λ_0, and hence proves (a)

To prove (b) we argue as follows. Suppose that for all $\varepsilon > 0$, every real solution of the equation $\tau\sigma = (\lambda_0 - \varepsilon)\sigma$ has only finitely many zeros in I. We wish to show that τ is finite below λ_0. By Theorem 28 and Theorem 4, it is sufficient to show that the restrictions of τ to $(a, c]$ and $[c, b)$, where $a < c < b$, are finite below λ_0.

That is, we may suppose without loss of generality that I is half open, say half open on the right so that $I = [a, b)$. The solution of $\tau\sigma = (\lambda_0 - \varepsilon/2)\sigma$ satisfying $f(a) = 0$, which is unique up to a constant factor, has only finitely many zeros. Let c be its largest zero, so that the solution of $\tau\sigma = (\lambda_0 - \varepsilon/2)\sigma$ satisfying $f(c) = 0$ has no zeros interior to $[c, b)$. By Theorem 28 and Lemma 29 the restriction of τ to $[c, b)$ is finite below $\lambda_0 - \varepsilon$ for each $\varepsilon > 0$ if and only if the restriction of τ to $[a, b)$ is finite below $\lambda_0 - \varepsilon$ for each $\varepsilon > 0$. That is, we may suppose without loss of generality that the solution of $\tau\sigma = (\lambda_0 - \varepsilon/2)\sigma$ satisfying $f(a) = 0$ has no zeros interior to I. By the comparison principle stated in Lemma 35, if $\lambda < \lambda_0 - \varepsilon/2$, the solution of $\tau\sigma = \lambda\sigma$ satisfying $f(a) = 0$ has no zeros interior to I. Hence, if c is any quantity such that $a < c < b$, the self adjoint operator T_c derived

from the restriction of τ to $[a, c]$ by imposition of the boundary conditions $f(a) = f(c) = 0$ has no eigenvalues below $\lambda - \varepsilon/2$. It follows immediately from Theorems 4.1, 4.2, and XII.7.2 that $(\tau f, f) = (T_c f, f) \geq (\lambda_0 - \varepsilon/2)|f|^2$ for f in $\mathfrak{D}(T_c)$. Since $\bigcup_{a<c<b}\mathfrak{D}(T_c) \supseteq \mathfrak{D}(T_0(\tau))$, $((\tau - (\lambda_0 - \varepsilon/2))f, f) \geqq 0$ for f in $\mathfrak{D}(T_0(t))$, so that $\tau - (\lambda_0 - \varepsilon/2)$ is formally positive, and, by Corollary 30, τ is finite below $\lambda_0 - \varepsilon/2$ for any $\varepsilon > 0$. Q.E.D.

40 THEOREM. *Let*

$$\tau = -\left(\frac{d}{dt}\right) p(t) \left(\frac{d}{dt}\right) + q(t)$$

be a real formally self adjoint formal differential operator defined on an interval $I = (a, b)$. Let $p(t) > 0$ for t in I. Then

(a) *if τ is not bounded below, all solutions of every equation $\tau\sigma = \lambda\sigma$ have infinitely many zeros in I, and conversely;*

(b) *if τ is bounded below, and if λ_0 is the smallest point in $\sigma_e(\tau)$ (so that $\lambda_0 > -\infty$ in the present case), then, for $\mu > \lambda_0$, every solution of $\tau\sigma = \mu\sigma$ has infinitely many zeros, while, for $\mu < \lambda_0$, no solution of $\tau\sigma = \mu\sigma$ has infinitely many zeros.*

PROOF. The operator τ is bounded below if and only if it is finite below λ for some finite λ. Thus statement (a) follows immediately from the preceding lemma.

To prove statement (b), note that τ is finite below λ_0, but not below any $\lambda > \lambda_0$. Statement (b) then follows immediately from the preceding lemma. Q.E.D.

Next we turn to the more detailed analysis of case (b) of the preceding theorem. Let τ be the formal differential operator of that theorem. Let λ_0 be the smallest number in $\sigma_e(\tau)$. Let τ have n_a $(= 0$ or 2, by Theorem 2.30) boundary values at a and n_b boundary values at b. Let T be a self adjoint restriction of $T_1(\tau)$ defined by a separated set of boundary conditions. By Corollary 2.32, these boundary conditions are real, and consist of $(1/2)n_a$ $(= 0$ or $1)$ boundary conditions at a, together with $(1/2)n_b$ $(= 0$ or $1)$ boundary conditions at b.

We shall make a detailed study of the solutions of $\tau\sigma = \lambda\sigma$ in $-\infty < \lambda < \lambda_0$. In order to simplify the formulations to follow, let

us first agree that if one of n_a, n_b is zero, we will make the change of variable $t \to -t$ if necessary, so as to be able to assume without loss of generality that n_b is zero. Thus, except in the cases when τ has no boundary values at all, τ has two boundary values at a, and, by Corollary 2.31, the set of boundary conditions defining T includes exactly one real boundary condition B at a.

Let t be any point in (a, b). Let τ_1 and τ_2 be the restrictions of τ to $(a, t]$ and $[t, b)$ respectively. By Theorem 2.20, any boundary value for τ_1 (for τ_2) at a (at b) may be regarded as a boundary value for τ at a (at b). If τ has any boundary values at a, then by Theorem 2.30 we may find two real boundary values G_1, G_2 for τ_1 at a, and two real boundary values E_1, E_2 for τ_1 at t, such that

$$\begin{aligned}(\tau_1 f, g) - (f, \tau_1 g) &= G_1(f)\overline{G_2(g)} - G_2(f)\overline{G_1(g)} \\ &\quad + E_1(f)\overline{E_2(g)} - E_2(f)\overline{E_1(g)}, \qquad f, g \in \mathfrak{D}(T_1(\tau_1)).\end{aligned}$$

By Corollary 2.23, we may write

$$E_1(f) = a_{11} f(t) + a_{12} f'(t),\ E_2(f) = a_{21} f(t) + a_{22} f'(t), \qquad f \in \mathfrak{D}(T_1(\tau_1)),$$

where, since E_1 and E_2 are real, the coefficients a_{ij} are real. Hence, we find

$$\begin{aligned}(\tau_1 f, g) - (f, \tau_1 g) &= G_1(f)\overline{G_2(g)} - G_2(f)\overline{G_1(g)} \\ &\quad + (a_{11}a_{22} - a_{21}a_{12})(f(t)\overline{g'(t)} - f'(t)\overline{g(t)}), \qquad f, g \in \mathfrak{D}(T_1(\tau_1)).\end{aligned}$$

If we choose for f and g functions which vanish in the neighborhood of a, and apply Green's formula (Theorem 2.4), we find

$$(\tau_1 f, g) - (f, \tau_1 g) = F_t(f, g) + G_1(f)\overline{G_2(g)} - G_2(f)\overline{G_1(g)}, \quad f, g \in \mathfrak{D}(T_1(\tau_1)),$$

where $F_t(f, g)$ is the boundary form for τ at the point t, given by the expression

$$F_t(f, g) = p(t)(f(t)\overline{g'(t)} - f'(t)\overline{g(t)}), \qquad a < t < b.$$

If τ_1 has no boundary values at a, we find in the same way

$$(\tau_1 f, g) - (f, \tau_1 g) = F_t(f, g), \qquad f, g \in \mathfrak{D}(T_1(\tau_1)).$$

In the same way, we have

$$(\tau_2 f, g) - (f, \tau_2 g) = -F_t(f, g), \qquad f, g \in \mathfrak{D}(T_1(\tau_2)),$$

if τ has no boundary values at b; while if τ has boundary values at b, we may find two real boundary values D_1, D_2 for T_2 at b, such that

$$(\tau_2 f, g) \quad (f, \tau_2 g) \quad D_1(f)\overline{D_2(g)} \quad D_2(f)\overline{D_1(g)} \quad F_t(f, g), \ f, g \in \mathfrak{D}(T_1(\tau_2)).$$

By Theorem 2.30 and Corollary 2.31, a set of boundary conditions defining a self adjoint restriction T of $T_1(\tau)$ is of the form

$$\begin{aligned} B(f) & \quad \alpha_1 G_1(f) + \alpha_2 G_2(f) = 0, \qquad \alpha_1^2 + \alpha_2^2 \neq 0, \qquad \alpha_1, \alpha_2 \text{ real}, \\ \hat{B}(f) & = \beta_1 G_1(f) + \beta_2 G_2(f) = 0, \qquad \beta_1^2 + \beta_2^2 \neq 0, \qquad \beta_1, \beta_2 \text{ real}, \end{aligned}$$

if τ has boundary values both at a and at b, and

$$B(f) \quad \alpha_1 G_1(f) + \alpha_2 G_2(f) \quad 0, \qquad \alpha_1^2 + \alpha_2^2 \neq 0, \qquad \alpha_1, \alpha_2 \text{ real},$$

if τ has boundary values at a but not at b. Consequently

(a) if $f, g \in \mathfrak{D}(T_1(\tau_1))$ satisfy the boundary condition $B(f) = 0$, or if τ has no boundary values at a and $f, g \in \mathfrak{D}(T_1(\tau_1))$, then

$$(\tau_1 f, g) - (f, \tau_1 g) = F_t(f, g);$$

(b) if S is the restriction of $T_1(\tau_1)$ defined by a boundary condition $f(t) + cf'(t) = 0$ and by the boundary condition $B(f) = 0$ (if τ has any boundary values at a), then S^* is the restriction of $T_1(\tau_1)$ defined by the boundary condition $f(t) + \bar{c}f'(t) = 0$ and by the boundary condition $B(f) \quad 0$ (if τ has any boundary values at a).

This follows from the above formulae, from Theorem 2.10, and from Theorem XII.4.28. We have similarly:

(c) If $\hat{S}$ is the restriction of $T_1(\tau_1)$ defined by the boundary conditions $f(t) \quad f'(t) = 0$ and $B(f) = 0$ (if τ has any boundary values at a) then $\hat{S}^*$ is the restriction of $T_1(\tau_1)$ defined by the boundary condition $B(f) = 0$ (if τ has any boundary values at a).

If τ has no boundary values at a, then, since $(\quad \infty, \lambda_0) \cap \sigma_e(\tau) \quad \phi$, it follows from Lemmas 6.7 and 6.9 that for $\lambda \in (\quad \infty, \lambda_0)$ there exists precisely one (up to a constant multiple) solution $\sigma(x, \lambda)$ of $\tau\sigma \quad \lambda\sigma$ which is square-integrable in the neighborhood of a.

If τ has two boundary values at a, there exists precisely one (up to a constant multiple) solution $\sigma(x, \lambda)$ of $\tau\sigma = \lambda\sigma$ which is square-integrable in the neighborhood of a and satifies the boundary condition B. Indeed, it is clear that since all the solutions of $\tau\sigma = \lambda\sigma$ are square-integrable in the neighborhood of a by Theorem 6.7, at

least one such solution must exist. On the other hand, if two linearly independent solutions of $\tau\sigma = \lambda\sigma$ satisfy the boundary condition B, it follows that all solutions of $\tau\sigma = \lambda\sigma$ satisfy B. By the remark (a) made above, it then follows that for any two solutions f, g of $\tau\sigma = \lambda\sigma$, and $a < c < b$, we have

$$\begin{aligned} 0 &= \int_a^c \{((\tau-\lambda)f)(t)\overline{g(t)} - f(t)\overline{((\tau-\lambda)g)(t)}\}dt \\ &= \int_a^c \{(\tau f)(t)\overline{g(t)} - f(t)\overline{(\tau g)(t)}\}dt \\ &= \quad p(c)(\overline{g(c)}f'(c) - f(c)\overline{g'(c)}). \end{aligned}$$

However, since solutions f and g of $\tau\sigma = \lambda\sigma$ such that $f(c) = 0$, $f'(c) = 1$, $g(c) = 1$, $g'(c) = 0$ exist, this is impossible.

We summarize the above remarks for future reference in the following lemma.

41 Lemma. *Let τ and the boundary condition B (if τ has any boundary values at a) be as above. Then for $-\infty < \lambda < \lambda_0$, the equation $\tau\sigma = \lambda\sigma$ has precisely one solution $\sigma(t, \lambda)$ (up to a constant multiple) which is square-integrable in the neighborhood of a and satisfies the boundary condition B (if any).*

42 Lemma. *The solution $\sigma(t, \lambda)$ defined above may be chosen so as to be real and infinitely differentiable in t and λ for $t \in I$ and $\lambda \in (-\infty, \lambda_0)$.*

Proof. The boundary condition B at a defining T (if any) is real by the remarks above. Hence, if f is a solution of $\tau\sigma = \lambda\sigma$ square-integrable at a and satisfying the boundary condition B, so is $\bar{f}$. Since, by the preceding lemma, only one such solution (up to a constant multiple) exists, we must have $\bar{f} = \alpha f$, where, since $|\bar{f}| = |f|$, $|\alpha| = 1$. Putting $\rho = \alpha^{1/2}$, we have $\bar{\rho} = \rho^{-1}$, so that $\bar{\rho}\bar{f} = \rho f$. Thus f is a non-zero multiple of a real solution of $\tau\sigma = \lambda\sigma$; so that, if $a < c < b$, $f'(c)/f(c)$ is real. Thus, if we let τ_1 be the restriction of τ to the interval $\tilde{I} = (a,c]$, and let T be the restriction of $T_1(\tau_1)$ defined by the boundary conditions B (if τ has boundary values at a) and $f(c) = if'(c)$, then $Tf = \lambda f$ has no solutions for any real λ. By Theorem 2.10, Lemma XII.1.6(a), and Definition XII.4.20, T is closed. By remark (b) preceding Lemma 41, the adjoint of T^* of T is the restriction of $T_1(\tau)$ defined by the boundary conditions B (if τ has boundary values at a)

and $f(c) = if'(c)$. Consequently, $T^*f = \lambda f$ has no solutions for any real λ. This means, according to Lemma XII.3.6(d), that $(T-\lambda)\mathfrak{D}(T)$ is dense in $L_2(\hat{I})$ for every real λ.

Now let $-\infty < \lambda < \lambda_0$. By Definition 6.1, $(T-\lambda I)\mathfrak{D}(T) = L_2(\hat{I})$. Thus, $\lambda I - T$ is a closed one-to-one mapping of $\mathfrak{D}(T)$ onto $L_2(\hat{I})$. Hence $(\lambda I - T)^{-1}$ is a closed everywhere-defined mapping. Thus, by the closed graph theorem (II.2.4) and by Lemma VII.9.2, the resolvent $R(\lambda; T)$ is an analytic function defined in a neighborhood V of $(-\infty, \lambda_0)$ in the complex plane. By Lemma 2.16, the expression

$$\varphi_\lambda(f) = (R(\lambda; T)f)(c)$$

defines a bounded linear functional on $L_2(\hat{I})$ for each λ in V. Consequently, there exists an element $g_{\bar{\lambda}}$ in $L_2(\hat{I})$ such that $\varphi_\lambda(f) = (f, g_{\bar{\lambda}})$ for each λ in V. It is evident that g_λ depends analytically on λ for $\bar{\lambda}$ in V.

Let $T_2 \subseteq T$ be the restriction of $T_1(\tau_1)$ determined by the boundary conditions B (if τ has boundary values at a) and $f(c) = f'(c) = 0$.

By Corollary 2.23 and by XII.4.28, T_2^* is the restriction of $T_1(\tau)$ determined by the single boundary condition B. If f is in $\mathfrak{D}(T_2)$ we have

$$\begin{aligned}((\lambda I - T_2)f, g_{\bar{\lambda}}) &= ((\lambda I - T)f, g_{\bar{\lambda}}) \\ &= R(\lambda; T)((\lambda I - T)f)(c) = f(c) = 0.\end{aligned}$$

Thus, $(\bar{\lambda} I - T_2^*)g_{\bar{\lambda}} = 0$ for λ in V. Consequently, for λ in $(-\infty, \lambda_0)$, g_λ is a solution of $\tau_1\sigma = \lambda\sigma$ lying in $L_2(\hat{I})$ and satisfying the boundary condition $B(g_\lambda) = 0$ (if τ has boundary values at a). As λ varies, g_λ and $\tau_1 g_\lambda - \lambda g_\lambda$ vary analytically in the norm of $L_2(\hat{I})$. Hence, by Lemma 2.16, $g_\lambda(c)$ and $g_\lambda'(c)$ vary analytically with λ as λ varies in $(-\infty, \lambda_0)$. If we let $\sigma(t, \lambda)$ be that unique solution of $\tau\sigma = \lambda\sigma$ defined by the initial conditions $\sigma(c, \lambda) = g_\lambda(c)$, $\sigma'(c, \lambda) = g_\lambda'(c)$, it is clear from Corollary 1.5 that $\sigma(t, \lambda)$ is a C^∞ function of t and λ for t in I and λ in $(-\infty, \lambda_0)$. Moreover, $\sigma(\cdot, \lambda)$ is evidently square-integrable at a, and evidently satisfies the boundary condition B (if τ has boundary values at a). Q.E.D.

We now let J_n be that subset of the λ-axis for which $\sigma(t, \lambda)$ has exactly n zeros interior to I. By Theorem 40 each set J_n lies in the region $\lambda < \lambda_0$, while the open interval $\lambda < \lambda_0$ is contained in the

union $\bigcup_{n=0}^{\infty} J_n$. The point λ_0 may or may not lie in some one of the intervals J_n. We will see below what significance this alternative has for the spectrum of T.

43 LEMMA. *If λ is in J_n and λ_1 is in J_{n+1}, then $\lambda < \lambda_1$.*

PROOF. Suppose this is false. Then $\lambda_1 < \lambda$. Since, by Lemma 35, $\sigma(t, \lambda)$ has a zero between every pair of zeros of $\sigma(t, \lambda_1)$, we have only to show that the interval $(a, z]$ between a and the smallest zero z of $\sigma(t, \lambda_1)$ contains a zero of $\sigma(t, \lambda)$, and we will have established that $\sigma(t, \lambda)$ has at least $n+1$ zeros, and consequently contradicted the fact that λ is in J_n.

To establish this, note that in the contrary case we may, by multiplying by suitable constants, assume that $\sigma(t, \lambda_1)$ and $\sigma(t, \lambda)$ are non-negative in (a, z). Since z is a zero of $\sigma(t, \lambda_1)$ but not one of $\sigma(t, \lambda)$, it follows immediately that $\sigma'(z, \lambda_1) < 0$, $\sigma(z, \lambda) > 0$. By the remark (a) made preceding Lemma 41,

$$
\begin{aligned}
0 < (\lambda - \lambda_1)\int_a^z \sigma(t, \lambda)\sigma(t, \lambda_1)dt \\
= \int_a^z \{(\tau\sigma)(t, \lambda)\sigma(t, \lambda_1) - \sigma(t, \lambda_1)(\tau\sigma)(t, \lambda)\}dt \\
= +p(z)\sigma(z, \lambda)\sigma'(z, \lambda_1) < 0.
\end{aligned}
$$

This contradiction establishes our assertion. Q.E.D.

44 COROLLARY. *The sets J_n are intervals of the real axis, and J_n lies to the left of J_{n+1}.*

PROOF. It follows immediately from the preceding lemma that if $\lambda_1, \lambda_2 \in J_n$, and $\lambda_1 < \lambda < \lambda_2$, then $\lambda \in J_n$. Thus J_n is an interval. Our second assertion follows immediately from the preceding lemma. Q.E.D.

45 LEMMA. *At most one eigenvalue of T lies in any interval J_n.*

PROOF. Suppose that this is false. Let λ_1 and λ_2 be points in J_n, both being eigenvalues of T. Then $\sigma(t, \lambda_1)$ and $\sigma(t, \lambda_2)$ are eigenfunctions of T. Let $\lambda_1 < \lambda_2$. By the Sturm comparison theorem (cf. Lemma 35), $\sigma(\cdot, \lambda_2)$ has a zero in the open interval between every two zeros of $\sigma(\cdot, \lambda_1)$. Let z_1 and z_2 be the smallest and the largest zeros of

$\sigma(\cdot, \lambda_1)$. If we can show that $\sigma(\cdot, \lambda_2)$ has a zero in $(a, z_1]$ and a zero in $[z_2, b)$, we will have established that $\sigma(\cdot, \lambda_2)$ has at least $n+1$ zeros in (a, b), contradicting the fact that λ_2 is in J_n.

It is sufficient to prove that $\sigma(\cdot, \lambda_2)$ has a zero in $(a, z_1]$, for then it will follow by symmetry that $\sigma(\cdot, \lambda_2)$ has a zero in $[z_2, b)$.

To establish this, note that in the contrary case, multiplying by suitable constants, we may assume without loss of generality that $\sigma(t, \lambda_1)$ and $\sigma(t, \lambda_2)$ are non-negative in $(a, z_1]$. Since z_1 is a zero of $\sigma(t, \lambda_1)$ but not of $\sigma(t, \lambda_2)$ it follows immediately that $\sigma'(z_1, \lambda_1) < 0$, $\sigma(z_1, \lambda_2) > 0$. By the remark (a) preceding Lemma 41,

$$\begin{aligned} 0 &< (\lambda_2 - \lambda_1)\int_a^{z_1} \sigma(t, \lambda_2)\sigma(t, \lambda_1)dt \\ &-\int_a^{z_1} \{(\tau\sigma)(t, \lambda_2)\sigma(t, \lambda_1) - \sigma(t, \lambda_2)(\tau\sigma)(t, \lambda_1)\}dt \\ &\quad p(z_1)\sigma(z_1, \lambda_2)\sigma'(z_1, \lambda_1) < 0. \end{aligned}$$

This contradiction establishes our assertion. Q.E.D.

46 Corollary. *The kth zero of $\sigma(t, \lambda)$ (the zeros being counted in ascending order) is a monotone decreasing function of λ.*

Proof. Let $\lambda_1 < \lambda_2$. By the comparison theorem, Lemma 35, $\sigma(\cdot, \lambda_2)$ has a zero in the open interval between every two zeros of $\sigma(\cdot, \lambda_1)$. Let z_1 be the smallest zero of $\sigma(\cdot, \lambda_1)$. If we can show that $\sigma(\cdot, \lambda_2)$ has a zero in $(a, z_1]$, our corollary will follow immediately. This, however, follows exactly as in the last paragraph of the proof of the preceding lemma. Q.E.D.

47 Lemma. *All the non-null intervals J_n, $n < \infty$, are open on the left.*

Proof. Let us suppose the contrary, so that there exists a λ_0 such that $\sigma(t, \lambda_0)$ has n zeros, but $\sigma(t, \lambda_0 - \varepsilon)$ has fewer zeros for each $\varepsilon > 0$. Now $\sigma(t, \lambda_0)$ changes sign at each of its zeros $z_1, \ldots, z_n$. Hence, we can find a δ sufficiently small so that $\sigma(z_i + \delta, \lambda_0)$ and $\sigma(z_i - \delta, \lambda_0)$ have opposite signs. Thus, (Lemma 42) for sufficiently small ε, $\sigma(z_i + \delta, \lambda_0 - \varepsilon)$ and $\sigma(z_i - \delta, \lambda_0 - \varepsilon)$ have opposite signs. But this means that $\sigma(t, \lambda_0 - \varepsilon)$ has a zero within δ of each of $z_1, z_2, \ldots, z_n$, so that $\sigma(t, \lambda_0 - \varepsilon)$ has at least n zeros. This contradiction proves the lemma. Q.E.D.

48 Lemma. *The projections $E(\lambda)$ corresponding to the eigenvalues $\lambda < \lambda_0$ have one-dimensional ranges.*

Proof. We have $(T-\lambda)E(\lambda)f = 0$ by Theorem XII.2.6. Thus, every element of the range of $E(\lambda)$ is a square-integrable solution of $\tau\sigma = \lambda\sigma$ satisfying the boundary conditions at a and b defining T. However, we saw in Lemma 41 above that any such solution must be a multiple of $\sigma(t, \lambda)$. Q.E.D.

49 Lemma. *If $\sigma(t, \lambda)$ has n zeros interior to I, $\sigma(T)$ contains at least n points which lie in the interval $\{\mu|\mu \leqq \lambda\}$.*

Proof. Suppose that this assertion is false. Let $z_1, \ldots, z_n$ be the zeros of $\sigma(t, \lambda)$, enumerated in ascending order. Put

$$\begin{aligned}
f_1 &= \sigma(t, \lambda), && a < t \leqq z_1, && \text{and zero elsewhere,}\\
f_2 &= \sigma(t, \lambda), && z_1 \leqq t \leqq z_2, && \text{and zero elsewhere,}\\
&\vdots\\
f_n &= \sigma(t, \lambda), && z_{n-1} \leqq t \leqq z_n, && \text{and zero elsewhere.}
\end{aligned}$$

Since $\sigma(T)$ contains at most p points $\lambda_1, \ldots, \lambda_p$, $p < n$, lying in the interval $\{\mu|\mu \leqq \lambda\}$, it follows from Lemma 48 that we can find an orthonormal basis $\varphi_1, \ldots, \varphi_p$ for

$$E((-\infty, \lambda])L_2 = E(\lambda_1)L_2 \oplus \ldots \oplus E(\lambda_p)L_2$$

consisting of at most $n-1$ vectors. Thus we can find a non-zero set of constants $\alpha_1, \ldots, \alpha_n$ such that $g = \sum_{i=1}^n \alpha_i f_i$ is orthogonal to the range of $E((-\infty, \lambda])$, so that $E((-\infty, \lambda])g = 0$. Since $\lambda < \lambda_0$, $\lambda \notin \sigma_e(T)$. Thus, λ must be either an isolated point of $\sigma(T)$ or a point of the resolvent of T. Thus, by Theorem XII.2.3, there exists an $\varepsilon > 0$ such that $E((-\infty, \lambda+\varepsilon))g = 0$. For $2 \leqq j \leqq n$, let $\{h_{jm}\}$ be a sequence of functions in $C^\infty(I)$, such that h_{jm} vanishes except when $z_{j-1} \leqq t \leqq z_j$, such that $|h'_{jm} - f'_j|_2 \to 0$, and such that $h_{jm} \to f_j$ uniformly as $m \to \infty$. These functions h_{jm} exist by the auxiliary principle stated and proved in the second paragraph of the proof of Lemma 39, and $|h_{jm} - f_j|_2 \to 0$. Let $a < c < z_1$. Let f_1 be decomposed into the sum of two functions $f_1 = f + \hat{f}$, both infinitely often differentiable in (a, z_1), where $f_1(t) = \hat{f}(t)$ in the neighborhood of z_1, and $\hat{f}(t) = 0$ in $(a, c]$. Using the auxiliary principle stated and proved in the second paragraph of the proof of Lemma 39, let $\hat{h}_m$ be a sequence of functions in C^∞ which

vanish outside (c, z_1), such that $|\hat{h}'_m - \hat{f}'|_2 \to 0$. Then $\hat{h}_m \to \hat{f}$ uniformly, so that $|\hat{h}_m - \hat{f}|_2 \to 0$. Thus, putting $h_{1m} = \hat{h}_m + f$, we have $h_{1m}(t) = f_1(t)$ for t in a neighborhood of a, $|h'_{1m} - f'_1|_2 \to 0$, and $|h_{1m} - f_1|_2 \to 0$. Let $g_m = \sum_{i=1}^{n} \alpha_i h_{im}$. Then $g_m \in \mathfrak{D}(T)$, $g_m \to g$, and $|g'_m - g'|_2 \to 0$. Moreover, $g_m(t) = g(t)$ for $t < c$, and $g_m(t) = 0$ for $t \geq z_n$. Thus

$$
\begin{aligned}
&((T-\lambda)g_m, g_m) \\
&= \int_c^{z_n} \{-(p(t)g'_m(t))' + (q(t)-\lambda)g_m(t)\}\overline{g_m(t)}\,dt \\
&= -p(c)g'_m(c)\overline{g_m(c)} + \int_c^{z_n} \{p(t)|g'_m(t)|^2 + (q(t)-\lambda)|g_m(t)|^2\}\,dt \\
&= -p(c)g'(c)\overline{g(c)} + \int_c^{z_n} \{p(t)|g'_m(t)|^2 + (q(t)-\lambda)|g_m(t)|\}^2\,dt \\
&\to -p(c)g'(c)\overline{g(c)} + \int_c^{z_n} \{p(t)|g'(t)|^2 + (q(t)-\lambda)|g(t)|^2\}\,dt \\
&= \int_c^{z_n} ((\tau-\lambda)g)(t)\overline{g(t)}\,dt = 0.
\end{aligned}
$$

Moreover, $E((-\infty, \lambda+\varepsilon))g_m \to E((-\infty, \lambda+\varepsilon))g = 0$. Thus, by Theorem XII.2.6,

$$|((T-\lambda)g_m, g_m) - \int_{\lambda+\varepsilon}^{\infty} (\mu-\lambda)(E(d\mu)g_m, g_m)| \to 0.$$

Consequently,

$$\int_{\lambda+\varepsilon}^{\infty} (\mu-\lambda)(E(d\mu)g_m, g_m) \to 0.$$

Since

$$
\begin{aligned}
\int_{\lambda+\varepsilon}^{\infty} (\mu-\lambda)(E(d\mu)g_m, g_m) &\geq \varepsilon \int_{\lambda+\varepsilon}^{\infty} (E(d\mu)g_m, g_m) \\
&= \varepsilon|E([\lambda+\varepsilon, \infty))g_m|^2,
\end{aligned}
$$

then we have $E([\lambda+\varepsilon, \infty))g_m \to 0$. Thus,

$$g_m = E((-\infty, \lambda+\varepsilon))g_m + E([\lambda+\varepsilon, \infty))g_m \to 0.$$

Since $g_m \to g$, it follows that $g = 0$. But we saw above that $g \neq 0$. This contradiction proves our theorem. Q.E.D.

On the basis of the above lemma, we are now able to state three theorems which give a picture of the relation between the spectral theory and the oscillation theory of a real formally self adjoint second order formal differential operator τ, in the case in which τ is

bounded below. Theorem 40(a) describes this connection in case τ is not bounded below.

50 THEOREM. *Let τ be a real second order formally self adjoint formal differential operator τ defined on an interval I. Let T be a self adjoint extension of $T_0(\tau)$ defined by a separated set of boundary conditions. Suppose that τ is bounded below, and that $\sigma_e(\tau)$ is void. Let the spectrum $\sigma(\tau)$ be enumerated as $\lambda_1 < \lambda_2 < \lambda_3 < \ldots$ in ascending order. Then $\lambda_n \to \infty$. There is a unique (up to constant multiple) eigenfunction φ_n of T associated with λ_n, and φ_n has precisely $n-1$ zeros interior to I.*

PROOF. Since $\sigma_e(\tau)$ is void, every point in $\sigma(T)$ is isolated by Theorem 6.5. Thus $\lambda_n \to \infty$ by Corollary 26 and Corollary 27. The uniqueness of φ_n follows immediately from Lemma 41. Since, by Lemma 45, eigenfunctions belonging to distinct eigenvalues have different numbers of zeros, and since by Corollary 44 this number increases with n. φ_n has at least $n-1$ zeros. Suppose, on the other hand, that φ_n has $p \geqq n$ zeros. By Lemma 47, for some sufficiently small $\varepsilon > 0$, $\varphi(t, \lambda_n - \varepsilon)$ has $p \geq n$ zeros so that, by Lemma 49, there are at least n eigenvalues in the interval $(-\infty, \lambda_n - \varepsilon]$ which is a contradiction. Q.E.D.

51 LEMMA. *If every solution of $\tau\sigma = \lambda\sigma$ has an infinite number of zeros, then an infinite number of points of $\sigma(T)$ lie below λ.*

PROOF. If $\lambda > \lambda_0$, $(-\infty, \lambda)$ contains points in $\sigma_x(T) - \sigma_e(\tau)$, so that our assertion is evident from Theorem 40. Hence, we may assume that $\lambda \leq \lambda_0$. Let $a < c < b$, and let $f(t, \mu)$ be the solution of $\tau\sigma = \mu\sigma$ satisfying the initial condition $f(c, \mu) = 0$, $f'(c, \mu) = 1$. Since $f(t, \mu)$ depends continuously on μ for each t in I, since $f(\cdot, \lambda)$ has infinitely many zeros, and since $f(\cdot, \lambda)$ has a non vanishing derivative (and hence changes sign) at each of its zeros, it follows immediately that as $\mu \to \lambda$, the number of zeros of $f(\cdot, \mu)$ approaches infinity. Hence, by Corollary 86, the number of zeros of $\sigma(\cdot, \mu)$ approaches infinity as $\mu \to \lambda$. Our assertion now follows immediately from Lemma 49.

52 LEMMA. *If τ has no boundary values at b, no point of J_n but its upper end point can be an eigenvalue of T.*

Proof. Let λ be a point of J_n other than its upper end point. Using Lemma 47, let $\varepsilon > 0$ be chosen so small that $\lambda-\varepsilon$ and $\lambda+\varepsilon$ are both in J_n. Let M be the largest zero of $\sigma(\cdot, \lambda+\varepsilon)$, and let m be the largest zero of $\sigma(\cdot, \lambda-\varepsilon)$. By Corollary 46, the largest zero of $\sigma(\cdot, \mu)$ lies between M and m for $|\mu-\lambda| \leqq \varepsilon$. Let $N > M$, let B denote the boundary condition (if any) at a defining T, and let B_N denote the boundary condition $f(N) = 0$. By remark (b) preceding Lemma 41, the operator T_N obtained from the restriction τ_N of τ to the interval $(a, N]$ by imposition of the boundary conditions B, B_N is self adjoint. By Lemma 41 the only square-integrable solutions of the equation $\tau_N \sigma = \mu\sigma$ satisfying the boundary condition B at a are the multiples of $\sigma(\cdot, \mu)$. By what we have observed above, if $|\lambda-\mu| < \varepsilon$, none of these solutions have a zero in $[M, \infty)$, so none satisfy the boundary condition B_N. That is, T_N has no eigenvalues μ with $|\lambda-\mu| \leqq \varepsilon$. Since the spectrum of T_N consists entirely of its point spectrum, the set $|\lambda-\mu| \leq \varepsilon$ does not intersect $\sigma(T_N)$. Thus, by Theorem XII.2.6, $|(\tau-\lambda)f| \geq \varepsilon|f|$ for every f in $\mathfrak{D}(T_N)$. That is, $|(\tau-\lambda)f| \geqq \varepsilon|f|$ for every function in $\mathfrak{D}(T_1(\tau))$ which satisfies the boundary condition B and vanishes in a neighborhood of b.

We maintain that this set Γ of functions is dense in $\mathfrak{D}(T)$, if $\mathfrak{D}(T)$ is made into a complete Hilbert space with the norm

$$|f|^* = \{|f|_2^2 + |Tf|_2^2\}^{1/2}.$$

To establish this, note that it follows from the Hahn-Banach theorem (cf. II.3.13) that it suffices to show that every continuous linear functional φ on $\mathfrak{D}(T_1(\tau))$ which vanishes on Γ vanishes on $\mathfrak{D}(T)$. Since $\Gamma \supseteqq \mathfrak{D}(T_0(\tau))$, φ is a boundary value for τ. Since, by hypothesis, τ has no boundary values at b, it follows from Theorem 2.19 that φ is a boundary value at a. Let f be in $\mathfrak{D}(T)$, and let g agree with f in a neighborhood of a and vanish in a neighborhood of b. Then g is in Γ and since $\varphi(f) = \varphi(g)$, $\varphi(f) = 0$. Thus Γ is dense in $\mathfrak{D}(T)$.

Consequently, the inequality $|(T-\lambda)f| > \varepsilon|f|$, which we have already verified for f in Γ, must hold for f in $\mathfrak{D}(T)$. It follows immediately that λ cannot be an eigenvalue of T. Q.E.D.

53 Theorem. *Let τ be a real second order formally self adjoint formal differential operator defined on an interval I. Let T be a self*

adjoint extension of $T_0(\tau)$ defined by a separated set of boundary conditions. Suppose that τ is bounded below and that $\sigma_e(\tau)$ is not void. Let λ_0 be the smallest point in $\sigma_e(\tau)$. Then, if every solution of the equation $\tau\sigma = \lambda_0\sigma$ has an infinite number of zeros in I, $\sigma(T)$ contains an infinite number of isolated points lying below λ_0. If these points are enumerated in ascending order as $\mu_1 < \mu_2 < \mu_3 < \ldots$, then $\mu_n \to \lambda_0$. Moreover, there is a unique (up to constant multiple) eigenfunction φ_n of T associated with μ_n, and φ_n has precisely $n-1$ zeros.

Proof. Since $\sigma_e(\tau) \cap (-\infty, \lambda_0)$ is void, then every point in $\sigma(T) \cap (-\infty, \lambda_0)$ is isolated by Corollary 6.3 and Theorem 6.5. By Lemma 51 there are an infinite number of points in $\sigma(T)$ below λ_0. Thus it follows from Corollary 24(c) and Lemma 21 that $\mu_n \to \lambda_0$. That φ_n is unique and has exactly $n-1$ zeros is proved just as in the preceding theorem. Q.E.D.

54 Corollary. *Under the hypotheses and with the notation of the preceding theorem, T is defined by at most one boundary condition, which is a boundary condition at an end point a of the interval I. Let $\lambda < \lambda_0$, and let $\sigma(t, \lambda)$ be a solution of the equation $\tau\sigma = \lambda\sigma$ which is square-integrable at a and satisfies the boundary condition (if any) at a. Then the number of points in $\sigma(T)$ which lie in the region $z < \lambda$ is precisely the number of zeros of $\sigma(t, \lambda)$ interior to I.*

Proof. Since $\sigma_e(\tau)$ is not void, it follows from Theorems 4.1 and 4.2 that the deficiency indices of τ cannot be (2,2). Thus, by the remarks on second order operators τ made at the end of Section 2, just before the statement of Theorem 2.30, there must be at least one end point b of I at which there are no boundary values. Moreover, τ will have at most two boundary values, both at the other end point of a of I, and every self adjoint extension T of $T_0(\tau)$ will be defined by at most one boundary condition, which is necessarily a boundary condition at a. By the preceding theorem and by Lemma 52, the number of points in $\sigma(T) \cap (-\infty, \lambda)$ is precisely equal to the number of intervals $J_0, J_1, \ldots, J_n$ which are contained entirely in $(-\infty, \lambda)$. If λ is in J_k, so that $\sigma(\cdot, \lambda)$ has k zeros, this number is clearly k. Q.E.D.

55 Theorem. *Let τ be a real second order formally self adjoint formal differential operator τ defined on an interval I. Let T be a self*

adjoint extension of $T_0(\tau)$. *Suppose that* τ *is bounded below and that* $\sigma_e(\tau)$ *is not void. Let* λ_0 *be the smallest point in* $\sigma_e(\tau)$. *Then, if a solution of the equation* $\tau\sigma = \lambda_0\sigma$ *has a finite number* k *of zeros in* I, *the part of* $\sigma(T)$ *lying below* λ_0 *consists of at least* $k-1$ *and at most* $k+2$ *points. If these points are enumerated in ascending order as* $\mu_1, \ldots, \mu_m$, *then there is a unique (up to a constant multiple) eigenfunction* φ_n *of* T *associated with* μ_n *and* φ_n *has precisely* $n-1$ *zeros.*

Proof. We establish, just as in the proof of Theorem 53, that if the eigenvalues μ_n of T lying in $(-\infty, \lambda_0)$ are enumerated in ascending order, then the eigenfunction φ_n corresponding to μ_n is unique and has exactly $n-1$ zeros. Let m be the number of μ_n in $(-\infty, \lambda_0)$. If m were infinity, it would follow from Lemma 35 that every solution of $\tau\sigma = \lambda_0\sigma$ has arbitrarily many zeros in I, contrary to assumption. Hence m is finite. Since φ_m has $m-1$ zeros, by Lemma 35, the number k of zeros of a solution σ of $\tau\sigma = \lambda_0\sigma$ must be at least $m-2$. Thus, $m \leq k+2$.

On the other hand, if σ has k zeros, let $a < c < b$. Then, for sufficiently small ε, the solution f of $\tau f = (\lambda_0-\varepsilon)f$ such that $f(c) = \sigma(c)$, $f'(c) = \sigma'(c)$ also has k zeros (cf. proof of Lemma 47). Hence by Lemma 35, $\sigma(t, \lambda_0 - \varepsilon)$ has at least $k-1$ zeros. By Lemma 49, at least $k-1$ eigenvalues of T lie below $\lambda_0 - \varepsilon$. Thus $m \geqq k-1$. Q.E.D.

56 Corollary. *Under the hypotheses and with the notation of the preceding theorem,* T *is defined by at most one boundary condition, which is a boundary condition at an end point* a *of the interval* I.

Let $\lambda < \lambda_0$, *and let* $\sigma(t, \lambda)$ *be a solution of the equation* $\tau\sigma = \lambda\sigma$ *which is square-integrable at* a *and satisfies the boundary condition (if any) at* a. *Then the number of points in* $\sigma(T)$ *which lie in the region* $z < \lambda$ *is precisely the number of zeros of* $\sigma(t, \lambda)$ *interior to* I.

Proof. This follows from the previous theorem in exactly the same way that Corollary 54 follows from Theorem 53. Q.E.D.

57 Corollary. *Let* τ *be a real self adjoint second order formal differential operator of the form*

$$\tau = -\left(\frac{d}{dt}\right)^2 + q(t)$$

defined on an interval $[a, \infty)$. *Suppose that* $\lim_{t\to\infty} q(t) = \lambda_0$. *Let* T *be a self adjoint extension of* $T_0(\tau)$. *Then* τ *is finite below* λ_0, *and* λ_0 *is the smallest number in* $\sigma_e(\tau)$. *If*

$$\limsup_{t\to\infty} t^2(q(t)-\lambda_0) < -\tfrac{1}{4},$$

$\sigma(T) \cap (-\infty, \lambda_0)$ *consists of an infinite sequence of isolated points converging to* λ_0, *while if*

$$\liminf_{t\to\infty} t^2(q(t)-\lambda_0) > -\tfrac{1}{4},$$

$\sigma(T) \cap (-\infty, \lambda_0)$ *consists of a finite number of points.*

Proof. This follows immediately from Corollary 37, Theorem 16, Theorem 6.5, Corollary 26 and Theorems 55 and 53, for, if $c = \sup_{a\leq t<\infty} |q(t)|$, $\tau + c$ is evidently positive, proving that τ is bounded below. Q.E.D.

The next theorem applies the methods we have just developed to give a useful result on the asymptotic distribution of the eigenvalues of a second order differential operator. Finer but similar results may be obtained by less elementary methods.

58 Theorem. *Let* τ *be the formal differential operator*

$$\tau = -\left(\frac{d}{dt}\right)^2 + q(t), \qquad 0 \leq t < \infty,$$

where q *is increasing and* q' *is positive and increasing. For each sufficiently large* λ, *let* $t(\lambda)$ *be the unique solution of the equation* $q(t(\lambda)) = \lambda$. *Let* T *be the self adjoint operator determined by* τ *and by the boundary condition*

$$f(0) + kf'(0) = 0, \qquad -\infty < k \leq +\infty.$$

Let $N(\lambda)$ *be the number of eigenvalues of* T *in the region* $\mu \leq \lambda$. *Then as* $\lambda \to \infty$ *we have the asymptotic relation*

$$N(\lambda) \sim \frac{1}{\pi}\int_0^{t(\lambda)} (\lambda-q(t))^{1/2}\, dt.$$

Proof. Put $\varphi_\lambda(t) = (\lambda-q(t))^{1/2}$, $0 \leq t \leq t(\lambda)$. The function $\lambda - q(t)$ is concave downward. Since $(f(t))^{1/2}$ is concave downward wherever f is positive and concave downward, φ_λ is decreasing and

concave downward. Let σ_λ be a solution of the equation $\tau\sigma_\lambda = \lambda\sigma_\lambda$, so that

$$\sigma_\lambda''(t) + \varphi_\lambda(t)^2\sigma_\lambda(t) = 0, \qquad 0 \leq t \leq t(\lambda),$$

and suppose also that $\sigma_\lambda(t)$ satisfies the boundary condition determining T. Then, by Corollary 54, $N(\lambda)$ is equal to the number $Z(\lambda)$ of zeros of σ_λ.

Next define a finite sequence of points as follows. Put $t_0 = 0$ and put $t_{i+1} = t_i + \pi\varphi_\lambda(t_i)^{-1}$ as long as $\varphi_\lambda(t_i)$ is defined, that is, as long as $t_i \leq t(\lambda)$. In this way, a finite sequence $t_0, \ldots, t_n$ of points (depending on λ) is defined. Since φ_λ is decreasing, it follows that the sequence $t_1 - t_0, t_2 - t_1, \ldots, t_n - t_{n-1}$ is increasing. Since φ_λ is decreasing, we may use Lemma 35 to compare the equation

$$\sigma_\lambda''(t) + \varphi_\lambda(t)^2\sigma_\lambda(t) = 0$$

to the equation

$$\sigma_0''(t) + \varphi_\lambda(t_i)^2\sigma_0(t) = 0,$$

concluding that σ_λ has at most one zero in the half-open interval $[t_{i-1}, t_i)$. Thus, σ_λ has at most n zeros in the interval $(0, t_n)$. Since by the comparison Lemma 35, σ_λ has at most one zero in the interval $[t_n, t(\lambda))$ and none in the interval $[t(\lambda), \infty)$, it follows that $Z(\lambda) \leq n+1$.

Now $t_{i+1} - t_i = \pi\varphi_\lambda(t_i)^{-1}$, so $\varphi_\lambda(t_i)(t_{i+1} - t_i) = \pi$. Hence

$$[*] \qquad n = \frac{1}{\pi}\sum_{i=0}^{n-1} \varphi_\lambda(t_i)(t_{i+1} - t_i).$$

Let $\varepsilon > 0$ be fixed, j be the smallest integer such that $\varphi_\lambda(t_{j+1}) \leqq (1-\varepsilon)\varphi_\lambda(t_j)$ if such a j exists. If not, let $j = n$. Then

$$\varphi_\lambda(t_j) - \varphi_\lambda(t_{j+1}) \geqq \varepsilon\varphi_\lambda(t_j).$$

Since φ_λ is concave downward, and $t_{i+1} - t_i$ is an increasing function of i, we have

$$\varphi_\lambda(t_i) - \varphi_\lambda(t_{i+1}) \geqq \varepsilon\varphi_\lambda(t_j)$$

for all $n-1 \geqq i \geqq j$. Hence

$$\varphi_\lambda(t_j) - \varphi_\lambda(t_{n-1}) \geqq (n-1-j)\varepsilon\varphi_\lambda(t_j).$$

Since $\varphi_\lambda(t_{n-1}) \geq 0$, it follows that $\varepsilon(n-1-j) \leq 1$, so that $(n-j) \leq 1/\varepsilon+1$. Consequently, since φ_λ is decreasing and since all the terms in the sum [*] are unity, we have

$$
\begin{aligned}
N(\lambda)-2 \leq n &= \frac{1}{\pi}\sum_{i=0}^{n-1} \varphi_\lambda(t_i)(t_{i+1}-t_i)-1 \\
&\leq \frac{1}{\pi}\sum_{i=0}^{j-1} \varphi_\lambda(t_i)(t_{i+1}-t_i)+1/\varepsilon \\
&\leq \frac{1}{\pi}(1-\varepsilon)^{-1}\sum_{i=0}^{j-1} \varphi_\lambda(t_{i+1})(t_{i+1}-t_i)+1/\varepsilon \\
&\leq \frac{1}{\pi}(1-\varepsilon)^{-1}\int_0^{t_j} \varphi_\lambda(t)dt+1/\varepsilon \\
&\leq \frac{1}{\pi}(1-\varepsilon)^{-1}\int_0^{t(\lambda)} (\lambda-q(t))^{1/2}dt+1/\varepsilon.
\end{aligned}
$$

Thus, since $N(\lambda) \to \infty$,

$$
\limsup_{\lambda\to\infty} \frac{N(\lambda)}{\dfrac{1}{\pi}\displaystyle\int_0^{t(\lambda)} (\lambda-q(t))^{1/2}\,dt} \leq 1.
$$

This proves half of our assertion. To prove the other half we proceed, in much the same way, as follows. If λ is large enough, we can define a finite sequence of points by letting s_0 be the unique root of $\varphi_\lambda(s_0) = \pi$, and then by putting $s_{i+1} = s_i - \pi\varphi(s_i)^{-1}$ as long as $s_i-\pi\varphi(s_i)^{-1} \geq 0$, while if $s_i - \pi\varphi(s_i)^{-1} < 0$, we put $s_{i+1} = 0$ and terminate the sequence. Then a decreasing sequence $s_0, \ldots, s_m = 0$ of points is defined, and as above we see that

$$
m-\theta = \frac{1}{\pi}\sum_{i=0}^{m} \varphi_\lambda(s_{i-1})(s_i-s_{i-1}),
$$

where θ is a non-negative quantity which is at most 1. Since φ_λ is decreasing, we can use Lemma 35 to compare the equation

$$
\sigma_\lambda''(t)+\varphi_\lambda(t)^2\sigma_\lambda(t) = 0
$$

to the equation

$$
\sigma_\lambda''(t)+\varphi_\lambda(s_i)^2\sigma_\lambda(t) = 0,
$$

and conclude as above that σ_λ has at least $m-1$ zeros in the interval $[0, \infty)$. Thus $Z(\lambda) \geq m-1$.

Next we let j be the largest integer less than m such that $\varphi_\lambda(s_j) \leq (1-\varepsilon)\varphi_\lambda(s_{j+1})$ or, if no such $j < m$ exists, put $j = m$. Arguing exactly as above, we conclude that $j \leq 1/\varepsilon - 1$. Since $s_j - s_{j+1}$ decreases with increasing j, and $s_0 - s_1 = \pi\varphi_\lambda(s_0)^{-1} = 1$, it follows that $s_0 - s_{j+1} \leq j+1 \leq \varepsilon^{-1}$. Since $q'(t)$ is positive and increasing, we can find a constant K such that $q'(t) \geq K^{-1}$. Then the solution s_0 of the equation $q(s_0) = \lambda - \pi^2$ must be related to the solution $t(\lambda)$ of the equation $q(t(\lambda)) = \lambda$ by $t(\lambda) \geq s_0 \geq t(\lambda) - K\pi^2$. Hence we have $s_{j+1} \geq t(\lambda) - K - \varepsilon^{-1}$. Let $K\pi^2 - \varepsilon^{-1} = K_\varepsilon$. Since K_ε is finite, it follows that for λ so large that $t(\lambda) > K_\varepsilon$, we have $s_j \neq 0$, so $j \neq m$. Then

$$
\begin{aligned}
N(\lambda)+1 &\geq m - \frac{1}{\pi}\sum_{i=0}^{m} \varphi_\lambda(s_{i-1})(s_{i-1}-s_i)+\theta \\
&\geq \frac{1}{\pi}(1-\varepsilon)\sum_{i=j+2}^{m} \varphi_\lambda(s_i)(s_i - s_{i-1}) \\
&\geq \frac{1}{\pi}(1-\varepsilon)\int_0^{s_{j+1}} \varphi_\lambda(t)\,dt \\
&\geq \frac{1-\varepsilon}{\pi}\left\{\int_0^{t(\lambda)} \varphi_\lambda(t)\,dt - \int_{t(\lambda)-K_\varepsilon}^{t(\lambda)} \varphi_\lambda(t)\,dt\right\}.
\end{aligned}
$$

Since φ_λ is decreasing,

$$
\frac{1}{t(\lambda)}\int_0^{t(\lambda)} \varphi_\lambda(t)\,dt \geq \frac{1}{K_\varepsilon}\int_{t(\lambda)-K_\varepsilon}^{t(\lambda)} \varphi_\lambda(t)\,dt.
$$

Hence, it follows that

$$
N(\lambda)+1 \geq \frac{(1-\varepsilon)}{\pi}\left(1 - \frac{K_\varepsilon}{t(\lambda)}\right)\int_0^{t(\lambda)} \varphi_\lambda(t)\,dt.
$$

This shows that

$$
\liminf_{\lambda\to\infty} \frac{N(\lambda)}{\dfrac{1}{\pi}\displaystyle\int_0^{t(\lambda)} (\lambda - q(t))^{1/2}\,dt} \geq 1,
$$

and concludes the proof of the theorem. Q.E.D.

Equations with periodic coefficients (which are fundamental in the quantum-mechanical theory of solid crystals) have a number of interesting and important properties. In the next few paragraphs τ denotes a formally self adjoint formal differential operator of order n, defined on the interval $R = \{-\infty < t < +\infty\}$. Let τ have the form

$$\tau = \sum_{j=0}^{n} a_j(t)\left(\frac{d}{dt}\right)^j.$$

and suppose that all the coefficients a_j are periodic and have the same period. We can assume without loss of generality that this period is 1; thus

$$a_j(t+1) = a_j(t), \qquad j = 0, \ldots, n.$$

If follows immediately that all the coefficients of τ are bounded; thus, by Theorem 6.35, it follows that *τ has no boundary values at $+\infty$ or at $-\infty$*. By Theorem 2.19, τ has no boundary values. Thus, by Definition 2.17 every linear functional on the Hilbert space $\mathfrak{D}(T_1(\tau))$ which vanishes on $\mathfrak{D}(T_0(\tau))$ vanishes identically. Hence $\mathfrak{D}(T_1(\tau))$ is the closure of $\mathfrak{D}(T_0(\tau))$, so that $T_1(\tau)$ is the closure of $T_0(\tau)$. Since $T_1(\tau^*) = T_0(\tau)^*$, we have $T_1(\tau^*) = T_1(\tau)^*$ by Lemma XII.1.5. In particular, if τ is formally self adjoint, $T = T_1(\tau)$ is self adjoint.

Our next observation is that *T has no point spectrum*. Indeed, let $\lambda \in \sigma_p(T)$. Let $\mathfrak{A}$ be the space of square-integrable solutions of the equation $(\tau f) = Tf = \lambda f$. Then $\mathfrak{A}$ is clearly finite dimensional. Let S denote the unit *shift operator*, so that $(Sf)(t) = f(t-1)$. Then, since the coefficients of τ are periodic, it follows evidently that $S\tau = \tau S$. The operator S is evidently unitary. If $\hat{S}$ denotes the restriction of S to the finite dimensional Hilbert space $\mathfrak{A}$, $\hat{S}$ is evidently a unitary map in $\mathfrak{A}$. Hence, since $\mathfrak{A} \neq \{0\}$, it follows that $\hat{S}$ must have an eigenvalue α. By Theorem X.4.2, $|\alpha| = 1$. Thus, there exists a non-zero function $f \in \mathfrak{A}$ such that $Sf = \alpha f$; i.e., $f(t-1) = \bar{\alpha}f(t)$. It follows that $|f(\cdot)|$ is periodic; hence $f \notin L_2(-\infty, +\infty)$. This contradiction proves the above assertion.

Let E^n denote n-dimensional unitary space. With each complex number λ, associate a linear transformation $B(\lambda)$ in E^n, as follows. If $[c_0, \ldots, c_{n-1}] \in E^n$, let σ denote the unique solution of $\tau\sigma = \lambda\sigma$ which satisfies $\sigma^{(i)}(0) = c_i$, $i = 0, \ldots, n-1$. Put $(B(\lambda)c)_i = \sigma^{(i)}(-1)$,

$i = 0, \ldots, n-1$. By Corollary 1.5, $B(\lambda)$ depends analytically on λ.

59 Lemma. *The spectrum, the essential spectrum, and the continuous spectrum of $T = T_1(\tau)$ are all identical.*

Proof. It has just been seen that $\sigma_p(T)$ is void. Since $T^* = T_1(\tau^*)$, and since τ^* is also a formal differential operator with coefficients of period 1, $\sigma_p(\tau^*)$ is also void. By Lemma XII.1.6, $\bar{\lambda}$ is the residual spectrum of T only if $\lambda \in \sigma_p(T_1(\tau^*))$. Hence the residual spectrum of T is void, and $\sigma(T) = \sigma_c(T)$. We have $\sigma_c(T) \subseteq \sigma_e(T) \subseteq \sigma(T) \subseteq \sigma_c(T)$. Q.E.D.

60 Lemma. *If λ is a complex number such that $B(\lambda)$ has an eigenvalue of modulus 1, then λ is in $\sigma(T)$.*

Proof. Suppose that $\lambda \notin \sigma(T)$. Then, by Lemma 3.2 (see especially the first paragraph of the proof of that lemma), since $\lambda \notin \sigma_p(T)$ it follows that the space Σ of all solutions of $\tau\sigma = \lambda\sigma$ is the direct sum of the space Σ_+ of all solutions of $\tau\sigma = \lambda\sigma$ which belong to $L_2(0, \infty)$, and of the space Σ_- of all solutions of $\tau\sigma = \lambda\sigma$ which belong to $L_2(-\infty, 0)$. It is clear that both Σ_+ and Σ_- are invariant under the shift operator S.

Let E be the projection on Σ defined by $E\Sigma_- = 0$, $Ef = f$, $f \in \Sigma_+$. Let S_+ and S_- denote the restriction of the shift operator S to Σ_+ and Σ_- respectively, and let $\hat{S}$ denote the restriction of S to Σ. If $\lambda \in \sigma(S_-)$, so that λ is an eigenvalue of S_-, there exists a non-zero $f \in \Sigma_-$ such that $S_-f = \lambda f$. Consequently, $\hat{S}f = \lambda f$, so that λ is an eigenvalue of $\hat{S}$, and thus $\lambda \in \sigma(\hat{S})$. This shows that $\sigma(S_-) \subseteq \sigma(\hat{S})$; and similarly $\sigma(S_+) \subseteq \sigma(\hat{S})$, so that $\sigma(S_-) \cup \sigma(S_+) \subseteq \sigma(\hat{S})$. On the other hand, let $\lambda \in \sigma(\hat{S})$, so that there exists a non-zero f in Σ such that $Sf = \lambda f$. Since $S\Sigma_+ \subseteq \Sigma_+$, $S\Sigma_- \subseteq \Sigma_-$, it follows immediately that $SE = ES$. Thus $SEf = \lambda Ef$, $S(I-E)f = \lambda(I-E)f$. Since $0 \neq f = Ef + (I-E)f$, either $Ef \neq 0$, in which case we have $\lambda \in \sigma(S_+)$; or $(I-E)f \neq 0$, in which case we have $\lambda \in \sigma(S_-)$. This shows that $\sigma(\hat{S}) = \sigma(S_+) \cup \sigma(S_-)$.

We shall now show that neither $\sigma(S_+)$ nor $\sigma(S_-)$ contains any points of modulus 1. First consider S_-. Make Σ_- into a finite dimensional Hilbert space with norm

$$|f| = \left(\int_{-\infty}^{0} |f(t)|^2\, dt\right)^{\frac{1}{2}}, \qquad f \in \Sigma_- .$$

Then we have

$$\begin{aligned} |S_- f| &= \left(\int_{-\infty}^{0} |f(t-1)|^2\, dt\right)^{\frac{1}{2}} \\ &= \left(\int_{-\infty}^{-1} |f(t)|^2\, dt\right)^{\frac{1}{2}}, \qquad\qquad f \in \Sigma_- , \qquad f \neq 0, \\ &< |f|. \end{aligned}$$

Thus all the eigenvalues of S_- have modulus less than 1. In the same way, if the space Σ_+ is made into a finite dimensional Hilbert space with the norm

$$|f| = \left\{\int_{0}^{\infty} |f(t)|^2\, dt\right\}^{\frac{1}{2}}, \qquad f \in \Sigma_+,$$

then we find that $|S_+ f| > |f|$, $f \in \Sigma_+$, $|f| \neq 0$. Consequently, all the eigenvalues of S_+ have modulus greater than 1.

It follows that none of the points in $\sigma(\hat{S})$ have modulus 1. The final step in the proof is to show that $\sigma(\hat{S}) = \sigma(B(\lambda))$, so we may conclude that $\sigma(B(\lambda))$ contains no point of modulus 1, contrary to assumption. To do this, we argue as follows. Let $c = [c_0, \ldots, c_{n-1}] \in E^n$, and let $\sigma = M(c)$ be the unique solution of $\tau\sigma = \lambda\sigma$ such that $\sigma^{(i)}(0) = c_i$, $i = 0, \ldots, n-1$. Then it is clear from the definition of $B(\lambda)$ that $B(\lambda) = M^{-1}\hat{S}M$. Hence $B(\lambda)$ and $\hat{S}$ are equivalent mappings so that $\sigma(B(\lambda)) = \sigma(\hat{S})$. Q.E.D.

Remark. The last two lemmas hold with much the same proofs without the hypothesis that τ is formally self adjoint.

61 Lemma. *Let τ be formally self adjoint. Suppose that $B(\lambda_1)$ has no eigenvalue of modulus 1. Then λ_1 does not belong to $\sigma(T)$.*

Proof. Let (cf. Definition VII.3.17 ff. for notations) $E_+(\lambda) = E(U_+\sigma(B(\lambda)); B(\lambda))$ and $E_-(\lambda) = E(U_-\sigma(B(\lambda)); B(\lambda))$, where $U_+ = \{z \mid |z| > 1\}$ and $U_- = \{z \mid |z| < 1\}$. Then, by Lemma VII.6.6, $E_+(\lambda)$ and $E_-(\lambda)$ are analytic in λ for λ in a neighborhood of the closure of an open circle N with center λ_1. It is clear that $E_+(\lambda) +$

$E_-(\lambda) = I$. Let $v_1, \ldots, v_k$ be a basis for $E_+(\lambda_1)E^n$, and $v_{k+1}, \ldots, v_n$ a basis for $E_-(\lambda_1)E^n$. Put $v_i(\lambda) = E_+(\lambda)v_i$ for $i = 1, \ldots, k$, $v_i(\lambda) = E_-(\lambda)v_i$ for $i = k+1, \ldots, n$. By the Hahn-Banach theorem, there exist functionals $u_1^*, \ldots, u_n^* \in (E^n)^*$ such that $u_i^*(v_j(\lambda_1)) = \delta_{ij}$, $i = 1, \ldots, n$. Hence there exists a circular neighborhood N_1 of λ_1 with center λ_1 such that $\det \{u_i^*(v_j(\lambda))\} \neq 0$, $\lambda \in N_1$. It is clear that $v_1(\lambda), \ldots, v_n(\lambda)$ is an independent set of vectors for $\lambda \in N_1$. Thus $v_1(\lambda), \ldots, v_n(\lambda)$ is a basis for E^n if $\lambda \in N_1$. It follows immediately that if $\lambda \in N_1$, $v_1(\lambda), \ldots, v_k(\lambda)$ is a basis for $E_+(\lambda)E^n$ and $v_{k+1}(\lambda), \ldots, v_n(\lambda)$ is a basis for $E_-(\lambda)E^n$. For $c = [c_0, \ldots, c_{n-1}] \in E^n$, let $M_\lambda(c)$ be the unique solution σ of $\tau\sigma = \lambda\sigma$ such that $\sigma^{(i)}(0) = c^i$, $i = 0, \ldots, n-1$. Put $\sigma_i(\cdot, \lambda) = M_\lambda(v_i(\lambda))$, $i = 1, \ldots, n$, $\lambda \in N_1$. By Lemma VII.8.4, we have that $|B(\lambda)^n v_i(\lambda)| = O((1-\varepsilon)^n)$ for some $\varepsilon = \varepsilon(\lambda) > 0$, $1 \leq i \leq k$, $\lambda \in N$. Thus $\sigma_i(t, \lambda)$ and all its derivatives decrease exponentially fast as $n \to \infty$, uniformly for $\lambda \in N$, and for $1 \leq i \leq k$. In particular, $\sigma_i(\cdot, \lambda) \in L_2(0, \infty)$, $1 \leq i \leq k$. In the same way we see that $\sigma_i(\cdot, \lambda) \in L_2(-\infty, 0)$, $k < i \leq n$.

Let $\sum_{i=1}^n \alpha_i \sigma_i(\cdot, \lambda)$ be square-integrable at $-\infty$. Then, since $\sigma_i(\cdot, \lambda)$ is in $L_2(-\infty, 0)$ for $i \leq k$, so is $\sum_{i=k+1}^n \alpha_i \sigma_i(\cdot, \lambda)$. On the other hand, $\sigma_i(\cdot, \lambda) \in L_2(0, \infty)$ for $i > k$. Consequently, $\sum_{i=k+1}^n \alpha_i \sigma_i(\cdot, \lambda) \in L_2(0, \infty) \cap L_2(-\infty, 0) = L_2(-\infty, +\infty)$. Since we have already shown that no solution of $\tau\sigma = \lambda\sigma$ belongs to $L_2(-\infty, +\infty)$, it follows that $\sigma_i = 0$ for $i > k$. Thus it follows that $\{\sigma_1(\cdot, \lambda), \ldots, \sigma_k(\cdot, \lambda)\}$ is a basis for the set of solutions of $\tau\sigma = \lambda\sigma$ which are square-integrable at $-\infty$. Similarly, $\{\sigma_{k+1}(\cdot, \lambda), \ldots, \sigma_n(\cdot, \lambda)\}$ is a basis for the set of solutions of $\tau\sigma = \lambda\sigma$ which are square-integrable at $+\infty$. Let $F_{ij}(\lambda)$ be defined in terms of the boundary form F_t of the formal differential operator τ (cf. Definition 2.1) by the equation

$$F_{ij}(\lambda) = F_t(\sigma_i(\lambda), \overline{\sigma_j(\bar{\lambda})}).$$

By Green's formula 2.4, this matrix is independent of t. Since for $\lambda \in N_1$, $\sigma_1, \ldots, \sigma_n$ form a basis for the space of solutions of $\tau\sigma = \lambda\sigma$, it follows from Lemma 2.2 that the matrix $F_{ij}(\lambda)$ is non-singular for $\lambda \in N_1$. Let the matrix $G_{ij}(\lambda)$ be its inverse. By Corollary 3.18, the resolvent $R(\lambda; T)$ of T is represented for $\lambda \in N_1$, $\mathscr{I}\lambda \neq 0$, by an integral kernel $K(t, s; \lambda)$ having the form

$$K(t, s; \lambda) = -\sum_{i=1}^{k} \sum_{j=k+1}^{n} G_{ij}(\lambda)\varphi_j(t, \lambda)\overline{\varphi_i(t, \bar{\lambda})}, \qquad s < t, \mathscr{I}\lambda = 0.$$

Since $G_{ij}(\lambda)$ is analytic everywhere in N_1, even in those points of N_1 lying along the real axis, the desired result follows from the Titchmarsh-Kodaira formula (5.18), and from Corollary 5.29. Q.E.D.

62 Definition. Let λ_1 be a real number such that $B(\lambda_1)$ has an eigenvalue of modulus 1. Let $\mu_1, \ldots, \mu_p$ be the collection of all such eigenvalues. By Theorem VII.6.9, there exists a neighborhood V of λ_1, neighborhoods U_i of μ_i, $i = 1, \ldots, p$, and integers k_i and $K_i \geqq 1$, $i = 1, \ldots, p$, such that if $i = 1, \ldots, p$ and $\lambda \in V$, then $U_i\sigma(B(\lambda))$ consists of k_i points, given as the various values of a fractional power series

$$\sum_{j=0}^{\infty} \alpha_{ilj}(\lambda - \lambda_1)^{j/k_i} \qquad l = 1, \ldots, k_i, \qquad i = 1, \ldots, p,$$

in $(\lambda - \lambda_1)^{1/k_i}$. The point λ_1 is called a *branching point* of τ if one of the integers k_i is greater than 1.

If the first two sentences of this definition are modified to read: "Let λ_1 be a number, and $\mu_1, \ldots, \mu_p$ the eigenvalues of $B(\lambda_1)$," the definition of a *branching point of τ in the extended sense* is obtained.

Remark. A fractional power series $\sum_{i=0}^{\infty}\beta_i z^{i/L}$ which cannot be written as a fractional power series $\sum_{i=0}^{\infty}\alpha_i z^{i/M}$ for some M dividing L evidently takes on at least L distinct values for arbitrarily small z. Moreover, $\sum_{i=0}^{\infty}\beta_i z^{i/L}$ can evidently be written as $\sum_{i=0}^{\infty}\gamma_i z^{i/M}$ for any M of which L is a factor. If λ_1 is not a branching point, the roots $\mu_1(\lambda), \ldots, \mu_p(\lambda)$ vary analytically with λ for λ sufficiently near λ_1. If λ_1 is a branching point, there evidently exists a neighborhood U of the unit circle such that for $\lambda \neq \lambda_1$ and sufficiently close to λ_1, the number of points in $U\sigma(B(\lambda))$ is constant and larger than the number of points in $U\sigma(B(\lambda_1))$. Thus, *the set of branching points of τ is evidently isolated.* The same remark may be made concerning branching points in the extended sense.

63 Lemma. *If τ is formally self adjoint, and if a real number λ_1 is in the boundary of $\sigma(T)$, it is a branching point of τ.*

PROOF. Suppose that λ_1 is in the boundary of $\sigma(T)$, but that λ_1 is not a branching point of τ. Let $\mu_1, \ldots, \mu_p$ be the eigenvalues of modulus 1 of $B(\lambda_1)$. Then, by Definition 62, and the remark following that definition, if $\varepsilon > 0$ is sufficiently small, there exists a circular neighborhood N of λ, and p analytic functions $\varphi_1, \ldots, \varphi_p$ such that if $\lambda \in N$,

$$\{\mu | \mu \in \sigma(B(\lambda)), \| |\mu| - 1| < \varepsilon\} = \{\varphi_1(\lambda), \ldots, \varphi_p(\lambda)\}.$$

Now, the function φ_1, being analytic, maps open sets onto open sets; in particular, then, there must exist a (necessarily infinite) set A of points λ in N such that $\varphi_1(A)$ covers a neighborhood on the unit circle of $\alpha_1 = \varphi_1(\lambda_1)$. By Lemma 60, A must lie on the real axis. Hence the real analytic function $|\varphi_1(\lambda)|^2 - 1$ vanishes infinitely often in the open interval I cut out of the real axis by N. Hence $|\varphi_1(\lambda)| = 1$ for λ in I, proving by Lemma 61 that all of I must belong to $\sigma(T)$. Thus $\lambda_1 \in I$ is not a boundary point of $\sigma(T)$. Q.E.D.

64 THEOREM. *Let τ be a formally self adjoint formal differential operator of order n defined on the interval $(-\infty, +\infty)$. Suppose that the coefficients of τ are all periodic with the same period. Then τ has no boundary values, so $T_0(\tau)$ has a unique self adjoint extension T. The spectrum $\sigma(T)$ of T consists of a sequence of disjoint intervals, whose end points tend to $-\infty$ or $+\infty$. All of $\sigma(T)$ is continuous spectrum. A point λ is in $\sigma(T)$ if and only if the matrix $B(\lambda)$ introduced above has an eigenvalue of modulus 1; i.e., if and only if the equation $\tau\sigma = \lambda\sigma$ has a bounded solution. Each boundary point of each of the intervals comprising $\sigma(T)$ is a branching point of τ.*

PROOF. All this follows immediately from the previous lemma, and from Lemmas 59, 60, and 61. Q.E.D.

Next we consider the individual intervals that make up the spectrum of T We will discuss the specific form that the spectral resolution of T, as given generally by the Weyl-Kodaira theorem (5.14), takes on in such an interval. By the remark following Definition 62, the branching points in the extended sense of τ are isolated and since the spectrum of T is purely continuous, the countable set b of all the branching points of τ has spectral measure $E(b; T) = 0$. In

our analysis, we may consequently neglect this set as well as any other countable subset of the spectrum. Next, consider an open interval I of $\sigma(T)$ which does not contain any branching points. Then, by Definition 62 and the remark following that definition, there exists a connected neighborhood U of I, such that for each λ in U (aside from, possibly, an isolated set e_0 of points) the matrix $B(\lambda)$ has a fixed number k of eigenvalues which are given by functions $\varphi_i(\lambda), \ldots, \varphi_k(\lambda)$ analytic in U. (The set e_0 is that isolated set of points in U in which two or more of these distinct analytic functions take on the same value.) For λ in $U-e_0$, the eigenvalues $\varphi_i(\lambda)$, $i = 1, \ldots, k$, are distinct spectral sets of the finite dimensional operator $B(\lambda)$, so that by Theorem VII.3.14. the spectral projections $E_i(\lambda) = E\,(\varphi_i(\lambda); B(\lambda))$, $i = 1, \ldots, k$, are analytic for $\lambda \in U-e_0$. Let λ_0 be any chosen point in $U-e_0$, and let $v_1, \ldots, v_{j_1}$ be a basis for the range of $E_1(\lambda_0)$, $v_{j_1+1}, \ldots, v_{j_2}$ be a basis for the range of $E_2(\lambda_0), \ldots,$ $v_{j_{k-1}+1}, \ldots, v_n(v_n = v_{j_k})$ be a basis for the range of $E_k(\lambda_0)$. Put $v_j(\lambda) = E_i(\lambda)v_j$ for $\lambda \in U-e_0$ and $j_{i-1} < j \leqq j_i$. Then the analytically varying vectors $v_1(\lambda), \ldots, v_n(\lambda)$ in Euclidean n-space, being linearly independent at the point $\lambda = \lambda_0$, and hence having a non-vanishing determinant there, have a non-vanishing determinant and hence are linearly independent at every point λ in U except at those points belonging to an isolated set e_1 including e_0. We now let $\sigma(t, \lambda)$ be the unique solution of the equation $\tau\sigma = \lambda\sigma$ which satisfies the initial conditions

$$\sigma_j^{(i)}(0, \lambda) = (v_j(\lambda))_i, \qquad i = 0, \ldots, n-1.$$

(Cf. paragraph immediately preceding Lemma 59). Then, for λ in $U-e_1$, the set $\sigma_1(t, \lambda), \ldots, \sigma_n(t, \lambda)$ is a basis for the set of solutions of $\tau\sigma = \lambda\sigma$. Moreover, by the paragraph preceding Lemma 59,

$$\sigma_j(t-1, \lambda) = \varphi_i(\lambda)\sigma_j(t, \lambda), \qquad j_{i-1} < j \leqq j_i, \qquad i = 1, \ldots, k.$$

Next let us consider those points λ_0 in U at which one of the analytic functions $\varphi_j(\lambda)$, say for the sake of definiteness $\varphi_1(\lambda)$, takes on a value of modulus 1. Then, by Lemma 60, λ_0 is real. Let the expansion of $\varphi_1(\lambda)$ in the neighborhood of λ_0 be

$$\varphi_1(\lambda) = \varphi_1(\lambda_0)+a_m(\lambda-\lambda_0)^m+a_{m+1}(\lambda-\lambda_0)^{m+1}+\ldots, \; a_m \neq 0.$$

If $m \neq 1$, then by the Weierstrass preparation theorem the inverse image of the unit circle under the map φ_1 includes a set of m analytic arcs all intersecting at equal angles π/m at the point λ_0. If $m > 1$, not all of these arcs can possibly lie along the real axis. Hence we must have $m = 1$, so that φ_1 is one-to-one in the neighborhood of $\lambda = \lambda_0$, and the inverse image under φ_1 of a small arc of the unit circle containing $\varphi_1(\lambda_0)$ is a small arc of the real axis containing λ_0. Since we have then $\varphi_1(\lambda)\overline{\varphi_1(\bar{\lambda})} = 1$ for λ in a real neighborhood of λ_0, this equation must hold identically in U. Consequently:

If one of the analytic functions $\varphi_i(\lambda)$ takes on a value of modulus 1, *then we have $|\varphi_i(\lambda)| = 1$ for all real λ in U, and for no other λ in U. In this case, φ_i is a homeomorphic map of the interval I onto an arc of the unit circle.*

Since mappings by analytic functions are orientation-preserving, it follows that if φ_1 maps I into the unit circle C in such a way that an increasing t corresponds to an increasing argument θ, then the points of U in the positive half-plane are mapped by φ_1 into the interior of the unit circle, and the points in the negative half-plane into the exterior of the unit circle; and vice-versa if φ_1 maps I into C in such a way that increasing t corresponds to decreasing θ.

We may consequently divide the functions φ_i into four natural categories.

(a) Those φ_i which map the real axis into the unit circle, increasing t corresponding to increasing θ.

(b) Those φ_i which map the real axis into the unit circle, increasing t corresponding to decreasing θ.

(c) Those φ_i which map U into the interior of the unit circle.

(d) Those φ_i which map U into the exterior of the unit circle.

Let us now suppose that for notational convenience each function φ_i is repeated a number of times equal to the dimension of the range of $E_i(\lambda)$ (which is independent of λ by Lemma VII.6.4), and that the resulting φ_j are renumbered so that we have

$$\sigma_j(t-1, \lambda) = \varphi_j(\lambda)\sigma_j(t, \lambda), \qquad 1 \leqq j \leqq n,$$

in terms of the basis $\varphi_1, \ldots, \varphi_n$ for solutions $\tau\sigma = \lambda\sigma$ introduced above, and finally, that the φ_j and σ_j are numbered in accordance with the categories (a)—(d) introduced above, so that $\varphi_1, \ldots, \varphi_{\nu_1}$

belong to category (a), $\varphi_{\nu_1+1}, \ldots, \varphi_{\nu_2}$ to category (b), $\varphi_{\nu_2+1}, \ldots, \varphi_{\nu_3}$ to category (c), and $\varphi_{\nu_3+1}, \ldots, \varphi_{\nu_4} = \varphi_n$ to category (d).

Then clearly the σ of category (c) (of category (d)) belong to $L_2(-\infty, 0)$ (to $L_2(0, \infty)$) for all λ in $U-e_1$, and the σ of category (a) (of category (b)) belong to $L_2(-\infty, 0)$ for λ in $U-e_1$ and for $\mathscr{I}\lambda > 0$ (for $\mathscr{I}\lambda < 0$), and to $L_2(0, \infty)$ for λ in $U-e_1$ and for $\mathscr{I}\lambda < 0$ (for $\mathscr{I}\lambda > 0$).

Let $F_{ij}(\lambda)$ be defined in terms of the boundary form F_t of the formal differential operator τ (cf. Definition 2.1) by the equation

$$F_{ij}(\lambda) = F_t(\sigma_i(\lambda), \overline{\sigma_j(\bar{\lambda})}).$$

By Green's formula 2.4, this matrix is independent of t. Since for λ in $U-e_0$, the functions $\sigma_1(\lambda), \ldots, \sigma_n(\lambda)$ form a basis for the space of solutions of $\tau\sigma = \lambda\sigma$, it follows from Lemma 2.2 that the matrix $F_{ij}(\lambda)$ is non-singular for λ in $U-e_0$. Let the matrix $G_{ij}(\lambda)$ be its inverse. By Corollary 3.18, the resolvent $R(\lambda; T)$ of T is represented for λ in $U-e_0$, $\mathscr{I}\lambda \neq 0$ by an integral kernel $K(t, s; \lambda)$ having the form

$$\begin{aligned} K(t, s; \lambda) &= -\sum_{i=1}^{n} \sum_{j=1}^{n} \varepsilon_{ij}^{+} G_{ij}(\lambda)\sigma_j(t, \lambda)\overline{\sigma_i(s, \bar{\lambda})}, \qquad t > s, \qquad \mathscr{I}\lambda > 0, \\ &= -\sum_{i=1}^{n} \sum_{j=1}^{n} \varepsilon_{ij}^{-} G_{ij}(\lambda)\sigma_j(t, \lambda)\overline{\sigma_i(s, \bar{\lambda})}, \qquad t > s, \qquad \mathscr{I}\lambda < 0. \end{aligned}$$

By Corollary 3.13, the quantities ε_{ij}^{+} and ε_{ij}^{-} are either zero or $+1$. More specifically, it follows from Corollary 3.18 that

(a′) $$\varepsilon_{ij}^{+} = \varepsilon_{ij}^{-}$$

for a pair i, j if neither σ_i nor σ_j goes from being square-integrable over $L_2(0, \infty)$ to being square-integrable over $L_2(-\infty, 0)$ as $\mathscr{I}\lambda$ goes from positive to negative;

(b′) $$\varepsilon_{ij}^{+} = 1 \quad \text{if} \quad 1 \leqq i,j \leqq \nu_1,$$

while $\varepsilon_{ij}^{+} = 0$ for all other pairs i,j in the range $1 \leqq i,j \leqq \nu_2$;

(c′) $$\varepsilon_{ij}^{-} = 1 \quad \text{if} \quad \nu_1 < i,j \leqq \nu_2,$$

while $\varepsilon_{ij}^{-} = 0$ for all other pairs i, j in the range $1 \leqq i, j \leqq \nu_2$.

It then follows immediately from the Titchmarsh-Kodaira theorem (5.18) and from Theorem 5.27 that $\varphi_1, \ldots, \varphi_{\nu_2}$ are a determining set for t and that the matrix measure of Theorem 5.18 is

$$\begin{aligned}\rho_{ij}(\lambda_1, \lambda_2) &= \frac{1}{\pi i}\int_{\lambda_1}^{\lambda_2} G_{ji}(\lambda)d\lambda, & 1 \leqq i, j \leqq \nu_1, \\ &= -\frac{1}{\pi i}\int_{\lambda_1}^{\lambda_2} G_{ji}(\lambda)d\lambda, & \nu_1+1 \leqq i, j \leqq \nu_2, \\ &= 0, & \textit{otherwise.}\end{aligned}$$

Summarizing, we have the following theorem.

65 THEOREM. *Let τ and T be as in Theorem 64 and let I be an interval of λ containing no branching point of τ. Then there is a set $\varphi_1, \ldots, \varphi_n$ of solutions of the equation $\tau\sigma = \lambda\sigma$, which are analytic in λ on a complex neighborhood of I, and integers ν_1, ν_2 such that for $1 \leqq i \leqq \nu_1$,*

$$\varphi_i(\cdot, \lambda) \in L_2(-\infty, 0) \quad \textit{for} \quad \mathscr{I}\lambda > 0 \quad \textit{and} \quad \varphi_i(\cdot, \lambda) \in L_2(0, \infty), \quad \textit{for} \quad \mathscr{I}\lambda < 0;$$

for $\nu_1 < i \leqq \nu_2$,

$$\varphi_i(\cdot, \lambda) \in L_2(0, \infty) \quad \textit{for} \quad \mathscr{I}\lambda > 0 \quad \textit{and} \quad \varphi_i(\cdot, \lambda) \in L_2(-\infty, 0), \quad \textit{for} \quad \mathscr{I}\lambda < 0;$$

and for $\nu_2 < i \leqq n$, either

$$\varphi_i(\cdot, \lambda) \in L_2(-\infty, 0) \quad \textit{for} \quad \mathscr{I}\lambda \gtreqless 0, \quad \textit{or} \quad \varphi_i(\cdot, \lambda) \in L_2(0, \infty) \quad \textit{for} \quad \mathscr{I}\lambda \gtreqless 0.$$

The functions φ_i may be chosen so as to form a basis for the set of solutions of $\tau\sigma = \lambda\sigma$ everywhere on I except at an isolated set of points. Let I_1 be a subinterval of I not containing any of these points. Put

$$F_{ij}(\lambda) = F_t(\varphi_i(\lambda), \overline{\varphi_j(\bar{\lambda})}), \qquad 1 \leqq i, j \leqq n,$$

(so that, by Green's formula 2.4, F_{ij} is independent of t). Then the matrix F_{ij} is non-singular. Let $G_{ij}(\lambda)$ be its inverse. Then if ρ_{ij} is the positive definite matrix measure associated with the basis $\varphi_1, \ldots, \varphi_n$ of solutions of $\tau\sigma = \lambda\sigma$ on the interval I, by Theorem 5.18, we have

$$\begin{aligned}\rho_{ij}(e) &= \frac{1}{\pi i}\int_e G_{ij}(\lambda)d\lambda, & e \subseteq I_1, & \quad 1 \leqq i, j \leqq \nu_1, \\ &= \frac{-1}{\pi i}\int_e G_{ij}(\lambda)d\lambda, & e \subseteq I_1, & \quad \nu_1 < i, j \leqq \nu_2, \\ &= 0, & & \quad \textit{otherwise.}\end{aligned}$$

An additional argument would show that the spectral multiplicity of the restriction of T to a subspace $E(e)\mathfrak{H}$ of Hilbert space $\mathfrak{H}$ is a fixed, finite constant $m(I)$ for each subset e of positive Lebesgue measure of an interval I not containing any branching points of τ. However, we shall not give this argument here.

In case τ is a second order real formally self adjoint operator,

$$\tau = -\left(\frac{d}{dt}\right) p(t) \left(\frac{d}{dt}\right) + q(t), \qquad p(t) > 0,$$

we may go somewhat further. Let $\varphi(t, \lambda)$ be that solution of $\tau\sigma = \lambda\sigma$ satisfying $\varphi(0, \lambda) = p(0)^{-1}$, $\varphi'(0, \lambda) = 0$, and let $\psi(t, \lambda)$ be that solution of $\tau\sigma = \lambda\sigma$ satisfying $\psi(0, \lambda) = 0$, $\psi'(0, \lambda) = 1$. Then φ, ψ form a basis for the set of solutions of $\tau\sigma = \lambda\sigma$. The matrix for the transformation S relative to this basis is evidently

$$\begin{pmatrix} \varphi(-1, \lambda), & \psi(-1, \lambda) \\ \varphi'(-1, \lambda), & \psi'(-1, \lambda) \end{pmatrix}.$$

The quadratic equation satisfied by the characteristic roots of $B(\lambda)$ is consequently

$$\alpha^2 - (\varphi(-1, \lambda) + \psi'(-1, \lambda))\alpha + 1 = 0;$$

here we have used the fact that $F_t^\tau(\varphi, \bar{\psi}) = p(t)(\varphi(t, \lambda)\psi'(t, \lambda) - \psi(t, \lambda)\varphi'(t, \lambda))$ is independent of t, so that since $p(-1) = p(0)$,

$$\begin{aligned} \varphi(-1, \lambda)\psi'(-1, \lambda) - \psi(-1, \lambda)\varphi'(-1, \lambda) \\ = \varphi(0, \lambda)\psi'(0, \lambda) - \psi(0, \lambda)\varphi'(-1, \lambda) = 1. \end{aligned}$$

Let $\beta(\lambda) = \varphi(-1, \lambda) + \psi'(-1, \lambda) = \text{trace } B(\lambda)$. Then this equation can be written as

$$\alpha^2 - \beta(\lambda)\alpha + 1 = 0.$$

Since $\beta(\lambda)$ is real, the roots of this equation, if complex, are complex conjugates of each other. Hence, since their product is one, both have modulus 1. That is; the roots of $\alpha^2 - \beta(\lambda)\alpha + 1$ are either both real, or are complex conjugates of modulus 1. Applying Theorem 64, we find: in the former case, i.e., the case $\beta(\lambda)^2 - 4 > 0$, λ is not in $\sigma(T)$; in the latter case, i.e., in case $\beta(\lambda)^2 - 4 \leq 0$, $\lambda \in \sigma(T)$. At a branching point of τ, at least two, and hence both eigenvalues of $B(\lambda)$ necessarily coincide. Hence, each branching point λ of τ satisfies $\beta(\lambda)^2 = 4$; i.e.,

$\beta(\lambda) = \pm 2$. Thus either both roots of $\alpha^2-\beta(\lambda)\alpha+1 = 0$ are $+1$, or both are -1. In the former case the matrix $B(\lambda)$ necessarily has an eigenvector belonging to the eigenvalue $+1$; in the latter case, to the eigenvalue -1. Thus, in the former case, $\tau\sigma = \lambda\sigma$ necessarily has a periodic solution, in the latter case, an anti-periodic solution, that is, a solution satisfying $\sigma(t+1, \lambda) = -\sigma(t, \lambda)$. Let us now set up the following two sets of boundary conditions for the operator τ on the finite closed interval $[0, 1]$.

First Set: $f(0) = f(1)$, $f'(0) = f'(1)$ (Periodic conditions)
Second Set: $f(0) = -f(1)$, $f'(0) = -f'(1)$ (Anti-periodic conditions.)

Then by Theorems XII.4.28, 4.1, and 4.2, these sets of boundary conditions determine self adjoint operators T_1 and T_2 whose spectra consist entirely of eigenvalues which, by Lemma 29 and Corollary 24, approach plus infinity. By Theorem 64, we now see that the discrete eigenvalues of those two problems are the only possible end points in the collection of intervals or "bands" comprising $\sigma(T)$.

The "bands" appearing in $\sigma(T)$ for second order τ have been investigated considerably, being known as the "stability bands" for "Hill's equation"

$$-(p(t)f'(t))'+q(t)f(t)-\lambda f(t) = 0.$$

It would take us too far afield to develop the results of all these investigations here; we shall mention only those results from this theory which are most germane in the present context. Let the eigenvalues determined by the periodic boundary conditions stated above be enumerated in increasing order, and repeated according to multiplicity, be $p_0, p_1, p_2, \ldots,$. Let the corresponding enumeration of the eigenvalues determined by the anti-periodic boundary values determined above be $a_0, a_1, a_2, \ldots$. Then a theorem of G. D. Birkhoff [4] tells us that

$$p_0 < a_0 \leqq a_1 < p_1 \leqq p_2 \leqq a_2 \leqq a_3 < p_3 \leqq \ldots \leqq a_{2n+1} < p_{2n+1} \leqq p_{2n+2} < a_{2n+2} \ldots.$$

The spectrum $\sigma(T)$ is made up of the successive intervals $[a_1, p_1]$, $[p_2, a_2]$, $[a_3, p_3]$, ... The eigenfunction φ_0 belonging to the eigen-

value p_0 of the self adjoint operator determined by the first set of boundary conditions has no zeros in $[0, 1]$; the eigenfunctions φ_{2n+1} and φ_{2n+2} belonging to the eigenvalues p_{2n+1} and p_{2n+2} of this operator have exactly $2n+2$ zeros in $[0, 1]$. The eigenfunctions $\tilde{\varphi}_{2n}$ and $\tilde{\varphi}_{2n+1}$ belonging to the eigenvalues a_{2n} and a_{2n+1} of the self adjoint operator determined by the second set of boundary conditions have $2n+1$ zeros in $[0, 1]$.

It may also be shown that if we make the normalization $\int_0^1 q(t)dt = 0$, then there exists a sequence ε_n of numbers going to zero such that for all sufficiently large n, every point in the interval $((n-1/2)\pi)^2 \leq \lambda \leq ((n+1/2)\pi)^2$ but not in $\sigma(T)$ belongs to the interval $(n\pi)^2 - \varepsilon_n \leq \lambda \leq (n\pi)^2 + \varepsilon_n$. Thus, the gaps in the spectrum of T become arbitrarily small, and the spectrum becomes more like that of the simple operator $-(d/dt)^2$, as $\lambda \to \infty$. For proofs of these results, the reader is referred to Coddington-Levinson [1], pages 214—218.

In concluding this section, let us note that by making a suitable change of variable, one can often improve the range of validity of a theorem on differential equations. We shall illustrate this principle by exhibiting improvements of some of the theorems established above. First, however, let us consider the effect on formal differential operators of changes of variable in general, and the way in which such changes of variable may be used to simplify the coefficients in an operator τ. Suppose that τ is a formally symmetric formal differential operator of order n. Let $a_n(t)$ be the leading coefficient in τ, so that the leading term in τ is $a_n(t)(d/dt)^n$. Then the leading term in τ^* is $(-1)^n\overline{a_n(t)}(d/dt)^n$; hence, $a_n(t) = (-1)^n\overline{a_n(t)}$. Thus, if n is even, $a_n(t)$ is real, while if n is odd, $a_n(t)$ is pure imaginary. Suppose we make the change of variable $t = h(s)$ and the corresponding unitary transformation of functions

$$f(\cdot) \to (Uf)(\cdot),$$

given by

$$(Uf)(s) = f(h(s))(h'(s))^{1/2},$$

where the function $h(s)$ will be chosen later. The choice will of course be such that h has an everywhere positive derivative. Corresponding to this transformation, I is transformed into $h^{-1}(I)$. The "formal transform" of the formal operator d/dt may be computed as follows:

$$\begin{aligned}\left(U\left(\frac{d}{dt}\right)f\right)(s) &= f'(h(s))(h'(s))^{1/2}\\ &= (h'(s))^{-1}\left\{\frac{d}{ds}[f(h(s))(h'(s))^{1/2}] - \tfrac{1}{2}f(h(s))h'(s)^{-1/2}h''(s)\right\}\\ &= (h'(s))^{-1}\frac{d}{ds}(Uf)(s) - \tfrac{1}{2}h''(s)h'(s)^{-2}(Uf)(s).\end{aligned}$$

Thus, we have, formally

$$\left(U\frac{d}{dt}U^{-1}f\right)(s) = \left\{h'(s)^{-1}\frac{d}{ds} - \tfrac{1}{2}h'(s)^{-2}h''(s)\right\}f(s).$$

Consequently, the change of variable sends the formal differential operator $\tau = \sum_{k=0}^{n} a_k(t)(d/dt)^k$ into the formal differential operator

$$\begin{aligned}U\tau U^{-1} &= \sum_{k=0}^{n} a_k(h(s))\left\{h'(s)^{-1}\frac{d}{ds} - \tfrac{1}{2}h'(s)^{-2}h''(s)\right\}^k\\ &= a_n(h(s))h'(s)^{-n}\left(\frac{d}{ds}\right)^n + \hat{\tau}_{n-1},\end{aligned}$$

$\hat{\tau}_{n-1}$ being an irregular formal differential operator of order at most $n-1$. If, in particular, τ is formally symmetric and we let $h(s)$ be a solution of the equation $h'(s) = |a_n(h(s))|^{1/n}$, the operator $U\tau U^{-1}$ will have a leading coefficient ± 1 (if n is even) or $\pm i$ (if n is odd). It is worth remarking at this point that it may easily be established that we have $(U\tau U^{-1})^* = U\tau^* U^{-1}$, $U\tau_1\tau_2 U^{-1} = U\tau_1 U^{-1}U\tau_2 U^{-1}$, $U(\tau_1+\tau_2)U^{-1} = U\tau_1 U^{-1} + U\tau_2 U^{-1}$, all these equations holding as formal equations. Hence our unitary transformation takes formally symmetric formal differential operators into formally symmetric formal differential operators, etc. It is also worth remarking that the solution of the equation $h'(s) = |a_n(h(s))|^{1/n}$ is the inverse function of the solution of the equation $g'(t) = |a_n(t)|^{-1/n}$; that is, h is the inverse function of the indefinite integral

$$g(t) = \int^t \frac{ds}{|a_n(s)|^{1/n}}.$$

Thus, for instance, Corollary 19 will yield a number of interesting results if we make a change of variable of the above type in the operator

$$\tau = -\frac{d}{dt}p(t)\frac{d}{dt} + q(t)$$

of that corollary. Let ψ be an arbitrary positive infinitely often differentiable function of t, and put $\varphi(t) = \psi(t)^{-1}(p(t)\psi'(t))'$. If we then make the change of variable $t = h(s)$, where $h(s)$ is the inverse function of the function

$$s(t) = \int_0^t \psi(\tau)^2 d\tau,$$

it follows according to the above remarks that τ is transformed into the operator

$$\tau_1 = -\frac{d}{ds}P(s)\frac{d}{ds} + Q(s),$$

where

$$P(s) = p(h(s))\psi(h(s))^2$$

and

$$Q(s) = q(h(s)) - \varphi(h(s)).$$

The case $\psi(t) = p(t)^{-1/4}$, which gives $P(s) \equiv 1$, is the case cited above. In this case we have

$$Q(s) = q(h(s)) + \tfrac{1}{4}\{p''(h(s)) - \tfrac{1}{4}[p(h(s))]^{-1}[p'(h(s))]^2\}.$$

We consequently obtain the following theorems from Theorems 16 and 17.

66 THEOREM. *Let τ be a real second order formally symmetric formal differential operator of the form*

$$\tau = -\frac{d}{dt}p(t)\frac{d}{dt} + q(t),$$

defined on an interval $[a, b)$. Suppose that

$$\int_a^b p(t)^{-1/2} dt = \infty.$$

Let

$$Q(t) = q(t) + \tfrac{1}{4}\{p''(t) - \tfrac{1}{4}[p(t)]^{-1}[p'(t)]^2\}.$$

Then

(a) *if* $Q(t) \to \infty$ *as* $t \to b$, $\sigma_e(\tau)$ *is void;*

(b) *if* $Q(t) \to c$ *as* $t \to b$, $\sigma_e(\tau) = \{\lambda | \lambda \geqq c\}$.

67 THEOREM. *Let* τ *be a real second order formally symmetric formal differential operator of the form*

$$\tau = -\frac{d}{dt}p(t)\frac{d}{dt} + q(t)$$

defined in an interval $[a, b)$. *Suppose that*

$$\int_a^b p(t)^{-1/2}dt < \infty.$$

Let $$Q(t) = q(t) + \tfrac{1}{4}(p''(t) - \tfrac{1}{4}[p(t)]^{-1}[p'(t)]^2).$$

Then

(a) *if* $q(t) \to \infty$ *as* $t \to b$, $\sigma_e(\tau)$ *is void;*

(*b*) *if* $\limsup_{t\to b} \left|\left\{\int_t^b p(t)^{-1/2}dt\right\}^2 Q(t)\right| < \frac{3}{4}$, $\sigma_e(\tau)$ *is void.*

As observed by K. O. Friedrichs, the various cases $\psi(t) = f(a(t))$, where $a(t) = \int^t p(s)^{-1}ds$ and f is a suitably chosen function, also lead to interesting results. Here we have

$$\varphi(t) = p(t)^{-1}f''(a(t))f(a(t))^{-1}.$$

A convenient choice of $f(t)$ is $f(t) = t^{1/2}$, which gives $\varphi(t) = -(1/4)p(t)^{-1}a(t)^{-2}$. Making the substitution in Corollary 19, we find the following result of Friedrichs:

68 COROLLARY. *Let*

$$\tau = -\frac{d}{dt}p(t)\frac{d}{dt} + q(t)$$

be a real formally self adjoint formal differential operator defined on an interval $I = [a, b)$. *Let* $p(t) > 0$ *for* t *in* I, *and put*

$$Z(t) = q(t) + \left(4p(t)\left(\int_a^t p(\tau)^{-1}\,d\tau\right)^2\right)^{-1}.$$

If $\liminf_{t\to b} Z(t) = K$, *then* $\sigma_e(\tau)$ *belongs entirely to the region* $\lambda \geqq K$ *of the real axis.*

In the same way, we find from Theorem 6.14:

69 Corollary. *Suppose, under the hypotheses of the preceding corollary and with the notation of that corollary, that* $\int_a^b p(t)^{-1}\,dt = \infty$. *Then if* Z *is bounded below,* τ *has no boundary values at* b.

Various other results may be obtained by making suitable changes of variable in Theorems 7—19 of the present section, and in Theorems 12—15, 18—22, and 33 of the previous section. A number of results of this type will be given at the end of the chapter as exercises.

Another useful change of variable is the unitary "change of dependent variable" defined by the equation

$$(Vf)(t) = \exp\,(ib(t))f(t),$$

b being some suitably chosen real function. It is easy to see that

$$V^{-1}\frac{d}{dt}V = \frac{d}{dt} + ib'(t).$$

Hence, if

$$\tau = \sum_{k=0}^{n} a_k(t)\left(\frac{d}{dt}\right)^k,$$

we have

$$\tau_1 = V^{-1}\tau V = \sum_{k=0}^{n} a_k(t)\left(\left(\frac{d}{dt}\right) + ib'(t)\right)^k.$$

In particular, τ and τ_1 have the same leading coefficient. The coefficient in τ_1 of the operator $(d/dt)^{n-1}$ is evidently

$$a_{n-1}(t) + ina_n(t)b'(t).$$

We can make this coefficient zero by placing

$$b'(t) = ia_{n-1}(t)(na_n(t))^{-1},$$

which amounts to taking

$$b(t) = \int^t ia_{n-1}(s)(na_n(s))^{-1}ds,$$

(which is real by the formal symmetry of τ).

Combining the two forms of transformation of variable discussed above, it is obvious that we can, if convenient, reduce each formally self adjoint formal differential operator to the normal form

$$i^n\left(\frac{d}{dt}\right)^n + \sum_{j=0}^{n-2} a_j(t)\left(\frac{d}{dt}\right)^j.$$

This leads to the normal form

$$i\frac{d}{dt}$$

for the operator of order 1, so that every formally self adjoint formal differential operator of order 1 can be reduced to a known form by elementary unitary changes of variables. In the same way, we find the normal form

$$-\left(\frac{d}{dt}\right)^2 + q(t), \qquad q(t) \text{ real},$$

for the formally self adjoint formal operator of second order. This explains the emphasis put on operators of this particular form in certain of the theorems of the present and of the preceding sections,

8. Examples

We now wish to illustrate the application of the preceding theory to specific differential equations. If the reader surveys the theory developed in the past few sections, he will find it evident that in applying the general methods to specific equations, we will need definite manageable expressions for the solutions of these equations, which may be used to investigate the integrability, Wronskian, limiting values, etc., of the solutions in question. To obtain such expressions for the solutions of a given differential equation will in most cases be quite difficult. For this reason we will confine ourselves to the treatment of a few simple cases of self adjoint operators arising from *second order linear differential equations with rational coefficients.* For these equations the general theory of differential equations in the complex domain gives a surprising amount of information. We will quote the relevant theorems from this theory, referring the reader to the books of Coddington and Levinson [1] and of Poole [1] for proofs.

Let

$$L = \left(\frac{d}{dz}\right)^2 + r_1(z)\left(\frac{d}{dz}\right) + r_2(z)$$

be a second order formal differential operator with rational coefficients r_1, r_2. A point z_0 in the complex plane at which r_1 and r_2 are analytic is called a *regular point* of the operator. In the neighborhood of a regular point z_0, there exists a unique analytic solution $f(z)$ of the equation $Lf = 0$ with specified initial values $f(z_0)$, $f'(z_0)$. A point z_0 which is not a regular point is called a *singular point* or *singularity* of the equation. If r_1 has at most a kth order pole at z_0, and r_2 has at most a $2k$th order pole at z_0, the formal differential operator is said to have a kth *order singularity* at z_0. A singularity of order 1 is often called *regular singularity*, a singularity of order greater than 1 is said to be an *irregular singularity*.

If z_0 is a regular singularity of the formal differential operator $L = (d/dz)^2 + r_1(z)(d/dz) + r_2(z)$, then in the neighborhood of z_0, r_1 and r_2 can be expanded as

$$r_1(z) = (z-z_0)^{-1}(a + a'(z-z_0) + \ldots)$$

and

$$r_2(z) = (z-z_0)^{-2}(b + b'(z-z_0) + \ldots).$$

The equation $\mu(\mu-1) + a\mu + b = 0$ is called the *indicial equation* of L at z_0; its roots e_1 and e_2 are called the exponents of L at z_0. If e_1 and e_2 do not differ by an integer, the equation $Lf = 0$ has two linearly independent solutions of the form

$$\sigma_1(z) = (z-z_0)^{e_1}(1 + \alpha(z-z_0) + \ldots),$$
$$\sigma_2(z) = (z-z_0)^{e_2}(1 + \beta(z-z_0) + \ldots),$$

the power series being convergent up to the next closest singularity of L. If e_1 and e_2 do differ by an integer, and $\mathscr{R}e_2 \leq \mathscr{R}e_1$, then there still exists a solution σ_1 of the indicated form, but a linearly independent solution σ_2 of the indicated form need not exist. However, there will always exist a linearly independent solution of the form

$$\sigma_2(z) = (z-z_0)^{e_2}(1 + \beta(z-z_0) + \ldots) + \gamma\sigma_1(z)\log(z-z_0).$$

(The constant γ may be zero.)

A regular point of a differential equation may be regarded as a special case of a regular singular point at which the exponents are zero and one.

If $Lf = 0$ is a differential equation with rational coefficients and a regular singularity z_0 with exponents e_1 and e_2, then the second order equation $L'f' = 0$ satisfied by $f'(z) = (z-z_0)^{\alpha}f(z)$ has rational coefficients also. The operator L' has a regular singularity at z_0, with exponents $e_1+\alpha$, $e_2+\alpha$. If $\rho(z)$ is a rational function which sends the point ζ_0 into the point z_0, and if $\rho(z)-z_0$ has a zero of order l at ζ_0, then the second order equation $L''f'' = 0$ satisfied by $f'' = f[\rho(z)]$ has rational coefficients also. The operator L'' has a regular singularity at ζ_0 with exponents le_1, le_2.

By passing from the equation $Lf = 0$ to the equation $L'''f''' = 0$ satisfied by $f'''(z) = f(1/z)$, we can extend the concepts of regular point, singular point, singular point of order k, regular singular point, indicial equation, and exponents to the point at infinity on the Gaussian sphere also. The form of the results is as follows: the formal differential operator L has a regular point at infinity if $r_1(z)-2z^{-1}$ vanishes to the second order and $r_2(z)$ vanishes to the second order at infinity. L has a singularity of order k at infinity if r_1 vanishes at least to the $(2-k)$th order and r_2 vanishes at least to the $2(2-k)$th order at infinity. (A function with a pole of order k is here considered to have a zero of order k.) If L has at worst a regular singularity, i.e., a singularity of order 1 at infinity, then we have Laurent expansions

$$r_1(z) = az^{-1}+\tilde{a}z^{-2}+\ldots,$$
$$r_2(z) = bz^{-2}+\tilde{b}z^{-3}+\ldots.$$

The indicial equation of L at ∞ is $\mu(\mu+1)-a\mu+b = 0$; its roots e_1 and e_2 are called the exponents of L at infinity. If e_1 and e_2 are the exponents of L at infinity, and e_1-e_2 is not an integer, then there exist two linearly independent solutions σ_1 and σ_2 of the equation $Lf = 0$ which have the expressions

$$\sigma_1(z) = z^{-e_1}(1+\alpha z^{-1}+\tilde{\alpha}z^{-2}+\ldots),$$
$$\sigma_2(z) = z^{-e_2}(1+\beta z^{-1}+\tilde{\beta}z^{-2}+\ldots).$$

If e_1-e_2 is an integer, and $\mathscr{R}e_2 \leq \mathscr{R}e_1$, then there still exists a

solution σ_1 of the indicated form, but a linearly independent solution σ_2 of the indicated form need not exist. However, there will always exist a linearly independent solution of the form

$$\sigma_3(z) = z^{-\varepsilon_2}(1 + \beta z^{-1} + \bar{\beta} z^{-2} + \dots) + \gamma\sigma_1(z) \log z.$$

The constant γ may be zero.

Now let us pass to specifics. For the sake of simplicity, we will consider formally symmetric differential operators L of second order with rational coefficients, on intervals (a, b), where both a and b are singularities of L. Moreover, we will at first confine ourselves to operators L such that for each λ, $L-\lambda$ has regular singularities only; later, examples involving irregular singularities will be considered.

It should be noted that the interval (a, b) is at our disposal. Indeed, if $(d/dt)p(t)(d/dt) + q(t)$ is an arbitrary formally symmetric linear differential operator (with rational coefficients), and if we make the "formally unitary" transformation $f(t) \to (Uf)(t) = |s'(t)|^{1/2} f(s(t))$, where s is a monotone continuously differentiable function of t, we find $(U^{-1}LUf)(s) = (d/ds)P(s)(d/ds)f(s) + Q(s)f(s)$, where

$$[*] \qquad \begin{aligned} P(s) &= p(t)(s'(t))^2|_{t=t(s)}, \\ Q(s) &= s'(t)^{-1/2}(d/dt)p(t)(d/dt)s'(t)^{1/2} + q(t)|_{t=t(s)}, \end{aligned}$$

$t = t(s)$ being the inverse function to $s = s(t)$. If $s(a) = A$, $s(b) = B$, U transforms functions f defined on the interval (A, B) into functions Uf defined on (a, b). If s is a fractional linear function of t, the operator $U^{-1}LU$ clearly has rational coefficients also. Moreover, if L has only regular singularities, $U^{-1}LU$ has only regular singularities. Since there exists a real fractional linear transformation transforming a, b into any two arbitrarily given points A, B, the assertion that the interval (a, b) is at our disposal is valid in an obvious sense.

Now consider the case in which there are exactly two regular singularities, a, b of $L-\lambda$ for each λ. Making use of the reduction of the preceding paragraph, put these singularities at zero and infinity. Since

$$\left(\frac{d}{dt}\right) p(t) \left(\frac{d}{dt}\right) + q(t) = p(t) \left(\frac{d}{dt}\right)^2 + p'(t) \left(\frac{d}{dt}\right) + q(t)$$

has regular singularities at zero and infinity only, it follows that

$p'(t)p^{-1}(t)$ is a rational function with only one pole, a first order pole at zero, and that $p'(t)p^{-1}(t)$ vanishes at ∞. Thus $p'(t)p^{-1}(t) = a/t$, and similarly $q(t)/p(t) = b/t^2$. Since $L-\lambda$ is assumed to have only two regular singularities for each λ, it follows in the same way that $q(t)-\lambda/p(t) = b(\lambda)/t^2$ for each λ. Thus $p(t) = \text{const.}\ t^{-2}$; so that, after multiplication by a constant and addition of a constant, the operator L becomes $-(d/dt)t^2(d/dt)$. If we make the unitary transformation $f \to Uf = t^{-1/2}f(\log t)$, then, by formula [*], the operator L is transformed into $-(d/ds)^2+1/4$, or subtracting the constant $1/4$, into $-(d/ds)^2$. The interval on which we consider this operator is $(-\infty, +\infty)$. The reader will have no difficulty in seeing that this formal differential operator has both deficiency indices zero, so that it leads to the single self adjoint operator $T = (id/ds)^2$, id/ds denoting the self adjoint operator discussed at the end of Section 5. Thus it follows immediately from the operational calculus of that section that the spectral resolution of T may be expressed in terms of that of id/ds, and hence, ultimately, in terms of the Fourier transform. Consequently, the case in which $L-\lambda$ has only two regular singularities does not lead to anything new.

We consequently turn to the next most complicated case; that of formal differential operators L such that $L-\lambda$ has at most three regular singularities for each λ. Here the operator L is fully specified by its singularities $\zeta_0, \zeta_1, \zeta_2$ and the corresponding exponents e_0, e_0'; e_1, e_1'; e_2, e_2'. Indeed, if this were not the case, we would have two different operators L_1, L_2 with these singularities and exponents. By making a fractional linear transformation of the independent variable sending $\zeta_0, \zeta_1, \zeta_2$ into $0, 1, \infty$, we would get two different operators with these same exponents at 0, 1, and infinity. If we then multiply the dependent variable by $z^{-e_0}(z-1)^{-e_1}$, we would get two operators L, L' with regular singularities at $0, 1, \infty$, and with exponents of the form $0, a$; $0, b$; c, d at these singularities. Let $L = D^2+r_1(z)D+r_2(z)$. Then r_1 is a rational function with simple poles at 0 and 1, vanishing at ∞. Thus $r_1(z)$ must have the form $\alpha/z+\beta/(z-1)$. Similarly, $r_2(z)$ must have the form $(\gamma z+\gamma')z^{-2}+(\delta(z-1)+\delta')(1-z)^{-2}$. If we now make use of the fact that the roots of the indicial equation at $0, 1, \infty$ are $0, a$; $0, b$; c, d, we find $\alpha = 1-a$, $\beta = 1-b$, $\gamma' = \delta' = 0$; $\gamma = \delta = cd$. Thus L is uniquely determined by a, b, c, d, which proves that

$L_1 = L_2$ and hence establishes the uniqueness of the formal differential operator with three specified regular singularities and three specified pairs of exponents. From the indicial equations we find also that $c + d = \alpha + \beta - 1$, so that $a + b + c + d = 1$. Thus the sum of the exponents of an equation with three singularities is 1, but the exponents are otherwise arbitrary.

The effect on the differential equation under consideration of the various changes of variable discussed in the preceding paragraph can be greatly clarified by the use of a suitable symbolic notation, first introduced by Riemann. We have just seen that there exists one and only one second order linear differential equation with regular singularities a, b, c and the corresponding pairs α_1, α_2; β_1, β_2; γ_1, γ_2 of exponents at these singularities. Suppose that we then use the symbol

$$[*] \qquad P\begin{pmatrix} a & b & c & \\ \alpha_1 & \beta_1 & \gamma_1 & ; z \\ \alpha_2 & \beta_2 & \gamma_2 & \end{pmatrix}$$

to denote an unspecified branch of the (in general multi-valued) function which satisfies the differential equation in question. In some cases we will wish to be more specific, and to indicate a particular branch of this (multi-valued) function, which will generally be that (uniquely determined) branch which has a given asymptotic form at one of the singularities. For this purpose, we proceed as follows. Suppose that we wish to indicate the branch of the multi-valued function [*] which has the asymptotic form $(z-a)^{\alpha_1}(1 + c_1(z-a) + \ldots)$ at the regular singularity $z = a$ of our equation. Then we will write

$$P^a_{\alpha_1}\begin{pmatrix} a & b & c & \\ \alpha_1 & \beta_1 & \gamma_1 & ; z \\ \alpha_2 & \beta_2 & \gamma_2 & \end{pmatrix}.$$

Similarly,

$$P^b_{\beta_2}\begin{pmatrix} a & b & c & \\ \alpha_1 & \beta_1 & \gamma_1 & ; z \\ \alpha_2 & \beta_2 & \gamma_2 & \end{pmatrix}$$

will indicate the unique branch of [*] which has the asymptotic form $(z-b)^{\beta_2}(1 + c_1(z-b) + \ldots)$ at $z = b$. If $c = \infty$, the symbol

$$P^{\infty}_{\gamma_1}\begin{pmatrix} a & b & \infty & \\ \alpha_1 & \beta_1 & \gamma_1 & ;\ z \\ \alpha_2 & \beta_2 & \gamma_2 & \end{pmatrix}$$

will indicate the unique branch of [*] which has the asymptotic form $z^{-\gamma_1}(1+\tilde{c}_1z^{-1}+\ldots)$ at $z=\infty$. It is nearly evident from the symbolic notation and the principle governing changes of variable explained in the preceding paragraphs that

$$P^{a}_{\alpha_1}\begin{pmatrix} a & b & c & \\ \alpha_1 & \beta_1 & \gamma_1 & ;\ z \\ \alpha_2 & \beta_2 & \gamma_2 & \end{pmatrix} = \left(\frac{z-a}{z-b}\right)^{\alpha_1} P^{a}_{0}\begin{pmatrix} a & b & c & \\ 0 & \beta_1+\alpha_1 & \gamma_1 & ;\ z \\ \alpha_2-\alpha_1 & \beta_2+\alpha_1 & \gamma_2 & \end{pmatrix},$$

$$P^{0}_{0}\begin{pmatrix} 0 & 1 & \infty & \\ 0 & 0 & \gamma_1 & ;\ \dfrac{z}{z-1} \\ 1-\alpha & \alpha-\gamma_1-\gamma_2 & \gamma_2 & \end{pmatrix} = (1-z)^{\gamma_1} P^{0}_{0}\begin{pmatrix} 0 & 1 & \infty & \\ 0 & 0 & \gamma_1 & ;\ z \\ 1-\alpha & \gamma_2-\gamma_1 & \alpha-\gamma_2 & \end{pmatrix},$$

etc. Such changes of variable will be used frequently in the following analysis. We shall not give the details of these changes of variable, but will leave them to the reader to work out. In each case they will follow readily if only the calculations are made in the symbolic notation that has just been explained.

The differential equation $Lf=0$, where L has three regular singularities at $0, 1, \infty$, exponents α, β at ∞, exponents $0, 1-\gamma$ at zero, and exponents $0, \gamma-\alpha-\beta$ at 1, was seen in the previous paragraph to have the form (after multiplication by $z(1-z)$)

[1] $$z(1-z)\left(\frac{d}{dz}\right)^2 f+(\gamma-(\alpha+\beta+1)z)\left(\frac{d}{dz}\right) f-\alpha\beta f=0.$$

This equation is the famous *hypergeometric equation* of Euler Gauss. If γ is not a non negative integer, it follows as explained above that the equation has a unique solution in the form $\sigma(z)=1+a_1z+\ldots$. Comparison of coefficients gives the Euler *Hypergeometric Series.*

[2] $$F(\alpha,\beta;\gamma;z)=1+\frac{\alpha\beta}{\gamma}z+\frac{\alpha(\alpha+1)\beta(\beta+1)}{2!\,\gamma(\gamma+1)}z^2+\ldots.$$

If we make use of Euler's β-function integral

[8] $$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt, \qquad \mathscr{R}x>0, \qquad \mathscr{R}y>0,$$

where Γ is the Γ-function, we find

$$F(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\sum_{n=0}^{\infty}\frac{\beta(\beta+1)\dots(\beta+n-1)}{n!}z^n\int_0^1 t^{\alpha-1+n}(1-t)^{\gamma-\alpha-1}\,dt$$

so that we have Euler's integral

$$[4] \qquad F(\alpha,\beta;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}(1-tz)^{-\beta}\,dt,$$

valid for $|z| < 1$, $\mathscr{R}\alpha > 0$, $\mathscr{R}(\gamma-\alpha) > 0$.

In the present case, a complete analysis parallelling that given above for the case of two regular singularities is possible: but the details are too lengthy to be given here. We shall confine ourselves to the treatment of a number of examples chosen to illustrate the various kinds of spectra that can arise and the methods used for treating them.

We begin with the formal differential operator

$$L = L^{\alpha,\beta} = -\left(\frac{d}{dt}\right)(1-t^2)\left(\frac{d}{dt}\right)+\frac{2\alpha^2}{1+t}+\frac{2\beta^2}{1-t},$$

on the interval $(-1, +1)$, α and β being assumed to be real and positive. For any λ, $L-\lambda$ has regular singularities at -1, $+1$, and infinity. It is readily seen from the indicial equations that the exponents at -1, $+1$, and ∞ are α, $-\alpha$; β, $-\beta$; and $1/2+\gamma$, $1/2-\gamma$; where $\lambda = \gamma^2-1/4$, respectively. Thus three cases arise.

Case 1. $\alpha \geqq 1/2$, $\beta \geqq 1/2$. Then $Lf = \pm if$ has one solution (asymptotic to $(x+1)^\alpha$) which is in $L_2(-1, 0)$, while no linearly independent solution is in $L_2(-1, 0)$. The same holds at 1, so that, by Corollary 2.25, the deficiency indices of L are both zero. Thus L gives rise to a unique self adjoint operator in Hilbert space, which we continue to denote by the letter L.

Case 2. $\alpha \geqq 1/2$ and $0 \leqq \beta < 1/2$, or $0 \leqq \alpha < 1/2$ and $\beta \geqq 1/2$. These cases are equivalent under the change of variables $x \to -x$. We see here in the same way that $Lf = \pm if$ has one solution in L_2 at one of the end points, while $Lf = \pm if$ has all solutions in L_2 at the other end point. Thus by Corollary 2.25 the deficiency indices of L are

both 1. Thus we must impose a single boundary condition to obtain a self adjoint operator. The form of this boundary condition will be studied more closely below.

Case 3. $0 \leq \alpha < 1/2$, $0 \leq \beta < 1/2$. Here all solutions of $Lf = \pm if$ are square-integrable at both end points. Hence $d_+ = d_- = 2$, and two boundary conditions must be imposed in order to obtain a self adjoint operator in Hilbert space.

By Theorems 4.1 and 4.2, $L^{\alpha,\beta}$ has no essential spectrum if $0 \leq \alpha < 1/2$, $0 \leq \beta < 1/2$. Since $L^{\alpha,\beta}$ increases with increasing α and β and is formally positive, it follows from Theorem 7.34 that $L^{\alpha,\beta}$ has no essential spectrum for any positive α, β. Thus, in all cases, we have discrete eigenvalues only.

Let us turn to a closer analysis of Case 1, where no boundary conditions enter to trouble us. It is clear first of all that since at most one solution of $(L-\lambda)f = 0$ is in $L_2(-1, +1)$, every eigenvalue of L has multiplicity at most one. (This remark holds also for the self adjoint operators derived from L in Case 2.) In order that λ shall be an eigenvalue, it is clearly necessary and sufficient that the solution $\sigma = \sigma_{-1}$ of $(L-\lambda)f = 0$ which has the form $(t+1)^{\alpha}(1+\ldots)$ shall be square-integrable at $+1$; i.e., shall be a constant multiple of the solution σ_{+1} of this equation which has the form $(t-1)^{\beta}(1+\ldots)$. If this is the case, then $(t+1)^{-\alpha}(t-1)^{-\beta}\sigma$ is analytic at $+1$ and -1. Since the differential equation $(L-\lambda)f = 0$ has no finite singularities other than ± 1, $(t+1)^{-\alpha}(t-1)^{-\beta}\sigma$ is analytic at all other points of the complex plane. Since $L-\lambda$ has a regular singularity at infinity, $(t+1)^{-\alpha}(t-1)^{-\beta}\sigma$ is of the form $O(|z|^{+N})$ at ∞ for some sufficiently large integer N. It follows by known elementary theorems of complex variable theory that $(t+1)^{-\alpha}(t-1)^{-\beta}\sigma$ is a polynomial. Thus, in order that λ be an eigenvalue of L, it is necessary and sufficient that $(L-\lambda)\sigma = 0$ have a solution of the form $(1+t)^{\alpha}(1-t)^{\beta}P(t)$, where P is a polynomial.

It follows by the transformations of dependent and independent variable outlined above that σ is a constant multiple of

$$(t+1)^{\alpha}(1-t)^{\beta}F\left(\alpha+\beta+\gamma+\tfrac{1}{2}, \alpha+\beta+\tfrac{1}{2}-\gamma; 1+2\alpha; \frac{t+1}{2}\right)$$

(where as above $\lambda = \gamma^2 - 1/4$). Thus λ is an eigenvalue if and only if F

is a polynomial, i.e., if and only if the hypergeometric series terminates. It is evident from [2] that this is the case if and only if $\alpha+\beta+n+1/2 = \pm\gamma$, n being a non-negative integer. That is: the eigenvalues of L are $\lambda_n = (n+\alpha+\beta+1)(n+\alpha+\beta)$, the corresponding orthonormal eigenfunctions are

$$\varphi_n(x) = c_n(1+t)^\alpha(1-t)^\beta F\left(-n, +n+2\alpha+2\beta+1; 1+2\alpha; \frac{t+1}{2}\right)$$

where c_n is a normalizing factor which has yet to be chosen. The polynomial occurring on the right side is usually denoted (after multiplication by $\binom{n+2\alpha}{n}$ as $P_n^{(2\alpha, 2\beta)}$, and is called a *Jacobi polynomial.* Special cases are: $\alpha = \beta$, where the polynomials are known as *ultraspherical polynomials*; $\alpha = \beta = \pm 1/4$, where the polynomials are known as *Tchebichef polynomials of the first and second kind*; and $\alpha = \beta = 0$, where the polynomials are known as *Legendre polynomials.*

We now normalize the function φ_n. This may be done as follows. Since $E((\lambda_n))$ is the orthogonal projection of $L_2(-1, +1)$ on the one dimensional linear space generated by the eigenfunction φ_n, and since $\{\varphi_i\}$ is a complete orthonormal set, $E((\lambda_n))$ must be given by the formula

$$E(\{\lambda_n\})f = (f, \varphi_n)\varphi_n,$$

that is, by the formula

$$(E((\lambda_n))f)(t) = \varphi_n(t)\int_{-1}^{+1} \varphi_n(s)f(s)ds.$$

Now, by Theorem 3.16, by Corollary 5.30, and by the remark following Theorem 5.16, $E(\{\lambda_n\})$ is given by

$$\left(E(\{\lambda_n\})f\right)(t) = k_n\,\varphi(t, \lambda_n)\int_{-1}^{+1} \psi(s, \lambda_n)f(s)ds,$$

where

$$\varphi(t, \lambda) = (t+1)^\alpha(1-t)^\beta F\left(\alpha+\beta+\gamma+\tfrac{1}{2}, \alpha+\beta+\tfrac{1}{2}-\gamma; 1+2\alpha: \frac{t+1}{2}\right),$$

$$\psi(t, \lambda) = (t+1)^\alpha(1-t)^\beta F\left(\alpha+\beta+\gamma+\tfrac{1}{2}, \alpha+\beta+\tfrac{1}{2}-\gamma; 1+2\beta; \frac{1-t}{2}\right),$$

$\gamma = \sqrt{\lambda+1/4}$, and k_n is the residue of $\{(1-t^2)W(\varphi(t,\lambda),\psi(t,\lambda))\}^{-1}$ at $\lambda=\lambda_n$, $W(f,g)$ denoting, as usual, the Wronskian determinant $f'g-g'f$ of the two functions f and g. Next note that if the hypergeometric functions on the right of the above formula are denoted by $F_\alpha((1+t)/2)$ and $F_\beta((1-t)/2)$ respectively, we have

$$\begin{aligned}(1-t^2)W(\varphi,\psi) &= (1+t)^{2\alpha+1}(1-t)^{2\beta+1}W\left(F_\alpha\left(\frac{1+t}{2}\right),F_\beta\left(\frac{1-t}{2}\right)\right)\\ &= 2^{2\alpha+2\beta+1}s^{2\alpha+1}(1-s)^{2\beta+1}W\big(F_\alpha(s),F_\beta(1-s)\big),\end{aligned}$$

where $s=(1+t)/2$. Now, by the analysis of the effect of change of variables given at the beginning of the present section, F_α and F_β are both solutions of the equation with regular singularities at 0, 1, ∞ and corresponding exponents $0, -2\alpha; 0, -2\beta; 1/2+\gamma+\alpha+\beta, 1/2-\gamma+\alpha+\beta$. The particular solution $F_\alpha(s)$ of this equation may be characterized as being regular and having the value 1 at zero, and $F_\beta(1-s)$ may be characterized as being regular and having the value 1 at one. Let $F_\alpha^\dagger$ be the unique solution of the equation with these same regular singularities and exponents which has the form $z^{-2\alpha}(1+z+\ldots)$ near $z=0$. Then, since F_α and $F_\alpha^\dagger$ together comprise a basis for the solutions of our equation, we have a relation $F_\beta=\kappa_1F_\alpha+\kappa_2F_\alpha^\dagger$. Consequently, $W(F_\alpha,F_\beta)=\kappa_2W(F_\alpha,F_\alpha^\dagger)$. Now, as follows immediately from Green's formula, $s^{2\alpha+1}(1-s)^{2\beta+1}W(F,F')$ is constant for any two solutions of our equation. We have, therefore, $W(F_\alpha,F_\alpha^\dagger)$ $=$ const. $s^{-2\alpha-1}(1-s)^{-2\beta-1}$, and since $W(F_\alpha,F_\alpha^\dagger)=F_\alpha'F_\alpha^\dagger-F_\alpha(F_\alpha^\dagger)'$ $=(+2\alpha)s^{-2\alpha-1}(1+\ldots)$ for s near zero, we must have $W(F_\alpha,F_\alpha^\dagger)=$ $2\alpha s^{-2\alpha-1}(1-s)^{-2\beta-1}$ and $(F_\alpha,F_\beta)=2\alpha\kappa_2s^{-2\alpha-1}(1-s)^{-2\beta-1}$. To calculate the coefficient κ_2, we may reason as follows. The function $z^{2\alpha}F_\alpha^\dagger$ is the unique solution of the equation with singularities 0, 1, ∞ and corresponding exponents 2α, 0; 0, -2β; $1/2+\gamma-\alpha+\beta$, $1/2-\gamma-\alpha+\beta$ which is regular and has the value 1 at zero; the function $z^{2\alpha}F_\beta$ is the unique solution of this equation which is regular and has the value 1 at 1. Thus we have

$$\begin{aligned}z^{2\alpha}F_\alpha^\dagger(z) &= F(\tfrac12+\gamma-\alpha+\beta,\tfrac12-\gamma-\alpha+\beta;1-2\alpha;z)\\ z^{2\alpha}F_\beta(z) &= F(\tfrac12+\gamma-\alpha+\beta,\tfrac12-\gamma-\alpha+\beta;1+2\beta;1-z).\end{aligned}$$

Putting $z=0$ in the equation $z^{2\alpha}F_\beta(z)=\kappa_1z^{2\alpha}F_\alpha(z)+\kappa_2z^{2\alpha}F_\alpha^\dagger(z)$,

we consequently find (since $\alpha > 0$),

$$F(\tfrac{1}{2}+\gamma-\alpha+\beta, \tfrac{1}{2}-\gamma-\alpha+\beta; 1+2\beta; 1) = \kappa_2.$$

By formula [4] then,

$$\begin{aligned}
\kappa_2 &= \frac{\Gamma(1+2\beta)}{\Gamma(\frac{1}{2}+\gamma-\alpha+\beta)\Gamma(\frac{1}{2}+\beta-\gamma+\alpha)}\int_0^1 t^{\gamma-\alpha+\beta-1/2}(1-t)^{2\alpha-1}dt \\
&= \frac{\Gamma(1+2\beta)}{\Gamma(\frac{1}{2}+\gamma-\alpha+\beta)\Gamma(\frac{1}{2}+\beta-\gamma+\alpha)}\,\frac{\Gamma(\gamma-\alpha+\beta+\frac{1}{2})\Gamma(2\alpha)}{\Gamma(\alpha+\gamma+\beta+\frac{1}{2})} \\
&= \frac{\Gamma(1+2\beta)\Gamma(2\alpha)}{\Gamma(\frac{1}{2}+\beta-\gamma+\alpha)\Gamma(\frac{1}{2}+\alpha+\beta+\gamma)}.
\end{aligned}$$

Thus

$$\begin{aligned}
s^{2\alpha+1}(1-s)^{2\beta+1}W(F_\alpha, F_\beta) &= \frac{2\alpha\Gamma(1+2\beta)\Gamma(2\alpha)}{\Gamma(\frac{1}{2}+\alpha+\beta-\gamma)(\frac{1}{2}+\alpha+\beta+\gamma)} \\
&= \frac{\Gamma(1+2\beta)\Gamma(1+2\alpha)}{\Gamma(\frac{1}{2}+\alpha+\beta-\gamma)\Gamma(\frac{1}{2}+\alpha+\beta+\gamma)}.
\end{aligned}$$

The residue of $\{(1-t^2)W(\varphi(t,\lambda), \psi(t,\lambda))\}^{-1}$ at $\lambda = \lambda_n$, $1/2+\alpha+\beta-\gamma_n = -n$, is consequently

$$\begin{aligned}
\kappa_2 &= 2^{-2\alpha-2\beta-1}\frac{\Gamma(1+2\alpha+2\beta+n)(-1)^n}{\Gamma(1+2\alpha)\Gamma(1+2\beta)n!}2\sqrt{\lambda+1/4} \\
&= 2^{-2\alpha-2\beta-1}\frac{\Gamma(1+2\alpha+2\beta+n)(2n+2\alpha+2\beta+1)(-1)^n}{\Gamma(1+2\alpha)\Gamma(1+2\beta)n!},
\end{aligned}$$

Here we have used the known fact that the residue of $\Gamma(z)$ at its simple pole $z = -n$ is $(-1)^n/n!$.

Since at $\lambda = \lambda_n$, $\varphi(t,\lambda)$ and $\psi(t,\lambda)$ become linearly dependent, we have $\varphi(t,\lambda_n) = \varepsilon_n\psi(t,\lambda_n)$, whence, putting $t=1$, we find

$$\begin{aligned}
\varepsilon_n &= F(\alpha+\beta+\gamma_n+\tfrac{1}{2}, \alpha+\beta-\gamma_n+\tfrac{1}{2}; 1+2\alpha; 1) \\
&= F(1+2\alpha+2\beta+n, -n; 1+2\alpha; 1).
\end{aligned}$$

Using [4] again we find

$$\varepsilon_n = \frac{\Gamma(1+2\alpha)\Gamma(-2\beta)}{\Gamma(-2\beta-n)\Gamma(1+2\alpha+n)} = (-1)^n\frac{\Gamma(1+2\alpha)\Gamma(1+2\beta+n)}{\Gamma(1+2\alpha+n)\Gamma(1+2\beta)};$$

here we have used the formula $\Gamma(z)\Gamma(1-z) = \pi \operatorname{cosec} \pi z$. Thus

$$\begin{aligned}\big(E(\{\lambda_n\})f\big)(t) &= \varepsilon_n^{-1} k_n \varphi(t, \lambda_n) \int_{-1}^{+1} \varphi(s, \lambda_n) f(s) ds \\ &= \varphi_n(t) \int_{-1}^{+1} \varphi_n(s) f(s) ds,\end{aligned}$$

where $\varphi(t, \lambda_n) = (t+1)^{\alpha}(1-t)^{\beta} F(-n, n+2\alpha+2\beta+1; 1+2\alpha; (t+1)/2)$ and $\varphi_n(t) = c_n \varphi_n(t, \lambda_n)$. Consequently

$$|c_n|^2 = k_n/\varepsilon_n = \frac{2^{-2\alpha-2\beta-1}\Gamma(1+2\alpha+2\beta+n)\Gamma(1+2\alpha+n)(2n+2\alpha+2\beta+1)}{(\Gamma(1+2\alpha))^2 \Gamma(1+2\beta+n) n!}$$

so that we have successfully normalized the functions φ_n.

Next we turn to an analysis of Case 2, assuming that $\alpha \geqq 1/2$ and $0 < \beta < 1/2$, so that by Theorem 2.30, $L^{\alpha,\beta}$ has two independent boundary values, both at $+1$. (Analysis of the cases $\beta = 0$ or $\alpha = 1/2$ is omitted for simplicity, and left to the reader as an exercise.) Let $\varphi_+(\varphi_-)$ be any C^∞ function defined on $(-1, +1)$, vanishing for $t < 0$, which is equal to $(t-1)^{\beta}$ (to $(t-1)^{-\beta}$) for t near 1. Then a moment's calculation gives

$$L^{\alpha,\beta}\varphi_\pm(t) = (1-t)^{\pm\beta}((-2\beta^2+2\beta^2)(1-t)^{-1} + \text{const.} + \dots),$$

so that $L\varphi_\pm \in L_2(-1, +1)$, and $\varphi_\pm \in \mathfrak{D}(T_1(L))$. Thus, by Definition XII.4.20, $\tilde{A}_\pm(f) = (Lf, \varphi_\pm) - (f, L\varphi_+)$ are boundary values for L. We have

$$\begin{aligned}\tilde{A}_\pm(f) &= \lim_{\varepsilon\to 0}\int_{-1}^{1-\varepsilon} -((1-t^2)f'(t))'\varphi_\pm(t) + f(t)((1-t^2)\varphi'_\pm(t))'dt \\ &= \lim_{\varepsilon\to 0} (1-t^2)(\varphi'_\pm(t)f(t) - \varphi_\pm(t)f'(t))\Big|_{-1}^{1-\varepsilon} \\ &= \lim_{\varepsilon\to 0} 2\varepsilon(\pm\beta\varepsilon^{\pm\beta-1} f(1-\varepsilon) - \varepsilon^{\pm\beta} f'(\varepsilon)) \\ &= \lim_{\varepsilon\to 0} -2(\varepsilon^{\pm\beta+1} f'(1-\varepsilon) \mp \beta\varepsilon^{\pm\beta} f(1-\varepsilon)).\end{aligned}$$

Thus, the limits

$$\begin{aligned}A_+(f) &= \lim_{\varepsilon\to 0} (\varepsilon^{\beta+1} f'(1-\varepsilon) - \beta\varepsilon^{\beta} f(1-\varepsilon)), \\ A_-(f) &= \lim_{\varepsilon\to 0} (\varepsilon^{-\beta+1} f'(1-\varepsilon) + \beta\varepsilon^{-\beta} f(1-\varepsilon)),\end{aligned}$$

exist for all f in $\mathfrak{D}(T_1(L))$ and define boundary values for L. We have, by a short calculation,

$$A_+(\varphi_+) = 0, \qquad A_-(\varphi_+) = 2\beta,$$
$$A_+(\varphi_-) = -2\beta, \qquad A_-(\varphi_-) = 0.$$

It follows that A_+ and A_- are independent boundary values. Moreover, the equation

$$\{f, g\} = \quad i\{(Lf, g)-(f, Lg)\} = (i\beta)^{-1}\{A_+(f)\overline{A_-(g)}-A_-(f)\overline{A_+(g)}\}$$

is valid if f and g are chosen from among φ_+ and φ_-; and consequently, by Theorem XII.4.24, is valid for all f and g in $\mathfrak{D}(T_1(L))$. By Corollary 2.81, the most general self adjoint restriction of $T_1(L)$ is the restriction of $T_1(L)$ to the domain determined by a single boundary condition $A_+(f) = kA_-(f)$, where k is a real number, $-\infty < k \leq \infty$.

If $k = 0$, the unique solution of the equation $Lf = \lambda f$ which is square-integrable at $t = +1$ and satisfies the given boundary condition $A_+(f) = 0$ can easily be seen to be that solution asymptotic to $(1-t)^\beta$ at $t = 1$; so that all our calculations run along precisely the same lines as in Case 1, and we are again led to the eigenvalues $\lambda_n = (n+\alpha+\beta+1)(n+\alpha+\beta)$ and orthonormal eigenfunctions

$$\varphi_n(t) = c_n(1+t)^\alpha(1-t)^\beta F\left(-n, n+2\alpha+2\beta+1; 1+2\alpha; \frac{t+1}{2}\right),$$

$$|c_n|^2 = \frac{2^{-2\alpha-2\beta-1}\Gamma(1+2\alpha+2\beta+n)\Gamma(1+2\alpha+n)}{(\Gamma(1+2\alpha))^2\Gamma(1+2\beta+n)n!}(2n+2\alpha+2\beta+1).$$

If $k = \infty$, so that our boundary condition is $A_-(f) = 0$, the unique solution of the equation $Lf = \lambda f$ which is square-integrable at $t = +1$ and satisfies the given boundary condition can be seen to be that solution asymptotic to $(1-t)^{-\beta}$ at $t = 1$. Substituting $\beta' = -\beta$ brings us back to the previous case, so that we have eigenvalues $\lambda_n = (n+\alpha-\beta+1)(n+\alpha-\beta)$ and orthonormal eigenfunctions

$$\varphi_n(t) = c_n(1+t)^\alpha(1-t)^{-\beta}F\left(-n, n+2\alpha-2\beta+1; 1+2\alpha; \frac{t+1}{2}\right),$$

$$|c_n|^2 = \frac{2^{-2\alpha+2\beta-1}\Gamma(1+2\alpha-2\beta+n)\Gamma(1+2\alpha+n)}{(\Gamma(1+2))^2\Gamma(1-2+n)n!}(2n+2\alpha-2\beta+1).$$

If $0 < |k| < \infty$, we cannot expect so close an analogy with the results of Case 1. In this case, the solution of $L\sigma = \lambda\sigma$ satisfying the boundary condition $A_+(f) = kA_-(f)$ at $t = 1$ may easily be seen to be $\rho = (\psi - k\tau)$, where ψ (where τ) is the unique solution of $L\sigma = \lambda\sigma$ of the form $(1+t)^\alpha(1-t)^\beta(1+\ldots)$ (of the form $(1+t)^\alpha(1-t)^{-\beta}(1+\ldots)$) in the neighborhood of $t = 1$. The eigenfunctions can consequently no longer be polynomials. If we put

$$\varphi(t,\lambda) = (t+1)^\alpha(1-t)^\beta F(\alpha+\beta+\gamma+1/2,\\ \alpha+\beta+1/2-\gamma;\, 1+2\alpha;\, (t+1)/2)$$

as in Case 1, so that $\varphi(t, \lambda)$ is the unique solution of $Lf = \lambda f$ square-integrable in the neighborhood of $t = -1$, we have, as in Case 1,

$$(1-t^2)W(\varphi(t,\lambda), \psi(t,\lambda)) = \frac{2^{2\alpha+2\beta+1}\Gamma(1+2\beta)\Gamma(1+2\alpha)}{\Gamma(\frac{1}{2}+\alpha+\beta-\gamma)\Gamma(\frac{1}{2}+\alpha+\beta+\gamma)}.$$

The solutions ψ and τ are conjugate under the substitution $\beta \to -\beta$; thus

$$(1-t^2)W(\varphi(t,\lambda), \tau(t,\lambda)) = \frac{2^{2\alpha-2\beta+1}\Gamma(1-2\beta)\Gamma(1+2\alpha)}{\Gamma(\frac{1}{2}+\alpha-\beta-\gamma)\Gamma(\frac{1}{2}+\alpha-\beta+\gamma)}.$$

The eigenvalues λ_n are consequently the successive roots of

$$\frac{2^{2\beta}\Gamma(1+2\beta)}{\Gamma(\frac{1}{2}+\alpha+\beta-\gamma_n)\Gamma(\frac{1}{2}+\alpha+\beta+\gamma_n)} = k2^{-\beta}\frac{\Gamma(1-2\beta)}{\Gamma(\frac{1}{2}+\alpha-\beta-\gamma_n)\Gamma(\frac{1}{2}+\alpha-\beta+\gamma_n)},$$

$\gamma_n = (\lambda_n+1/4)^{1/2}$. The corresponding orthonormal eigenfunctions are $\varphi_n(t) = c_n\varphi(t, \lambda_n)$, where c_n is a normalization constant still to be determined. Reasoning as in Case 1, we see that $|c_n|^2 = k_n/\varepsilon_n$, where k_n is the residue of $\{(1-t^2)W(\varphi, \rho)\}^{-1}$ at $\lambda = \lambda_n$, and where ε_n is the coefficient in the equation $\varphi(t, \lambda_n) = \varepsilon_n\rho(t, \lambda_n)$. If

$$(1-t^2)W(\varphi(t,\lambda), \rho(t,\lambda)) = W(\lambda),$$

then $k_n = (W'(\lambda_n))^{-1}$; since

$$W(\lambda) = 2^{2\alpha+1}\Gamma(2\alpha+1)\left\{\frac{2^\beta\Gamma(1+2\beta)}{\Gamma(\frac{1}{2}+\alpha+\beta-\gamma)\Gamma(\frac{1}{2}+\alpha+\beta+\gamma)}\right.\\ \left. - \frac{k2^{-\beta}\Gamma(1-2\beta)}{\Gamma(\frac{1}{2}+\alpha-\beta-\gamma)\Gamma(\frac{1}{2}+\alpha-\beta+\gamma)}\right\},$$

we have

$$
\begin{aligned}
W'(\lambda_n) & \\
= 2^{2\alpha}\Gamma(2\alpha+1)\gamma_n^{-1}\Bigg[& 2^{\beta}\frac{\Gamma(1+2\beta)}{\Gamma(\frac{1}{2}+\alpha+\beta-\gamma_n)\Gamma(\frac{1}{2}+\alpha+\beta+\gamma_n)} \\
& \qquad \{\xi(\tfrac{1}{2}+\alpha+\beta-\gamma_n)-\xi(\tfrac{1}{2}+\alpha+\beta+\gamma_n)\} \\
& k2^{-\beta}\frac{\Gamma(1-2\beta)}{\Gamma(\frac{1}{2}+\alpha-\beta-\gamma_n)\Gamma(\frac{1}{2}+\alpha-\beta+\gamma_n)} \\
& \qquad \{\xi(\tfrac{1}{2}+\alpha-\beta-\gamma_n)-\xi(\tfrac{1}{2}+\alpha-\beta+\gamma_n)\}\Bigg],
\end{aligned}
$$

where $\xi(z) = \Gamma'(z)/\Gamma(z)$ is the logarithmic derivative of the Γ-function. Thus the constant $k_n = (W'(\lambda_n))^{-1}$ is determined. To determine the constant ε_n note that we have

$$
\begin{aligned}
\varphi(t, \lambda_n) &= \varepsilon_n\psi(t, \lambda_n) - \varepsilon_n k\tau(t, \lambda_n), \\
\psi(t, \lambda_n) &= (1+t)^{\alpha}(1-t)^{\beta}F_{\beta}\left(\frac{1-t}{2}\right), \\
\tau(t, \lambda_n) &= (1+t)^{\alpha}(1-t)^{\beta}F_{\beta}^{\dagger}\left(\frac{1-t}{2}\right)
\end{aligned}
$$

in terms of the notations used in Case 1. Thus ε_n can be determined from the knowledge of the coefficient ε in the equation $F_{\alpha}(t) = \varepsilon F_{\beta}(1-t)+\varepsilon' F_{\beta}^{\dagger}(1-t)$. To determine this coefficient, suppose momentarily that $\beta < 0$, so that $F_{\beta}^{\dagger}(0) = 0$, and hence by [4]

$$
\begin{aligned}
\varepsilon = F_{\alpha}(1) &= F(\alpha+\beta+\gamma+\tfrac{1}{2}, \alpha+\beta-\gamma+\tfrac{1}{2}; 1+2\alpha; 1) \\
&= \frac{\Gamma(1+2\alpha)\Gamma(-2\beta)}{\Gamma(\frac{1}{2}+\alpha-\beta-\gamma)\Gamma(\frac{1}{2}+\alpha-\beta+\gamma)};
\end{aligned}
$$

by analytical continuation, this must hold for $\beta > 0$ also. Thus

$$
\varepsilon_n = \frac{\Gamma(1+2\alpha)\Gamma(-2\beta)}{\Gamma(\frac{1}{2}+\alpha-\beta-\gamma_n)\Gamma(\frac{1}{2}+\alpha-\beta+\gamma_n)}.
$$

We have consequently expressed the orthonormal eigenfunctions φ_n in closed form in terms of hypergeometric functions, the roots γ_n of the equation determining the eigenvalues, the gamma-function and its derivatives.

Finally, let us make a brief examination of Case 8. Here we have

four boundary values, two at $+1$ and two at -1. It is easily seen that if, for $0 < \alpha < \frac{1}{2}$, we take the boundary values

$$B_+(f) = \lim_{\varepsilon \to 0} \{\varepsilon^{\alpha+1} f'(\varepsilon - 1) - \alpha\varepsilon^{\alpha} f(\varepsilon - 1)\},$$
$$B_-(f) = \lim_{\varepsilon \to 0} \{\varepsilon^{-\alpha+1} f'(\varepsilon - 1) + \alpha\varepsilon^{\alpha} f(\varepsilon - 1)\},$$

in addition to the boundary values A_+ introduced in the course of analyzing Case 2, we have a complete set of boundary values. It may be seen by arguments similar to those used in Case 2, i.e., by use of XII.4.24, that

$$\begin{aligned}(f, g) = {} & (i\beta)^{-1}\{A_+(f)\overline{A_-(g)} - A_-(f)\overline{A_+(g)}\} \\ & + (i\alpha)^{-1}\{B_+(f)\overline{B_-(g)} - B_-(f)\overline{B_+(g)}\}.\end{aligned}$$

In this case the set of all self adjoint restrictions of $T_1(L)$ depends on an arbitrary 2×2 unitary matrix, and hence on four real parameters. We shall consequently not do more than examine a few of the simpler special cases which arise.

First of all, if we impose the condition $B_+(f) = 0$, and a second condition at $t = +1$, the unique solution of $Lf = \lambda f$ satisfying the given boundary condition at -1 is, just as in Case 2,

$$\varphi(t, \lambda) = (t+1)^{\alpha}(1-t)^{\beta} F\left(\tfrac{1}{2} + \alpha + \beta + \gamma, \tfrac{1}{2} + \alpha + \beta - \gamma; 1 + 2\alpha; \frac{1+t}{2}\right),$$

$\gamma = (\lambda + 1/4)^{1/2}$. Thus the calculations for the boundary condition $B_+(f) = 0$ are much the same as the calculations for Case 2. In particular, the boundary conditions $B_+(f) = 0$, $A_+(f) = 0$ give us eigenvalues $\lambda_n = (n + \alpha + \beta + 1)(n + \alpha + \beta)$, and normalized eigenfunctions $c_n\varphi(t, \lambda_n)$, $|c_n|^2$ being given by the same formulas as in Case 2.

All the above sets of boundary conditions are separated. An interesting self adjoint set of mixed boundary conditions is the set $A_+(f) = B_-(f)$, $A_-(f) = B_+(f)$, which we study in the case $\alpha = \beta$. Let the corresponding restriction of $T_1(L)$ be denoted as T. We have $A_+(f) = -B_-(g)$ and $A_-(f) = -B_+(g)$ if $f(t) = g(-t)$. Thus if f is in $\mathfrak{D}(T)$, the function $g(t) = f(-t)$ is also in $\mathfrak{D}(T)$. Since $Lf = Lg$, the even and odd parts of an eigenfunction of T are eigenfunctions of T. An even eigenfunction of T satisfies $A_+(f) = -B_-(f)$ and $A_-(f) = -B_+(f)$ in addition to the boundary conditions $A_+(f) = B_-(f)$,

$A_-(f) = -B_+(f)$; thus $A_+(f) = 0 = B_-(f)$. Conversely, if $A_+(f) = 0 = B_-(f)$ and f is even, both boundary conditions defining $\mathfrak{D}(T)$ are satisfied. Thus the even eigenfunctions of T are the even functions satisfying the self adjoint set $A_+(f) = 0 = B_-(f)$ of boundary conditions and the equation $Lf = \lambda f$ for some λ. By our previous analysis, these are precisely the even functions among the set of all functions

$$\varphi_n(t) = c_n(1+t)^{\alpha}(1-t)^{\alpha} F\left(-n, n+4\alpha+1; 1+2\alpha; \frac{1+t}{2}\right),$$

$$|c_n|^2 = \frac{2^{-4\alpha-1}\Gamma(1+4\alpha+n)(2n+4\alpha+1)}{(\Gamma(1+2\alpha))^2\, n!}.$$

Now, if $A_+(f) = 0 = B_-(f)$ and $g(t) = f(-t)$, then $A_+(g) = 0 = B_-(g)$; thus, if f is an eigenfunction of the self adjoint restriction S of $T_1(L)$ determined by this latter pair of boundary conditions, so is g. Since, as we have seen, every eigenvalue of S is simple, it follows that every eigenfunction of S, i.e., every function φ_n in the formula displayed above, is either even or odd. Since the polynomial $F(-n, n+4\alpha+1, 1+2\alpha, (1+t)/2)$ is of order n, it must consequently be even for even n and odd for odd n. Thus, φ_n is an eigenfunction of T for even n, but not for odd n. A similar analysis of the odd eigenfunctions f of T shows them to satisfy $A_-(f) = 0 = B_+(f)$, and consequently to be precisely the functions

$$\bar{\varphi}_n(t) = \bar{c}_n(1+t)^{-\alpha}(1-t)^{-\alpha} F\left(-n, n-4\alpha+1; 1-2\alpha; \frac{1+t}{2}\right),$$

$$|\bar{c}_n|^2 = \frac{2^{4\alpha-1}\Gamma(1-4\alpha+n)(2n-4\alpha+1)}{(\Gamma(1-2\alpha))^2\, n!},$$

for n odd, $n \geqq 1$. Thus, the eigenvalues of T break into two sequences

$$\lambda_n = (n+4\alpha+1)(n+4\alpha), \qquad n \text{ even}, \ n \geqq 0$$
$$\bar{\lambda}_n = (n-4\alpha+1)(n-4\alpha), \qquad n \text{ odd}, \ n \geqq 0,$$

with corresponding orthonormal functions φ_n (n even) and $\bar{\varphi}_n$ (n odd).

Next we turn to the analysis of the operator

$$L_1 = -L = -\left(\frac{d}{dt}\right)(t^2-1)\left(\frac{d}{dt}\right) - \frac{2\alpha^2}{1+t} - \frac{2\beta^2}{1-t}$$

on the interval $(1, \infty)$. We assume here that β is positive but let α be

real or pure imaginary. As above, we see that the exponents of $L_1 - \lambda$ at -1, $+1$, ∞ are $\pm\alpha$, $\pm\beta$, and $1/2 \pm \gamma$ respectively, where $\lambda = 1/4 - \gamma^2$. Putting $\gamma_0 = 1/2 + i$ so that $\lambda_0 = 1 + i$, we see that the equation $(L_1 - \lambda_0)f$ has one solution of the order of t^{-1-i} as $t \to \infty$ and another which behaves like t^i as $t \to \infty$. The solution at $\lambda_0 = 1 - i$ is exactly similar. Thus, by Theorem XII.4.19, $L_1 - \lambda$ has precisely one solution belonging to $L_2(2, \infty)$ for each non-real λ. Thus two cases arise.

Case 1. $\beta \geq 1/2$. Here the deficiency indices of L_1 are zero, so that no boundary condition need be imposed, and L_1 leads to a unique self adjoint operator in $L_2(1, \infty)$, which we continue to denote by the symbol L_1.

Case 2. $0 \leq \beta < 1/2$. The deficiency indices of L_1 are 1, and a single boundary condition at $+1$ must be imposed in order to get a self adjoint operator in Hilbert space.

Let us consider Case 1. If γ is pure imaginary, $\gamma = i\rho$, i.e, if $\lambda \geqq 1/4$, then $(L_1 - \lambda)f$ has two linearly independent solutions which behave at ∞ like $t^{-1/2 \pm i\rho}$; thus, no solution of $(L - \lambda)f = 0$ is in $L_2(1, \infty)$. Consequently, the point spectrum of L_1 is confined entirely to the region $\lambda < 1/4$. In this region, $(L_1 - \lambda)f = 0$ has one solution of the order of $t^{-1/2-\gamma}$ and one of the order of $t^{-1/2+\gamma}$; so that there is a one dimensional subspace of solutions square-integrable at infinity. By the reasoning applied above to the operator L on the interval $(-1, +1)$, then λ is an eigenvalue if and only if $(L_1 - \lambda)f = 0$ has a solution σ of the form $(t-1)^\beta(1+t)^{-1/2-\gamma-\beta}P((1+t)/2)$, γ being taken as the positive square root of $1/4 - \lambda$, and P being a polynomial. It follows by the transformations of dependent and independent variables outlined at the beginning of this section that σ is a constant multiple of

$$(t-1)^\beta(1+t)^{-1/2-\gamma-\beta}F\left(\beta - \alpha + \gamma + \tfrac{1}{2}, \beta + \alpha + \gamma + \tfrac{1}{2}; 2\gamma + 1; \frac{1+t}{2}\right).$$

Thus λ is an eigenvalue if and only if $\beta - \alpha + \gamma + 1/2 = -n$, n being a non-negative integer, and γ a positive real number. (Note that $\beta + \alpha + \gamma + 1/2$ can never be a non-positive integer, since $\alpha, \beta, \gamma > 0$.) If α is non-real, this is impossible; if α is real, we have orthonormal eigenfunctions

$$c_n(t-1)^{\beta}(1+t)^{n-\alpha}F\left(-n,\ -n+2\alpha;\ 2\alpha-2\beta-2n;\ \frac{2}{1+t}\right),$$

where $0 \leq n < \alpha-\beta-1/2$, and where c_n is a normalizing constant still to be determined. Thus, we have only a finite number of eigenvalues $(\alpha-\beta-n)(n+1+\beta-\alpha)$, $0 \leq n < \alpha-\beta-1/2$. The normalizing constants c_n could be determined directly by the general method used in studying L on the interval $(-1, +1)$; but in the present case they may also be determined from the information gained from the study of L by the following elementary substitution. The constant $|c_n|^2$ is the reciprocal of the integral

$$I = \int_1^{\infty} (t-1)^{2\beta}(1+t)^{2n-2\alpha}F\left(-n,\ -n+2\alpha;\ 2\alpha-2\beta-2n;\ \frac{2}{1+t}\right)^2 dt.$$

Make the substitution

$$\frac{2}{1+t} = \frac{1+s}{2}, \qquad t-1 = \frac{2(1-s)}{1+s}, \qquad \frac{-4ds}{(1+s)^2} = dt,$$

to obtain

$$I = 2^{2\beta+4n-4\alpha+2} \times$$
$$\int_{-1}^{+1}(1-s)^{2\beta}(1+s)^{2\alpha-2\beta-2n-2}F\left(-n,\ -n+2\alpha;\ 2\alpha-2\beta-2n;\ \frac{1+s}{2}\right)^2 ds.$$

In normalizing the eigenfunctions of L above, we established the equation

$$\int_{-1}^{+1}(1+s)^{2a}(1-s)^{2b}F\left(-n, n+2a+2b+1; 1+2a; \frac{s+1}{2}\right)^2 ds$$
$$= \frac{2^{2a+2b+1}(\Gamma(1+2a))^2\Gamma(1+2b+n)n!}{\Gamma(1+2a+2b+n)\Gamma(1+2a+n)(2n+2a+2b+1)}.$$

Thus, putting $2\alpha-2\beta-2n = 2a+1$ and $2n+2a+2b+1 = 2\alpha$, we find

$$I = \frac{2^{2\beta+2n-2\alpha+1}(\Gamma(2\alpha-2\beta-2n))^2\Gamma(1+2\beta+n)n!}{\alpha\Gamma(2\alpha-n)\Gamma(2\alpha-2\beta-n)}; \qquad |c_n|^2 = I^{-1}.$$

Thus the discrete spectrum of L is fully analyzed, and we turn to the analysis of the continuous spectrum.

First let us determine the location of the continuous spectrum. By Corollary 7.4, we may consider the end points $t = 1$ and $t = \infty$ separately. Since our operator has two boundary values at $t = 1$ for $0 \leq \beta < 1/2$ and is evidently semi-bounded below on any interval $(1, c]$, $c < \infty$, and increases with increasing β, it follows from the comparison theorem (7.84), and from Theorems 4.1 and 4.2, that the end point $t = 1$ contributes nothing to the essential spectrum for any β. Since in the present case $\int_2^\infty p(t)^{-1/2}\, dt = \infty$ and

$$\lim_{t\to\infty} q(t)+\tfrac{1}{4}(p''(t) - \tfrac{1}{4}[p(t)]^{-1}[p'(t)]^2)$$
$$= \lim_{t\to\infty} \tfrac{1}{4}(2 - \tfrac{1}{4}(t^2 - 1)^{-1}(2t)^2) = \tfrac{1}{4},$$

it follows from Theorem 7.66 that the essential spectrum of L_1 is the region $\lambda \geq 1/4$ of the real axis. Thus the spectrum of L_1 consists of the eigenvalues listed above, which lie in the range $\lambda < 1/4$, and a continuous spectrum covering the range $\lambda \geqq 1/4$. We wish to find explicit formulas for that part of the spectral resolution of L corresponding to the continuous spectrum.

Let us agree that the quantity $\gamma = \gamma(\lambda) = (1/4 - \lambda)^{1/2}$ is to be taken for $\mathscr{R}(\lambda) > 1/4$ as that square root with $\mathscr{I}\gamma > 0$. Thus $\gamma(\lambda)$ is an analytic function of λ in the half-plane $\mathscr{R}(\lambda) > 1/4$. We have $\mathscr{R}(\gamma) > 0$ if $\mathscr{I}\lambda > 0$, $\mathscr{R}(\gamma) < 0$ if $\mathscr{I}\lambda < 0$. Let $S_\pm$ be the unique solution of the differential equation $L_1\sigma = \lambda\sigma$ which is of the form $t^{-1/2\pm\gamma}(1 + c/t + \ldots)$ in the neighborhood of $t = \infty$, and let S_β be the unique solution of this differential equation which is of the form $2^\alpha(t-1)^\beta(1 + c'(t-1) + \ldots)$ in the neighborhood of $t = 1$. We have, by the changes of dependent and independent variables outlined at the beginning of this section,

$$S_\beta = (t+1)^\alpha(t-1)^\beta F\left(\alpha+\beta+\gamma+\tfrac{1}{2}, \alpha+\beta-\gamma+\tfrac{1}{2}; 1+2\beta; \frac{1-t}{2}\right)$$

$$S_+ = (t-1)^\beta(1+t)^{-1/2-\gamma-\beta} F\left(\beta-\alpha+\gamma+\tfrac{1}{2}, \beta+\alpha+\gamma+\tfrac{1}{2}; 2\gamma+1; \frac{2}{1+t}\right)$$

$$S_- = (t-1)^\beta(1+t)^{-1/2+\gamma-\beta} F\left(\beta-\alpha-\gamma+\tfrac{1}{2}, \beta+\alpha-\gamma+\tfrac{1}{2}; 1-2\gamma; \frac{2}{1+t}\right).$$

All the solutions S_β, S_+, S_- consequently depend analytically on λ for $\mathscr{R}(\lambda) > 1/4$. Since any one of these three solutions is linearly dependent

on the other two, we have a relation $S_- = k_+(\lambda)S_+ + k_\beta(\lambda)S_\beta$, the coefficients k_+ and k_β being analytic functions of λ for $\mathscr{R}(\lambda) > 1/4$. Let us use S_α and S_+ as a standard basis for the solutions of $L_1\sigma = \lambda\sigma$.

Now, S_+ is square-integrable in the neighborhood of $t = \infty$ for $\mathscr{I}\lambda > 0$, and S_- is square-integrable in the neighborhood of $t = \infty$ for $\mathscr{I}\lambda > 0$. Thus, by Theorem 3.16, the Green's kernel K_λ is given by

$$K_\lambda(t, s) = \frac{S_\beta(t, \lambda)\overline{S_+(s, \bar{\lambda})}}{(t^2-1)W(S_\alpha, \bar{S}_+)}, \quad t < s, \qquad \mathscr{I}\lambda > 0, \qquad \mathscr{R}\lambda > 1/4,$$

$$= \frac{S_\alpha(t, \lambda)\overline{S_-(s, \bar{\lambda})}}{(t^2-1)W(S_\alpha, \bar{S}_-)}, \quad t < s, \qquad \mathscr{I}\lambda < 0, \qquad \mathscr{R}\lambda > 1/4.$$

In terms of the basis S_β, S_+ we have

$$K_\lambda(t, s) = \frac{\overline{k_+(\bar{\lambda})}S_\beta(t, \lambda)\overline{S_+(s, \bar{\lambda})}}{(t^2-1)\overline{k_+(\bar{\lambda})}W(S_\beta, \bar{S}_+)} + \frac{\overline{k_\beta(\bar{\lambda})}S_\beta(t, \lambda)\overline{S_\beta(s, \bar{\lambda})}}{(t^2-1)W(S_\beta, \bar{S}_-)},$$

$$t < s, \qquad \mathscr{I}\lambda < 0, \qquad \mathscr{R}\lambda > 1/4.$$

Thus, in the notations of the Titchmarsh-Kodaira theorem (5.18),

$$\gamma^-_{\beta,+}(\lambda) = \frac{1}{(t^2-1)W(S_\beta, \bar{S}_+)}, \qquad \gamma^-_{\beta,\beta}(\lambda) = 0,$$

$$\gamma^-_{+,+}(\lambda) = 0, \qquad \gamma^-_{+,\beta}(\lambda) = 0, \qquad \mathscr{I}\lambda > 0, \qquad \mathscr{R}\lambda > 1/4;$$

$$\gamma^-_{\beta,+}(\lambda) = \frac{1}{(t^2-1)W(S_\beta, \bar{S}_+)}, \qquad \gamma^-_{\beta,\beta}(\lambda) = \frac{\overline{k_\beta(\bar{\lambda})}}{(t^2-1)W(S_\beta, \bar{S}_-)},$$

$$\gamma^-_{+,+}(\lambda) = 0, \qquad \gamma^-_{+,\beta}(\lambda) = 0, \qquad \mathscr{I}\lambda < 0, \qquad \mathscr{R}\lambda > 1/4.$$

Thus from Theorem 5.27 and the fact that all our quantities are analytic even on the real axis we have $\rho_{++} = 0$, $\rho_{+,\beta} = 0$ (so that $\rho_{\beta,+} = \overline{\rho_{+,\beta}} = 0$ also), and

$$\rho_{\beta,\beta}(e) = \frac{1}{2\pi i}\int_e \frac{\overline{k_\beta(\bar{\lambda})}}{(t^2-1)W(S_\beta, \bar{S}_-)}\, d\lambda$$

$$= \frac{1}{2\pi i}\int_e \frac{(\overline{k_\beta(\bar{\lambda})})^2}{(t^2-1)\overline{k_+(\bar{\lambda})}W(\bar{S}_+, \bar{S}_-)}\, d\lambda.$$

Now we know that $(t^2-1)W(S_+, S_-)$ is constant, and since in the neighborhood of $t = \infty$ we have $(t^2-1)W(S_+S_-) \sim -2\gamma$, we have $(t^2-1)W(\bar{S}_+, \bar{S}_-) = -2\bar{\gamma}$. To evaluate the constant k_β we argue as follows. By the transformations of variables outlined at the beginning of the present section, the function

$$(t-1)^\beta(t+1)^{-\beta-\gamma-\frac{1}{2}}\, F\left(\tfrac{1}{2}+\gamma+\beta+\alpha, \tfrac{1}{2}+\gamma+\beta-\alpha; 1+2\beta; \frac{t-1}{t+1}\right) = S(t)$$

is a solution of the equation $L_1\sigma = \lambda\sigma$. It is, moreover, regular and of the form $2^{-\beta-\gamma-\frac{1}{2}}(t-1)^\beta(1+c(t-1)+\ldots)$ in the neighborhood of $t = 1$. Thus $2^{\alpha+\beta+\gamma+\frac{1}{2}}S = S_\beta$. We have $S = l_+S_+ + l_-S_-$, where, for $\mathscr{R}\gamma < 0$ we may put $t = \infty$ to obtain

$$l_+ = F(1/2+\gamma+\beta+\alpha, 1/2+\gamma+\beta-\alpha; 1+2\beta; 1);$$

by analytical continuation, this must hold for $\mathscr{R}\gamma > 0$ also. The involution $\gamma \to -\gamma$ interchanges S_+ and S_- and maps S_β into itself. Since $2^{\alpha+\beta+\gamma+\frac{1}{2}}S = S_\beta$, this involution multiplies S by $2^{2\gamma}$. Consequently, we must have

$$l_- = 2^{-2\gamma} F(\tfrac{1}{2}-\gamma+\beta+\alpha, \tfrac{1}{2}-\gamma+\beta-\alpha; 1+2\beta; 1).$$

Since $k_\beta S_\beta + k_+S_+ = S_-$, we have $k_\beta = (2^{+\alpha+\beta+\gamma+\frac{1}{2}}l_-)^{-1}$, $k_+ = -l_+l_-$; so that $-k_\beta^2/k_+ = (2^{+2\alpha+2\beta+2\gamma+1}l_-l_+)^{-1}$. Since for λ real, $\lambda > 1/4$, γ is pure imaginary and $\mathscr{I}\gamma > 0$, we have

$$2^{2\alpha+2\beta+2\gamma+1}l_-l_+ = 2^{2\alpha+2\beta+1}|F(\tfrac{1}{2}+\gamma+\beta+\alpha, \tfrac{1}{2}+\gamma+\beta-\alpha; 1+2\beta; 1)|^2,$$

$$\rho_{\beta,\beta}(e) = \frac{2^{2\alpha+2\beta+1}}{(2\pi\Gamma(1+2\beta))^2}\int_e \frac{|\Gamma(\frac{1}{2}+\beta-\gamma+\alpha)\Gamma(\frac{1}{2}+\beta-\gamma-\alpha)|^2 \sinh 2\pi\sqrt{\lambda-\frac{1}{4}}}{(\lambda-\frac{1}{4})}\, d\lambda,$$

and $\gamma = i\sqrt{\lambda-1/4}$, for subsets of the continuous spectrum $\lambda > 1/4$. This completes our analysis of the operator L_1 in Case 1. It is, however, interesting to summarize the result that has been obtained and to rephrase it somewhat.

For simplicity in statement, we consider only the subcases of Case 1 in which L_1 has a pure continuous spectrum but no point spectrum. The reader will have no trouble in formulating the general result pertaining to Case 1.

THEOREM. *Let L_1 be the formal differential operator*

$$L_1 = -\frac{d}{dt}(t^2-1)\frac{d}{dt} - \frac{2\alpha^2}{1+t} - \frac{2\beta^2}{1-t},$$

on the interval $(1, \infty)$. Suppose that $\beta \geqq \frac{1}{2}$ and either that α is pure imaginary or that $\alpha < \beta + 3/2$. Then L_1 has no boundary values, and hence determines a unique self adjoint operator which we continue to denote by the symbol L_1. If $\psi(t, \lambda)$ denotes the function

$$\psi(t, \lambda) = 2^{\alpha+\beta-\frac{1}{2}}\pi^{-1}\Gamma(1+2\beta)^{-1}|\Gamma(\tfrac{1}{2}+\beta-\gamma+\alpha)\Gamma(\tfrac{1}{2}+\beta-\gamma-\alpha)|^2 \times \left(\frac{\sinh 2\pi(\lambda-1/4)^{1/2}}{\lambda-1/4}\right)^{1/2}$$

$$\times (t+1)^{\alpha}(t-1)^{\beta}\, F\left(\alpha+\beta+\gamma+\tfrac{1}{2}, \alpha+\beta-\gamma+\tfrac{1}{2}; 1+2\beta; \frac{1-t}{2}\right),$$

then the limit in mean

$$(Vf)(\lambda) = \underset{k\to\infty}{\text{l.i.m.}} \int_1^k \psi(t, \lambda)f(t)dt$$

exists for each $f \in L_2(1, \infty)$, and determines a unitary mapping of $L_2(1, \infty)$ onto $L_2(1/4, \infty)$. The inverse of V is given by the formula

$$(V^{-1}f)(t) = \underset{\mu\to\infty}{\text{l.i.m.}} \int_{\frac{1}{4}}^{\mu} \psi(t, \lambda)f(\lambda)d\lambda.$$

If $F(L_1)$ is any Borel function of the self adjoint operator L_1, then $VF(L_1)V^{-1}$ is the operation of multiplication by $F(\cdot)$ in the space $L_2(1/4, \infty)$.

Rather than continuing to give a study of the formal differential operator L in Case 2, we shall pass to the study of another equation. In the Euler integral [4], put $z = \zeta/\beta$ and let $\beta \to \infty$, obtaining in the limit the entire function

$$\text{[5]} \qquad \Phi(\alpha, \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^1 t^{\alpha-1}(1-t)^{\gamma-\alpha-1}e^{tz}\,dt,$$

$$\mathscr{R}\alpha > 0, \qquad \mathscr{R}(\gamma-\alpha) > 0;$$

the so-called *confluent hypergeometric function*. This same process applied to the series [2] yields

[6] $$\Phi(\alpha, \gamma; z) = 1 + \frac{\alpha}{\gamma} z + \frac{\alpha(\alpha+1)}{2!\gamma(\gamma+1)} z^2 + \ldots;$$

so that $\Phi(\alpha, \gamma; z)$ is analytic in all three of its arguments for γ not a negative integer. The same process applied to the hypergeometric equation [1] shows that $\Phi(\alpha, \gamma; z)$ satisfies the *confluent hypergeometric equation*

[7] $$z\left(\frac{d^2}{dz^2}\right)\Phi + (\gamma - z)\frac{d}{dz}\Phi - \alpha\Phi = 0.$$

This equation has singularities at zero and infinity. The singularity at zero is regular, with exponents 0 and $1-\gamma$. The singularity at infinity is irregular, of type 2. To deal with the confluent hypergeometric equation we must consequently use some of the theory of equations with irregular singular points. It is most convenient to take the equation $Lf = (d/dz)^2 f + p(z)(d/dz)f + q(z)f = 0$ to have an irregular singularity of order (or type) $k > 1$ at infinity; the resulting theory can then be carried over to irregular singularities at a finite point z_0 by the substitution $z = z_0 + 1/w$. Let $p(z) = z^{k-2}P(z)$, $q(z) = z^{2(k-2)}Q(z)$, so that the quantities $P(z)$, $Q(z)$ are regular at infinity. Then the equation $((k-1)\alpha)^2 + P(\infty)(k-1)\alpha + Q(\infty) = 0$ is called the *characteristic equation* of $D^2 f + p(z)Df + q(z)f = 0$ at infinity; the roots $\zeta_1^{(1)}, \zeta_2^{(1)}$ of the characteristic equation are called the *first characteristics*. We assume in all that follows that $\zeta_1^{(1)} \neq \zeta_2^{(1)}$. The lines through the origin along which the quantity $(\zeta_1^{(1)} - \zeta_2^{(1)})z^{k-1}$ is pure imaginary are called the *Stokes lines* of the differential equation. In any closed angle not containing more than one of the Stokes lines, the differential equation has a solution whose behavior at infinity is expressed by an asymptotic series

$$(\exp(\zeta_1^{(1)}z^{k-1} + \zeta_1^{(2)}z^{k-2} + \ldots + \zeta_1^{(k-1)}z))z^{-\sigma_1}\left\{1 + \frac{\alpha}{z} + \frac{\beta}{z^2} + \ldots\right\}$$

and a second solution whose behavior at infinity is expressed by an asymptotic series

$$(\exp(\zeta_2^{(1)}z^{k-1} + \zeta_2^{(2)}z^{k-2} + \ldots + \zeta_2^{(k-1)}z))z^{-\sigma_2}\left\{1 + \frac{\alpha'}{z} + \frac{\beta'}{z^2} + \ldots\right\}.$$

These asymptotic relations hold as $|z| \to \infty$ uniformly in any closed angle not containing more than one of the Stokes lines. They can consequently be differentiated arbitrarily often. The quantities $\zeta_1^{(2)}, \zeta_1^{(3)}, \ldots, \zeta_1^{(k-1)}, e_1$ are called the *second, third,* etc., *characteristics,* belonging to the first characteristic $\zeta_1^{(1)}$, and the *exponent* belonging to the first characteristic $\zeta_1^{(1)}$, respectively. The quantities $\zeta_2^{(2)}, \zeta_2^{(3)}, \ldots, e_2$ are denoted similarly. The ordered sets $[\zeta_i^{(1)}, \zeta_i^{(2)}, \ldots, \zeta_i^{(k-1)}, e_i]$, $i = 1, 2$, are called the *characteristic sets* for the irregular singularity at infinity of the differential equation. The characteristic sets and the coefficients in the corresponding asymptotic series are uniquely determined by the differential equation, and can be found simply by substituting the asymptotic series into the differential equation, and solving the resulting sequence of algebraic equations for the coefficients. The first of these algebraic equations, which is simply the characteristic equation of the differential equation, is quadratic; all the succeeding equations are linear. If we find the differential equation $L'f' = 0$ satisfied by

$$f' = (\exp(\zeta^{(1)}z^{k-1} + \ldots + \zeta^{(k-1)}z))z^{-e}f,$$

f being a solution of the original differential equation $Lf = 0$, we find that $L'f'$ has rational coefficients, and an irregular singularity at infinity with characteristic sets $[\zeta_i^{(1)} + \zeta^{(1)}, \ldots, \zeta_i^{(k-1)} + \zeta^{(k-1)}, e_i + e]$, $i = 1, 2$. If $z(w)$ is a rational function of w of the form

$$aw^s\left(1 + \frac{c}{w} + \frac{d}{w^2} + \ldots\right), \qquad s \geq 1,$$

in the neighborhood of infinity, then the differential equation $L''f'' = 0$ satisfied by $f''(w) = f(z(w))$ also has rational coefficients, and has an irregular singularity of order ks at infinity, whose characteristics may easily be determined by substituting $z = z(w)$ in the asymptotic series for f. In particular, if $z = aw$, then $ks = k$ and the characteristics of L'' are

$$[a^{k-1}\zeta_i^{(1)}, \ldots, a\zeta_i^{(k-1)}, e_i], \qquad i = 1, 2.$$

It should be noted that in any closed angle not including any of the Stokes lines of our differential equation, $\mathscr{R}(\zeta_1^{(1)}z^{k-1})$ and $\mathscr{R}(\zeta_2^{(1)}z^{k-1})$ will stand in a fixed inequality to each other: say $\mathscr{R}(\zeta_1^{(1)}z^{k-1}) <$

$\mathscr{R}(\zeta_2^{(1)}z^{k-1})$. Then in this angle any solution of $Lf = 0$ whose asymptotic expansion begins with the factor $\exp(\zeta_1^{(1)}z^{k-1})$ is exponentially small (as $|z| \to \infty$ in the angle) relative to any solution whose asymptotic expansion begins with the factor $\exp(\zeta_2^{(1)}z^{k-1})$. Thus, a solution ("small solution") with the first kind of asymptotic expansion is uniquely determined by its asymptotic expansion; while a solution ("large solution") with the second kind of asymptotic expansion is not, since we may add any small solution to it without affecting the asymptotic expansion.

The confluent hypergeometric equation has the characteristic equation $\alpha^2-\alpha = 0$, so that $\zeta_1^{(1)} = 0$, $\zeta_2^{(1)} = 1$. Thus the Stokes lines for this equation are the positive and negative imaginary axes. Trying solutions of the form $z^{-e_1}(1+c/z+\dots)$ and $e^z z^{-e_2}(1+c'/z+\dots)$ we find $e_1 = a$, $e_2 = \gamma-\alpha$. Let an arbitrary differential equation $Lf = 0$ with one regular and one irregular singularity be given; let the characteristic roots of $Lf = 0$ at its irregular singularity be distinct. We will show that specification of the regular singularity and its exponents, and the irregular singularity and its characteristics and exponents, determines the equation uniquely. First note that, making an appropriate fractional linear transformation of the independent variable, we may suppose without loss of generality that the regular singularity is at zero and the irregular singularity at infinity. Then, multiplying the dependent variable by a factor $e^{\lambda z}z^k$, we may suppose one of the exponents at the regular singularity and one of the first characteristics at the irregular singularity is zero. Then if our equation is

$$D^2f+\left(\beta+\frac{\gamma}{t}\right)Df+\left(\delta-\frac{\alpha}{t}+\frac{\varepsilon}{t^2}\right)f = 0,$$

we must have $\delta = \varepsilon = 0$. Making the change $z = kw$ of independent variable, we may suppose that the first characteristics at infinity are zero and 1; so that $\beta = -1$, and our equation is the confluent hypergeometric equation $xD^2f+(\gamma-x)Df-\alpha f = 0$, whose characteristics are $(0, \alpha)$, $(1, \gamma-\alpha)$. This proves our assertion. It follows, moreover, that the sum of the four exponents of an equation with one regular and one irregular singularity, and with distinct first characteristics at the irregular singularity, is one; but that the exponents and characteristics are otherwise arbitrary.

The fact that there is just one second order linear differential equation with regular singularity a having exponents α_1, α_2 and irregular singularity b of type 2 having characteristic sets $[\zeta_1, e_1]$, $[\zeta_2, e_2]$ enables us to introduce the symbolic notation

$$\Phi\begin{pmatrix} a & b & \\ \alpha_1 & [\zeta_1, e_1]; & z \\ \alpha_2 & [\zeta_2, e_2] & \end{pmatrix}$$

to denote an unspecified branch of the (in general multivalued) function which satisfies the differential equation in question. If we wish to indicate a specific branch of this equation, we may write

$$\Phi^{a}_{\alpha_1}\begin{pmatrix} a & b & \\ \alpha_1 & [\zeta_1, e_1]; & z \\ \alpha_2 & [\zeta_2, e_2] & \end{pmatrix}$$

for the (uniquely determined) branch which has the asymptotic form $(z-a)^{\alpha_1}(1+c_1(z-a)+\ldots)$ at the regular singularity $z=a$ of our equation, and

$$\Phi^{b}_{\zeta_1}\begin{pmatrix} a & b & \\ \alpha_1 & [\zeta_1, e_1]; & z \\ \alpha_2 & [\zeta_2, e_2] & \end{pmatrix}$$

for that branch which has the asymptotic form

$$\exp[\zeta_1(z-b)^{-1}](z-b)^{e_1}(1+\hat{c}_1(z-b)+\ldots)$$

in some previously specified angular section of the neighborhood of $z=b$ not containing more than one Stokes line. If the angular sector contains at least one Stokes line, this branch is uniquely determined. If the angular sector contains no Stokes line, then the real parts of $\zeta_1(z-b)^{-1}$ and $\zeta_2(z-b)^{-1}$ will stand in a fixed relationship of inequality throughout the sector; in this case, only the branch corresponding to the asymptotic expansion containing that one of these two expressions with the smaller real part is uniquely determined.

It is nearly evident from the symbolic notation and the principles governing changes of variable explained in the preceding paragraphs that

$$\Phi\begin{pmatrix} & 0 & \infty & \\ & 0 & [0, \gamma] & ; z \\ 1 & \alpha & [1, \alpha-\gamma] & \end{pmatrix} \quad e^z\Phi\begin{pmatrix} & 0 & \infty & \\ & 0 & [-1, \gamma] & ; z \\ 1 & \alpha & [0, \alpha-\gamma] & \end{pmatrix}$$

$$- e^z\Phi\begin{pmatrix} & 0 & \infty & \\ & 0 & [0, \alpha-\gamma]: & z \\ 1 & \alpha & [1, \gamma] & \end{pmatrix},$$

etc. Such changes of variable will be used frequently (although implicitly) in what follows; in each case they will follow readily if only the calculations are made in the symbolic notation that has just been explained.

If $\mathscr{R}\alpha > 0$, $\mathscr{R}(\gamma-\alpha) > 0$, so that the integral formula [5] is valid, we may make direct asymptotic evaluations of $\Phi(\alpha, \gamma; z)$ as follows. If $|z| \to \infty$, $(-z)$ remaining in a closed angle $|\arg(-z)| \leq \pi/2-\varepsilon$ contained in the open right half-plane, the integral [5] differs by an exponentially small term, i.e., by $\int_{1/2}^{1} t^{\alpha-1}(1-t)^{\gamma-\alpha-1}e^{tz}\,dt$, which is dominated by

$$e^{1/2\mathscr{R}z}\int_{1/2}^{1} |t^{\alpha-1}(1-t)^{\gamma-\alpha-1}|\,dt = O(\exp(\tfrac{1}{2}|z| \sin \varepsilon)),$$

from the integral

$$\frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\alpha)}\int_0^{1/2} t^{\alpha-1}(1-t)^{\gamma-\alpha-1} e^{-t(-z)}\,dt$$
$$= \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\alpha)}\int_0^{1/2} t^{\alpha-1}e^{-t(-z)}\,dt + A_1,$$

where $A_1 = O(\int_0^\infty t^\alpha e^{-t|z| \sin \varepsilon}\,dt) = O((|z| \sin \varepsilon)^{-\alpha-1}) = O(|z|^{-\alpha-1})$.

Similarly, the integral on the right of the equation last displayed above differs by an exponentially small term from

$$\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)}\int_0^\infty t^{\alpha-1} e^{-t(-z)}\,dt = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)}(-z)^{-\alpha};$$

thus

$$[8] \quad \Phi(\alpha, \gamma; z) \sim \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)}(-z)^{-\alpha}, \qquad |z| \to \infty, \qquad |\arg(-z)| \leq \frac{\pi}{2}-\varepsilon,$$

in any closed angle of the open left half-plane, provided $\mathscr{R}\alpha > 0$ and $\mathscr{R}(\gamma-\alpha) > 0$. We wish to remove these restrictions. To do this, note that it follows immediately from [5] that

$$[9] \qquad \frac{d}{dz}\Phi(\alpha, \gamma; z) = \frac{\alpha}{\gamma}\Phi(\alpha+1, \gamma+1; z).$$

Thus, if $\Phi(\alpha, \gamma; z)$ were exponentially small in the left half-plane for any values of α, γ; $\Phi(\alpha+n, \gamma+n; z)$ would also be exponentially small in the left half-plane; and, if $\gamma-\alpha > 0$, we know from our asymptotic formula [8] that this is impossible. Thus we must have

$$\Phi(\alpha, \gamma; z) \sim \frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)} k(\alpha, \gamma)(-z)^{-\alpha}$$

in the left half-plane for $\mathscr{R}(\gamma-\alpha) > 0$; where $k(\alpha, \gamma) = 1$ for $\mathscr{R}\alpha > 0$, but must still be determined for $\alpha \leq 0$. Using [9] we get

$$\frac{\alpha}{\gamma}\Phi(\alpha+1, \gamma+1; z) \sim \alpha\frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)} k(\alpha, \gamma)(-z)^{-\alpha-1};$$

thus $k(\alpha+1, \gamma+1) = k(\alpha, \gamma)$; so that $k(\alpha, \gamma) = 1$, and the asymptotic formula [8] is valid for $\mathscr{R}(\alpha-\gamma) > 0$. The function $e^z\Phi(\gamma-\alpha, \gamma; -z)$ is regular at the origin, and is a solution of the equation with exponents $0, 1-\gamma$ at zero, and characteristic sets $(1, \gamma-\alpha), (0, \alpha)$ at infinity; i.e., of the confluent hypergeometric equation. Thus

$$[10] \qquad \Phi(\alpha, \gamma; z) = e^z\Phi(\gamma-\alpha, \gamma; -z).$$

Using [10] and [8] together we find

$$[11] \qquad \Phi(\alpha, \gamma; z) \sim \frac{\Gamma(\gamma)}{\Gamma(\alpha)} e^z z^{\alpha-\gamma}, \qquad |z| \to \infty, |\arg z| \leq \frac{\pi}{2} - \varepsilon,$$

provided that $\mathscr{R}\alpha > 0$. Then, using [9] just as above, we find that the condition $\mathscr{R}\alpha > 0$ is superfluous, so that [11], and hence [8], is valid for all α, γ; provided only that γ is not a negative integer.

Let us now take the formally symmetric formal differential operator

$$B = -\left(\frac{d}{dt}\right)^2 + \frac{k^2-1/4}{t^2},$$

and subject it to a spectral analysis.

The equation $(B-\lambda)f = 0$ has a regular singularity at $t = 0$ with exponents $1/2 \pm k$, and an irregular singularity at ∞ with characteristics $[\pm i\sqrt{\lambda}, 0]$. We have, consequently, two boundary values at zero if $\mathscr{R}k < 1$, and no boundary values at zero if $\mathscr{R}k \geq 1$. For simplicity, we shall confine ourselves entirely to the latter case. For the sake of simplicity we shall also assume that $2k$ is not an integer, so that $(B-\lambda)f = 0$ has solutions $S_\pm$ of the form $t^{1/2\pm k}(1+\ldots)$ at $t \to 0$. By the transformations of dependent and independent variables outlined above we have

$$S_+(t, \lambda) = t^{1/2+k}e^{-i\mu t}\Phi(\tfrac{1}{2}+k, 1+2k; 2i\mu t),$$
$$S_-(t, \lambda) = t^{1/2-k}e^{-i\mu t}\Phi(\tfrac{1}{2}-k, 1-2k; 2i\mu t),$$

where $\mu = \lambda^{1/2}$. Let $\mathscr{I}\lambda > 0$; then, by the asymptotic formula [8],

$$S_+(t, \lambda) \sim \frac{\Gamma(1+2k)}{\Gamma(\frac{1}{2}+k)} e^{-i\mu t}(-2i\mu)^{-1/2-k},$$

$$S_-(t, \lambda) \sim \frac{\Gamma(1-2k)}{\Gamma(\frac{1}{2}-k)} e^{-i\mu t}(-2i\mu)^{-1/2+k},$$

where $-2i\mu$ is in the right half-plane and we take the principal value of its logarithm. Thus, not all solutions of $(B-\lambda)f = 0$ are square-integrable at infinity, so that by Lemma XII.4.21, Corollary 2.23, and Theorem XII.4.18, B has no boundary values at infinity. Thus $T_0(B)$ has one single self adjoint extension, which is simultaneously its closure and its adjoint: $T_1(B)$. For simplicity in notation, we shall write $T_1(B)$ simply as B where no confusion can arise. For f in $\mathfrak{D}(T_0(B))$, we have

$$(Bf, f) = \int_0^\infty \left\{ |f'(t)|^2 + \frac{k^2 - 1/4}{t^2}|f(t)|^2 \right\} dt \geq 0;$$

since $T_1(B)$ is the closure of $T_0(B)$, we have $(Bf, f) \geq 0$ for all f in $\mathfrak{D}(T_1(B))$. Thus $T_1(B)$ is positive, so that by Lemma XII.7.2, $\sigma(B)(= \sigma(T_1(B)))$ lies entirely on the non-negative portion of the real axis. If $\lambda = 0$, the solutions of the equation $(B-\lambda)f = 0$ are $t^{1/2\pm k}$; so that no solution is in $L_2(0, \infty)$, and $\lambda = 0$ is not in the point spectrum. Hence, the spectral measure $E((0))$ of the single point 0 is zero. Thus, to give a complete spectral analysis of B we need only

consider that part of the spectrum lying on the set $\lambda > 0$. From the asymptotic expansions for S_+ and S_- given in the second display formula above, it follows that the solution

$$\begin{aligned}\Sigma(t, \lambda) &= \frac{\Gamma(1/2+k)}{\Gamma(1+2k)}(-2i\mu)^{1/2+k}S_+(t, \lambda)\\ &\quad -\frac{\Gamma(1/2-k)}{\Gamma(1-2k)}(-2i\mu)^{1/2-k}S_-(t, \lambda)\\ &= c_+(\lambda)S_+(t, \lambda)+c_-(\lambda)S_-(t, \lambda)\end{aligned}$$

of the equation $(B-\lambda)\Sigma = 0$ is $O(e^{-i\mu t})$ as $t \to \infty$. Since the characteristics of $(B-\lambda)f = 0$ at infinity are $[\pm i\mu, 0]$, so that every solution of this equation is either asymptotic to a constant multiple of $e^{i\mu t}$ or to a constant multiple of $e^{-i\mu t}$ (if μ is not real), we must have $\Sigma(t, \lambda) = O(e^{i\mu t})$ as $t \to \infty$; thus $\Sigma(t, \lambda)$ is square-integrable at infinity. Thus the Green's kernel $K_\lambda(t, \lambda)$ is given for $\mathscr{I}\lambda > 0$ and $t < s$ by

$$K_\lambda(t, s) = \frac{S_+(t, \lambda)\overline{\Sigma(s, \bar{\lambda})}}{W(S_+, \Sigma)} = \frac{S_+(t, \lambda)\Sigma(s, \lambda)}{W(S_+, \Sigma)},$$

according to Theorem 8.16. Now, $W(S_+(t, \lambda), S_-(t, \lambda))$ is constant; and since

$$W(S_+(t, \lambda), S_-(t, \lambda)) = (\tfrac{1}{2}+k)t^{k-1/2}t^{1/2-k} - (\tfrac{1}{2}-k)t^{k+1/2}t^{-k-1/2} \sim 2k$$

at $t = 0$, we have $W(S_+, S_-) = 2k$. Thus, by Theorem 5.27, we have $\rho_{+-} = \rho_{-+} = \rho_{--} = 0$,

$$\begin{aligned}\rho_{++}(\Delta) &= \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_\Delta \mathscr{I}\left\{\frac{c_+(\lambda+i\varepsilon)}{c_-(\lambda+i\varepsilon)W(S_+(t, \lambda+i\varepsilon), S_-(t, \lambda+i\varepsilon))}\right\}d\lambda\\ &= \lim_{\varepsilon\to 0}\frac{1}{2\pi k}\frac{\Gamma(1/2+k)\Gamma(1-2k)}{\Gamma(1/2-k)\Gamma(1+2k)}\int_\Delta \mathscr{I}\{-2i\mu(\lambda+i\varepsilon)\}^{2k}\,d\lambda\\ &= \frac{\sin k\pi}{2\pi k}\frac{\Gamma(1/2+k)\Gamma(1-2k)}{\Gamma(1/2-k)\Gamma(1+2k)}\int_\Delta (2\mu)^{2k}\,d\lambda.\end{aligned}$$

Using the formulae $\Gamma(x)\Gamma(1-x) = \pi/\sin \pi x$, and

$$\Gamma(2z) = \frac{1}{(2\pi)^{1/2}}2^{2z-1/2}\Gamma(z)\Gamma(z+1/2),$$

we have

$$\rho_{++}(\Delta) = \frac{1}{(2^{k+1/2}\Gamma(k+1))^2}\int_\Delta \lambda^k\,d\lambda.$$

It is interesting to rephrase this result somewhat. If we put $\mathscr{J}_k(\mu t) = \mathscr{J}_k(t, \mu) = \mu^{k+1/2}/2^k\Gamma(k+1)\ S_+(t, \lambda)$, then Theorems 5.23 and 5.24 show that

$$(H_k f)(\mu) = \underset{t\to\infty}{\text{l.i.m.}} \int_0^\infty \mathscr{J}_k(t, \mu) f(t)\,dt$$

determines an isometric isomorphism of $L_2(0, \infty)$ onto itself whose inverse is also H_k. If we let $t^{-1/2}\mathscr{J}_k(t) = J_k(t)$ to bring our results into conformity with those of Theorem XI.8.34, we obtain the following result.

THEOREM. *Let f be in $L_2(0, \infty)$ with respect to Lebesgue measure. Then the limit*

$$g(s) = \underset{R\to\infty}{\text{l.i.m.}} \int_0^R (rs)^{1/2} J_k(rs) f(r)\,dr$$

exists in the norm of the space $L_2(0, \infty)$, and the (Hankel) transform $H_k: f \to g$ is a unitary mapping of $L_2(0, \infty)$ into itself whose inverse map is also H_k.

We learn from Theorems 5.23 and 5.24 that $H_k B H_k^{-1}$ is the operation of multiplication by μ in $L_2(0, \infty)$, which makes it evident that B has no point spectrum, but has a continuous spectrum covering the interval $[0, \infty)$.

The function J_k is (by the change of dependent variable outlined above) the unique solution of the *Bessel equation*

$$B_k(J) = t^2\left(\frac{d}{dt}\right)^2 J + \left(t\left(\frac{d}{dt}\right) + (t^2 - k^2)\right) J = 0$$

which is of the form $(t/2)^k\{\Gamma(k+1)\}^{-1}(1 + \ldots)$ near the regular singularity at zero. Applying the differential operator B_k to the function $(1/2\pi)\int_0^{2\pi} e^{i(n\theta - z\sin\theta)}d\theta$ of Theorem XI.8.23, we get

$$\frac{1}{2\pi}\int_0^{2\pi} e^{i(n\theta - z\sin\theta)}(z^2\cos^2\theta \quad iz\sin\theta \quad n^2)d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{d}{d\theta}\{i(n+z\cos\theta)e^{i(n\theta - z\sin\theta)}\}\, d\theta = 0.$$

Expanding $(1/2\pi)\int_0^{2\pi} e^{i(n\theta - z\sin\theta)}d\theta$ in a power series, we find its lowest non-vanishing term to be

$$\frac{1}{2\pi n!}(-z)^n \int_0^{2\pi}\left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^n e^{in\theta}\, d\theta \quad \frac{1}{n!}(z/2)^n;$$

so that

$$\frac{1}{2\pi}\int_0^{2\pi} e^{i(n\theta - z\sin\theta)}\, d\theta = J_n(z),$$

and the above theorem is a direct generalization of Theorem XI.8.23 to non-integral values of the parameter k.

As a final example we shall give an analysis of the formal differential operator $H = -d/dt^2 + t^2$ on the interval $(\quad\infty, +\infty)$. Since $H \quad \lambda$ is invariant under the substitution $z \to \quad z$, we are tempted to write its solutions $f(z)$ as $g(z^2)$; doing this we find that

$$-4z^2g''(z^2) \quad 2g'(z^2) + (z^2 \quad \lambda)g(z^2) = 0$$

so $g(w)$ satisifes the confluent hypergeometric equation

$$4wg'' - 2g' + (w-\lambda)g \quad 0,$$

with exponents 0, 1/2 at zero and characteristics $\pm 1/2$, $1/4(1+\lambda)$. Thus, $(H-\lambda)f = 0$ has a solution asymptotic to $e^{t^2/2}t^{-(1/2)(\lambda+1)}$ as $t \to \infty$, so not all solutions of $(H-\lambda)f \quad 0$ are square-integrable in the neighborhood of $t = \infty$, and there are no boundary values at infinity. By the symmetry $t \to -t$ there are no boundary values at $-\infty$; so that the sole self adjoint extension of $T_0(H)$ is its closure, which we denote by H. Since

$$(Hf, f) = \int_{-\infty}^{+\infty} \{|f'(t)|^2 + t^2|f(t)|^2\}dt \geq 0, \qquad f \in \mathfrak{D}(T_0(H)),$$

it follows (as above) that H is non-negative, so that $\sigma(H)$ lies entirely on the positive real axis; by Theorem 7.16(a), $\sigma(H)$ consists of a

sequence of discrete eigenvalues approaching infinity. By the transformations of dependent variables outlined above, we see that $(H-\lambda)f = 0$ has solutions

$$S_e(t,\lambda) = e^{-t^2/2}\,\Phi\left(\frac{1}{4}-\frac{\lambda}{4},\frac{1}{2};t^2\right)$$

$$S_o(t,\lambda) = e^{-t^2/2}\,t\,\Phi\left(\frac{3}{4}-\frac{\lambda}{4},\frac{3}{2};t^2\right);$$

the first being even, the second odd. Since

$$S_e(t,\lambda) \sim \frac{\Gamma(1/2)}{\Gamma(1/4-\lambda/4)}\,e^{t^2/2}\,t^{-(1/2)(1+\lambda)}, \qquad \lambda \neq 1+4n,$$

as $t \to \pm\infty$ and

$$S_o(t,\lambda) \sim \frac{\Gamma(3/2)}{\Gamma(3/4-\lambda/4)}\,e^{t^2/2}\,t^{-(1/2)(3+\lambda)}, \qquad \lambda \neq -1+4n,$$

as $t \to \pm\infty$ by the asymptotic formula [11], the only possible eigenvalues are $\lambda = 4n+1$, $n \geq 0$ and $\lambda = 4n-1$, $n \geq 1$; and since $\Phi(-k, a; z)$ is a polynomial if k is an integer, these values are eigenvalues, with corresponding orthonormal eigenfunctions

$$\begin{aligned} &c_n e^{-t^2/2}\Phi(-n,\tfrac{1}{2},t^2), && \lambda = 4n+1, && n \geq 0,\\ &c_n e^{-t^2/2}\Phi(1-n,\tfrac{3}{2},t^2), && \lambda = 4n-1, && n \geq 1, \end{aligned}$$

To determine the normalization constants c_n, note that

$$S_+ = \frac{\Gamma(1/4-\lambda/4)S_e}{\Gamma(1/2)} - \frac{\Gamma(3/4-\lambda/4)S_o}{\Gamma(3/2)}$$

is $O(e^{-t^2/2})$, for $t \to \infty$, and

$$S_- = \frac{\Gamma(1/4-\lambda/4)S_e}{\Gamma(1/2)} + \frac{\Gamma(3/4-\lambda/4)S_o}{\Gamma(3/2)}$$

is $O(e^{-t^2/2})$ for $t \to -\infty$. Thus, since $W(S_o, S_e) = 1$, the Green's kernel K_λ is given for $t < s$ by the formula

$$K(t,s) = \frac{S_-(t)S_+(s)}{W(S_-,S_+)} = \frac{\Gamma(1/4-\lambda/4)}{\Gamma(-1/4-\lambda/4)} S_e(t)S_e(s)$$
$$-\frac{\Gamma(-1/4-\lambda/4)}{4\Gamma(1/4-\lambda/4)} S_o(t)S_o(s) - \tfrac{1}{2}S_e(t)S_o(s) + \tfrac{1}{2}S_o(t)S_e(s).$$

Using the remarks following Theorem 5.16 we have, consequently, for the normalization constants c_n and c_n' the formulas

$$|c_n|^2 = \frac{4(-1)^n}{n!\,\Gamma(-n-1/2)} = \frac{4\Gamma(n+3/2)}{\pi n!},$$
$$|c_n'|^2 = \frac{(-1)^n}{n!\,\Gamma(1/2-n)} = \frac{\Gamma(n+1/2)}{\pi n!}.$$

9. Exercises

In the present section, we give a number of connected sets of exercises dealing with various aspects of the spectral theory of ordinary differential operators. The present section is divided into ten subsections; for the convenience of the reader, we list here the topics dealt with in the problems of each of these subsections: (A) General; (B) Nonselfadjoint Operators; (C) Semibounded Operators; (D) Minimax Principles; (E) Differential Operators in the Spaces $L_p(I)$; (F) The Sturm-Liouville Operator, I; (G) The Sturm-Liouville Operator, II; (H) The Sturm-Liouville Operator with Integrable Coefficient; (I) Spectral Analysis of Special Differential Operators; (J) Miscellaneous.

A. *General*

A1 Let τ be a regular formal differential operator on an interval I. Prove that $T_1(\tau)$ is a closed operator.

A2 Prove that every extension of the operator $T_0(\tau)$ obtained by the imposition of boundary conditions is a closed operator.

A3 Let τ be a regular formal differential operator on an interval I. Prove that the point 0 in the complex plane fails to belong to the essential spectrum of τ if and only if every closed extension of $T_0(\tau)$ maps bounded closed sets into closed sets.

A4 Let τ be a regular differential operator on the interval $[0, \infty)$. Prove that a complex number λ belongs to the essential spectrum of τ if and only if there exists a sequence $\{f_n\}$ of functions in $\mathfrak{D}(T_0(\tau))$ such that $|f_n| = 1$, f_n vanishes in the interval $[0, n]$, and

$$|(\lambda - \tau)f_n| \to 0$$

as $n \to \infty$.

A5 Let τ be a regular formal differential operator on an interval I. Suppose that for every finite set $f_1, f_2, \ldots, f_n$ in $\mathfrak{D}(T_0(\tau))$ and for every $\varepsilon > 0$ we can find a function g in $\mathfrak{D}(T_1(\tau))$ which is orthogonal to the functions f_i and such that

$$|(\lambda - \tau)g| < \varepsilon|g|.$$

Then the point λ belongs to the essential spectrum of τ.

A6 Let τ be a regular formally symmetric formal differential operator on $[0, \infty)$ with equal deficiency indices, and let λ be a real number. Prove that the distance from λ to the essential spectrum of τ is less than or equal to K if and only if there exists a sequence f_n in $\mathfrak{D}(T_0(\tau))$ such that $|f_n| = 1$, f_n vanishes on the interval $[0, n]$, and

$$|(\lambda - \tau)f_n| \leqq K.$$

A7 Let τ be a formally symmetric formal differential operator on $[0, \infty)$ with equal deficiency indices. Prove that the essential spectrum of τ is void if and only if for every sequence $\{f_n\}$ in $\mathfrak{D}(T_0(\tau))$ such that $|f_n| = 1$ and such that f_n vanishes in the interval $[0, n]$ we have

$$|\tau f_n| \to \infty$$

as $n \to \infty$.

A8 Let τ_1 and τ_2 be formal differential operators on an interval I, all of whose coefficients are identical except on a compact subinterval of I. Prove that the essential spectra of τ_1 and τ_2 coincide.

A9 Let τ be a formal differential operator on the interval $[0, \infty)$, and let C_n be the smallest closed convex set containing all the values

$$\{(\tau f, f) | f \in \mathfrak{D}(T_0(\tau)),\ f(t) = 0,\ 0 \leqq t \leqq n\}.$$

Prove that the essential spectrum of τ is contained in the set

$$\bigcap_{n=1}^{\infty} C_n.$$

A10 Let τ be a regular formal differential operator on an interval I, and let B be a compact operator in $L_2(I)$. Prove that the essential spectrum of τ coincides with the essential spectrum of the operator $T_1(\tau)+B$.

A11 Let τ be a regular formal differential operator on an interval I, and let B be a linear operator in $L_2(I)$ defined in $\mathfrak{D}(T_1(\tau))$ which is a compact operator from $\mathfrak{D}(T_1(\tau))$ to $L_2(I)$. Prove that the essential spectrum of τ coincides with the essential spectrum of $T_1(\tau)+B$.

B. *Nonselfadjoint Operators*

B1 Given the formal differential operator

$$\tau = -(d/dt)p(t)(d/dt)+q(t)$$

on the interval $[0, \infty)$, where $\mathscr{R}p(t) > 0$ and $\mathscr{R}q(t)$ is bounded below, show that τ has no boundary values at infinity.

B2 Let τ be a formal differential operator on I whose essential spectrum lies in the strip $|\mathscr{I}(\lambda)| \leq K$, and let q be a bounded measurable function on I. Show that every boundary value for τ is a boundary value for $\tau+q$ and conversely.

B3 Let τ be a formal differential operator of the form

$$\tau = \sum_{k=0}^{n} a_k(t)(d/dt)^k$$

on the interval $[a, \infty)$, and suppose that the essential spectrum of τ lies in the strip $|\mathscr{I}(\lambda)| \leq K$. Assume that the function $|a_n(\cdot)|$ is bounded away from zero, and that all functions $|a_k(\cdot)|$ are bounded. Show that the operator τ has no boundary values at infinity.

B4 Let τ be a formal differential operator on an interval I, and suppose that for all functions f in $\mathfrak{D}(T_0(\tau))$,

$$\mathscr{R}(\tau f, f) \geq 0.$$

Show that the essential spectrum of τ lies in the half-plane $\{\lambda | \mathscr{R}\lambda > 0\}$.

B5 Given the Sturm-Liouville operator

$$\tau = -(d/dt)p(t)(d/dt) + q(t)$$

on the interval $[0, \infty)$ (p positive, q real), suppose that

$$\limsup_{t\to\infty} |(p(t)q'(t))'|q(t)^2 < 1.$$

(a) Show that if the function f lies in $\mathfrak{D}(T_1(\tau))$, then the function qf is square-integrable

(b) Let r be a function such that $r(t) = o(q(t))$ as $t \to \infty$. Show that the essential spectrum of the operator τ coincides with the essential spectrum of the operator $\tau + r$.

B6 Given the differential operator

$$\tau = -(d/dt)^2 + q(t)$$

on the interval $[0, \infty)$, where q is not necessarily a real function, let

$$\tau_1 = a(t)(d/dt)^2 + b(t)(d/dt) + c(t),$$

where $c(t) = o(|q(t)|)$, $a(t) = o(1)$, $b(t) = o(1)$ as $t \to \infty$. Show that the essential spectrum of τ coincides with the essential spectrum of $\tau + \tau_1$.

B7 Let τ be a formally symmetric formal differential operator on an interval I, and let B be a bounded operator in $L_2(I)$ of norm at most K. Suppose that a point λ in the complex plane is at a distance greater than K from the essential spectrum of the operator τ. Show that λ does not belong to the essential spectrum of the operator $T_1(\tau) + B$.

B8 Let τ be a differential operator of the form

$$\tau = -(d/dt)p(t)(d/dt) + q(t)$$

on an interval $[a, b)$, where p is positive, and

$$\int_0^\infty p(t)^{-1}\, dt = \infty.$$

Let the function Q be defined as in Theorem 7.66. Suppose that $\mathscr{R}(Q(t)) \to \infty$ as $t \to b$. Show that the essential spectrum of the operator τ is void.

B9 Under the hypotheses and with the notation of the preceding exercise, suppose that $\mathscr{R}Q(t)$ is bounded below. Show that the operator τ has no boundary values at the end point b.

C. *Semibounded Operators*

C1 Let τ be a formally self adjoint formal differential operator of the form

$$\tau = \sum_{j=0}^{n} p_j(t)(d/dt)^j$$

on an interval $[a, \infty)$. Suppose that $\lim_{t\to\infty} p_j(t) = q_j$ exists for $j = 0, 1, \ldots, n$, and that $(-1)^{n/2}q_n \geq 0$. Show that the operator τ is bounded below.

C2 Suppose that the operator τ on an interval $[a, b)$ is bounded below, and that there exists a constant M such that

$$|f^{(k)}|^2 < M((\tau f, f) + |f|^2), \qquad f \in \mathfrak{D}(T_0(\tau)).$$

Let τ_1 be a (regular or irregular) formal differential operator of the form

$$\tau_1 = \sum_{j=0}^{k} a_j(t)(d/dt)^j$$

where $\lim_{t\to b} a_j(t) = 0$, $0 \leq j \leq k$. Then the operator $\tau + \tau_1$ is bounded below if it has a non-vanishing leading coefficient.

C3 Let the formal differential operator τ on the interval $[a, b)$ be bounded below, and satisfy the hypothesis of the preceding exercise and let τ_1 be a (regular or irregular) formal differential operator of the form

$$\tau_1 = \sum_{j=0}^{k} \left(\frac{d}{dt}\right)^j a_j(t) \left(\frac{d}{dt}\right)^j$$

where $\lim_{j\to b} a_j(t) = 0$, $0 \leq j \leq k$. Then the operator $\tau + \tau_1$ is bounded below if it has a non-vanishing leading coefficient.

C4 Let T_1 and T_2 be closed operators in Hilbert space such that $\mathfrak{D}(T_2) = \mathfrak{D}(T_1) + \mathfrak{N}$, where $\mathfrak{N}$ is a finite-dimensional subspace, and for which

$$\mathscr{R}(T_1 x, x) \leq \mathscr{R}(T_2 x, x), \qquad x \in \mathfrak{D}(T_1).$$

Show that $\mathscr{R}(T_1x, x)$ is bounded below for all x in $\mathfrak{D}(T_1)$ if and only if $\mathscr{R}(T_2x, x)$ is bounded below for all x in $\mathfrak{D}(T_2)$.

C5 Let τ be a formal differential operator of even order in the finite closed interval $[0, 1]$, whose leading coefficient a_{2n} satisfies the inequality $(-1)^n\mathscr{R}(a_{2n}(t)) \geqq 0$. Show that there exists a constant K such that

$$\mathscr{R}(\tau f, f) \geqq K(f, f),$$

for each f such that $f \in \mathfrak{D}(T_1(\tau))$ and $f(0) = \ldots = f^{(n-1)}(0) = f(1) = \ldots = f^{(n-1)}(1) = 0$.

C6 Let τ be a formally symmetric formal differential operator of the form

$$\tau = -(d/dt)p(t)(d/dt) + q(t)$$

on an interval $[a, b)$. Suppose that

$$\int_a^b p(t)^{-1/2}\,dt = \infty,$$

and let $Q(t)$ be defined as in Theorem 7.66 and be bounded below. Show that the operator τ is bounded below.

C7 Let the operator τ and the function Q be defined as in the preceding exercise and let $[a, b) = [0, \infty)$. Suppose that Q is integrable. Show that the operator τ is bounded below.

D. *Minimax Principles*

D1 Let T be a symmetric operator which is not bounded below, and let A be a real number. Then there exists an infinite dimensional subspace $\mathfrak{H}_0$ of $\mathfrak{D}(T)$ such that

$$(Tx, x) \leq A(x, x), \qquad x \in \mathfrak{H}_0.$$

(Hint: Adapt the method of proof of Lemma 7.22).

D2 (J. Berkowitz). Let T be an (unbounded) self adjoint operator in Hilbert space $\mathfrak{H}$. Let $\mathscr{S}^{(n)}$ denote the family of all n-dimensional subspaces of $\mathfrak{H}$. If $\mathfrak{H}_0$ is a subspace of $\mathfrak{H}$, let $P_{\mathfrak{H}_0}$ denote the orthogonal projection whose range is $\mathfrak{H}_0$. Let the sequences of real numbers $\{\lambda_n^{(i)}\}$ $(1 \leqq i \leqq 4)$ be defined as follows:

$$(1)\quad \lambda_n^{(1)} = \inf_{\mathfrak{H}_0 \in \mathcal{S}^{(n)}} \ \sup_{\substack{u \in \mathfrak{H}_0 \\ u \neq 0,\, u \in \mathfrak{D}(T)}} (u, Tu)/(u, u),$$

$$(2)\quad \lambda_n^{(2)} = \sup_{\mathfrak{H}_0 \in \mathcal{S}^{(n-1)}} \ \inf_{\substack{(u, \mathfrak{H}_0) = 0 \\ u \neq 0,\, u \in \mathfrak{D}(T)}} (u, Tu)/(u, u),$$

$$(3)\quad \lambda_n^{(3)} = \sup_{f_1, \ldots, f_{n-1} \in \mathfrak{H}} \ \inf_{\substack{u \neq 0 \\ u \in \mathfrak{D}(T)}} \Big[(u, Tu) + \sum_{i=1}^{n-1} |(u, f_i)|^2\Big]/(u, u)$$

$$(4)\quad \lambda_n^{(4)} = \sup_{\substack{\mathfrak{H}_0 \in \mathcal{S}^{(n-1)} \\ a > 0}} \ \inf_{\substack{u \neq 0 \\ u \in \mathfrak{D}(T)}} ((T + aP_{\mathfrak{H}_0})u, u)/(u, u).$$

(a) Prove that $\lambda_n^{(1)} = \lambda_n^{(2)} = \lambda_n^{(3)} = \lambda_n^{(4)}$ for all n.

(b) Calling λ_n the common value, show that the sequence $\{\lambda_n\}$ is non-decreasing.

(c) If $\lambda_n \neq -\infty$ for some n, then $\lambda_n \neq -\infty$ for all n, and this is the case if and only if the operator T is bounded below.

(d) Let $\lambda_\infty = \lim_{n\to\infty} \lambda_n$. Show that the operator T is finite below λ_∞, but not below any number greater than λ_∞.

(e) If $\lambda_n < \lambda_\infty$, then $\lambda_1, \lambda_2, \ldots, \lambda_n$ is an enumeration of the n smallest eigenvalues of T, each repeated a number of times equal to its multiplicity.

(Hint: Use Exercise D1, Lemma 7.22 and the spectral theorem.)

D3 Let S be a densely defined symmetric operator in Hilbert space $\mathfrak{H}$. Assume that S is bounded below, let T be a self adjoint extension of S, and let T_1 be the Friedrichs extension (cf. XII.10) of S. Let $\lambda_n(T)$ and $\lambda_n(T_1)$ be the numbers defined in Exercise D2, for the operators T and T_1 respectively. Show that, in the notation of Exercise D2,

$$\lambda_n(T) \leq \lambda_n(T_1) = \sup_{\mathfrak{H}_0 \in \mathcal{S}^{(n-1)}} \ \inf_{\substack{u \neq 0 \\ (u, \mathfrak{H}_0) = 0,\, u \in \mathfrak{D}(S)}} (u, Su)/(u, u).$$

D4 In the notation and with the hypotheses of the preceding exercise, show that for suitable S all numbers $\lambda_n(T_1)$ may be non-negative, whereas $\lambda_n(T) = -\infty$ for some other self adjoint extension T.

For suitable S, the essential spectrum of the operator T_1 may be void, whereas the continuous spectrum of T may range over the entire real axis.

D5 Let τ be a formally self adjoint formal differential operator, and suppose that τ is bounded below. Then the Friedrichs extension T of $T_0(\tau)$ is determined by τ and by a separated set of boundary conditions.

D6 Let τ be a formally self adjoint formal differential operator on a half-open interval $[a, b)$, and let τ be bounded below. Then τ is of even order $2n$. If the function f lies in the domain of the Friedrichs extension T of $T_0(\tau)$, then $f(a) = f'(a) = \ldots, = f^{(n-1)}(a) = 0$ is an enumeration of all the boundary conditions at a in the separated set of boundary conditions determining T. (Hint: see the preceding exercise.)

D7 (Friedrichs). Let τ be a real, second order, formally self adjoint formal differential operator defined on an interval $I = (a, b)$. Suppose that the operator is of the form

$$\tau = -(d/dt)p(t)(d/dt)+q(t),$$

where $p(t) > 0$ for t in I, and $q(t)$ is bounded below by a constant q. Let us agree to say that the endpoint b (or the endpoint a) is of type A relative to τ if the integral

$$\int_a^b [p(t)|f'(t)|^2+q(t)|f(t)|^2]dt$$

converges for each function f in $\mathfrak{D}(T_1(\tau))$ which vanishes in a neighborhood of b (or of a), and that b (or a) is of type B relative to τ otherwise. Prove the following:

(a) If τ has no boundary values at b, it is of type A at b.

(b) If τ is of type A at b, then either τ has no boundary values at b, or $b < \infty$,

$$\int^b p(t)^{-1}dt < \infty$$

and the limits

$$f(b) = \lim_{t\to b} f(t), \quad p(b)f'(b) = \lim_{t\to b} p(t)f'(t)$$

exist and define linearly independent boundary values for τ, in terms of which we have

$$\int_a^b [p(t)f'(t)g'(t)+q(t)f(t)g(t)]dt \quad [p(b)f'(b)]g(b) = (\tau f, g)$$

for each pair of functions f and g in $\mathfrak{D}(T_1(\tau))$ which vanish in a neighborhood of a.

(c) The Friedrichs extension T of $T_0(\tau)$ is the restriction of $T_1(\tau)$ to the set of functions defined by the conditions

(I) $\int_a^b [p(t)f'(t)^2+q(t)f(t)^2]dt$ converges,

(II) $f(b) \quad 0$ if b is of type A and τ has boundary values at b,

(III) $f(a) = 0$ if a is of type A and τ has boundary values at a,

D8 Suppose that the hypotheses of the preceding exercise are satisfied. Let $\mathfrak{D}_0$ be the subspace of $\mathfrak{D}(T_1(\tau))$ consisting of all those functions for which the integral

$$L(f) = \int_a^b [p(t)|f'(t)|^2+q(t)|f(t)|^2]dt$$

converges.

For each of the end points a, b which is of type A and at which τ has boundary values, let a real number w be chosen.

Let $T(w)$ be the restriction of $T_1(\tau)$ determined by the conditions

(I) $\int_a^b [p(t)|f'(t)|^2+q(t)|f(t)|^2]dt = L(f)$ converges,

(II) $p(a)f'(a)+wf(a) = 0$.

Let $T(\infty)$ be the restriction of $T_1(\tau)$ determined by the condition (I) and the condition

(III) $f(a) \quad 0$.

(a) Show that $T(w)$ is self adjoint for every w in the interval $-\infty < w \leqq \infty$.

Let $\lambda_n(T(w))$ be the numbers defined for the self adjoint operator $T(w)$ as in Exercise D2. Let $\mathfrak{D}_0$ be the set of functions in $\mathfrak{H}^{(1)}(a, b)$ defined by condition (I).

(b) Show that for every function f in $\mathfrak{D}_0$, the limit

$$f(a) = \lim_{t \to a} f(t)$$

exists.

(c) Show that for $-\infty < w < \infty$,

$$\lambda_n(T(w)) = \sup_{\mathfrak{H}_0 \in \mathscr{S}^{(n-1)}} \inf_{\substack{u \neq 0 \\ u \in \mathfrak{D}_0 \\ (u, \mathfrak{H}_0) = 0}} [w|u(a)|^2 + L(u)]/(u, u).$$

(d) Show that

$$\lambda_n(T(\infty)) = \sup_{\mathfrak{H}_0 \in \mathscr{S}^{(n-1)}} \inf_{\substack{u \neq 0 \\ u(a) = 0 \\ (u, \mathfrak{H}_0) = 0 \\ u \in \mathfrak{D}_0}} L(u)/(u, u).$$

(e) Show that for any w and w_1 not equal to ∞,

$$\lim_{n \to \infty} \lambda_n(T(w)) = \lim_{n \to \infty} \lambda_n(T(w_1)).$$

(f) Show that as w ranges monotonically over the real line, $\lambda_n(T(w))$ ranges monotonically between $\lambda_{n-1}(T(\infty))$ and $\lambda_n(T(\infty))$, and that $\lambda_n(T(w)) \neq \lambda_n(T(w'))$ if $w \neq w'$ and $\lambda_{n-1}(T(\infty)) \neq \lambda_n(T(\infty))$.

(g) Show that $\lambda_{n-1}(T(\infty)) \neq \lambda_n(T(\infty))$ unless $\lambda_m(T(\infty)) = \lambda_n(T(\infty))$ for all $m \geq n$.

(Hint: Use the oscillation theory of Section 7, Exercises D7, D8, D2 and the method of proof of Theorem 6.10.)

D9 Let τ and $\hat{\tau}$ be second order differential operators of the form

$$\tau = -(d/dt)p(t)(d/dt) + q(t),$$
$$\hat{\tau} = -(d/dt)\hat{p}(t)(d/dt) + \hat{q}(t),$$

defined in an interval $I = (0, b)$. Suppose that $p(t) \geq \hat{p}(t) > 0$ and that $q(t) \geq \hat{q}(t)$ for t in I, and that neither τ nor $\hat{\tau}$ has boundary values at b. Let v and w be real numbers, and let T and $\hat{T}$ be the restrictions of $T_1(\tau)$ and $T_1(\hat{\tau})$ determined by a non-trivial boundary condition $vf(0) + wf'(0) = 0$.

Let $\lambda_n(T)$ and $\lambda_n(\hat{T})$ be the numbers defined for the self adjoint operators T and $\hat{T}$ as in Exercise D2. Prove that $\lambda_n(T) \geq \lambda_n(\hat{T})$, $n \geq 1$.

D10 Let S and $\hat{S}$ be two densely defined symmetric operators in Hilbert spaces. Suppose that $\mathfrak{D}(S) \subseteq \mathfrak{D}(\hat{S})$, that $\hat{S}$ is bounded below, and that $S - \hat{S}$ is positive. Let T and $\hat{T}$ be the Friedrichs

extensions of S and $\hat{S}$ respectively, and let $\lambda_n(T)$ and $\lambda_n(\hat{T})$ be the numbers defined for the self adjoint operators T and $\hat{T}$ as in Exercise D2. Show that $\lambda_n(T) \geq \lambda_n(\hat{T})$, $n \geqq 1$.

D11 Let T_1 be a self adjoint operator in Hilbert space $\mathfrak{H}_1$, and let T_2 be a self adjoint operator in Hilbert space $\mathfrak{H}_2$. Define the operator T in $\mathfrak{H} = \mathfrak{H}_1 \oplus \mathfrak{H}_2$ by setting $\mathfrak{D}(T) = \mathfrak{D}(T_1) \oplus \mathfrak{D}(T_2)$ and

$$Tx = T(x_1 \oplus x_2) = T_1x_1 \oplus T_2x_2, \qquad x \in \mathfrak{D}(T).$$

Show that the operator T is self adjoint. Show that the operator T is bounded below if and only if both T_1 and T_2 are bounded below.

Let $\lambda_n(T_1)$, $\lambda_n(T_2)$, and $\lambda_n(T)$ be the numbers defined in Exercise D2, for the operators T_1, T_2, and T respectively.

Suppose that $\lambda_\infty(T_1) = \lim_{n\to\infty}\lambda_n(T_1) \leqq \lim_{n\to\infty}\lambda_n(T_2)$. Show that the sequence $(\lambda_n(T))$ is the rearrangement in increasing order of all the numbers $\lambda_n(T_1)$, taken together with all the numbers $\lambda_n(T_2)$ which satisfy $\lambda_n(T_2) < \lambda_\infty(T_1)$.

D12 Let τ be a formally symmetric formal differential operator defined on an interval I, and suppose that τ is bounded below, so that (cf. Lemma 7.29) τ is of even order $2n$. Let c be a point interior to I, and let τ^+ (let τ^-) be the restriction of τ to $I \cap [c, \infty)$ (or to $I \cap (-\infty, c]$). Let T be a self adjoint extension of $T_0(\tau)$ defined by a separated set B of boundary conditions.

(a) Let T_1 and T_2 be the extensions of $T_0(\tau^+)$ and $T_0(\tau^-)$ defined respectively by the boundary conditions $f(c) = f'(c) = \ldots = f^{(n-1)}(c) = 0$ and by the boundary conditions in the set B at the right and at the left endpoints of I respectively. Show that the operators T_1 and T_2 are self adjoint. Let $\lambda_n(T)$, $\lambda_n(T_1)$ and $\lambda_n(T_2)$ be the numbers of Exercise D2 as defined for the operators T, T_1 and T_2 respectively. Suppose that $\lambda_\infty(T_1) = \lim_{n\to\infty}\lambda_n(T_1) \leq \lim_{n\to\infty}\lambda_n(T_2)$. Let $\{\mu_m\}$ be the rearrangement in increasing order of all the numbers $\lambda_n(T_1)$, taken together with all the numbers $\lambda_m(T_2)$ which satisfy $\lambda_n(T_2) < \lambda_\infty(T_1)$. Show that $\lambda_m(T) \leqq \mu_m$, $n \geqq 1$.

(b) Let $\hat{T}_1$ and $\hat{T}_2$ be the extensions of $T_0(\tau^+)$ and $T_0(\tau^-)$ respectively, defined by the boundary conditions $f^{(n)}(c) = \ldots = f^{(2n-1)}(c)$ and by the boundary conditions in the set B at the right and left endpoints of I respectively. Show that $\hat{T}_1$ and $\hat{T}_2$ are self adjoint. Let $\lambda_n(\hat{T}_1)$ and $\lambda_n(\hat{T}_2)$ be the numbers defined in Exercise D2 for the

operators $\hat{T}_1$ and $\hat{T}_2$ respectively. Suppose that $\lambda_\infty(\hat{T}_1) = \lim_{n\to\infty}\lambda_n(\hat{T}_1) \leq \lim_{n\to\infty}\lambda_n(\hat{T}_2)$. Let (β_n) be the rearrangement in increasing order of all the numbers $\lambda_m(T_1)$, taken together with all the numbers $\lambda_m(\hat{T}_2)$ which satisfy $\lambda_m(\hat{T}_2) < \lambda_\infty(\hat{T}_1)$. Show that $\beta_m \leqq \lambda_m(T)$, $n \geq 1$.

E. *Differential Operators in the Banach Spaces $L_p(I)$*

The following set of problems deals with the generalization of the notions introduced in Sections 1, 2, 3 and 6 to the spaces $L_p(I)$, for $1 < p < \infty$. While the exercises below are limited to a few results of a transparent nature, the interested reader will no doubt notice that several results of a more involved character can be suitably, if less trivially, generalized. For example, several of the statements below are true in the spaces $L_1(I)$, $L_\infty(I)$ and $C(I)$, although the proofs are often more involved.

In all the following exercises it will be understood that the indices p and q are real numbers greater than one.

E1 Given a regular formal differential operator τ of order n on an interval I, define an operator $T_1(\tau, p)$ in the Banach space $L_p(I)$ as follows: $\mathfrak{D}(T_1(\tau, p))$ is to be the set of all functions f in $A^n(I)$ such that f and τf belong to $L_p(I)$. For f in $\mathfrak{D}(T_1(\tau, p))$ set

$$(T_1(\tau, p)f)(t) = (\tau f)(t), \qquad t \in I.$$

Prove the analogue of Lemma 2.9.

E2 Let the operator $T_0(\tau, p)$ be the restriction of the operator $T_1(\tau, p)$ to the set of functions in its domain which vanish outside a (variable) compact subset of I. Prove that $T_0(\tau, p)^* = T_1(\tau^*, q)$, where $p^{-1} + q^{-1} = 1$.

E3 Prove that $T_1(\tau, p)$ is a closed operator.

E4 Generalize Lemma 2.16 to the operator $T_1(\tau, p)$.

E5 Generalize the definition of boundary value, boundary condition and extension to differential operators in $L_p(I)$. Prove the analogues of Theorems 19, 20 and 27 and Corollaries 21, 23, 28 in Section 2.

E6 Develop a representation of the resolvent of an extension of $T_0(\tau, p)$ analogous to that given in Section 3 for Hilbert space.

E7 Suppose that for some (real or complex) λ every solution of the equation $(\lambda-\tau)f=0$ is of class $L_p(I)$ and every solution of the equation $(\bar{\lambda}-\tau^*)f=0$ is of class $L_q(I)$ $(p^{-1}+q^{-1}=1)$. Prove that the essential spectrum of the operator τ in $L_p(I)$ is the empty set.

E8 (Bellman) Suppose that every solution of the equation $\tau f=0$ is of class $L_p(I)$ and that every solution of the equation $\tau^* f=0$ is of class $L_q(I)$ $(p^{-1}+q^{-1}=1)$. Prove that for every (real or complex) λ every solution of the equation $(\lambda-\tau)f=0$ is of class $L_p(I)$.

E9 Let T be any closed extension of the operator $T_0(\tau, p)$. Prove that the essential spectrum of T coincides with the essential spectrum of $T_1(\tau, p)$.

E10 Prove that the essential spectrum of the operator $T_1(\tau, p)$ coincides with the essential spectrum of the operator $T_1(\tau^*, q)$.

E11 Let

$$\tau=\left(\frac{1}{i}\frac{d}{dt}\right)^n$$

on the interval $[0, \infty)$. Prove that the essential spectrum of the operator $T_1(\tau, p)$ is the positive semi-axis if n is even, and the entire real axis if n is odd.

E12 Let K be any constant which is not of the form $-(i\lambda)^n$ for any real λ. Prove that there exists a positive constant C, depending only on p and on K, such that

$$\left(\int_{-\infty}^{\infty}|Kf(t)+f^{(n)}(t)|^p\,dt\right)\Big/\left(\int_{0}^{\infty}|f(t)|^p\,dt\right)\geqq C$$

for all functions in $A^n(-\infty, +\infty)$ for which the numerator and denominator of the above expression are defined and finite.

E13 Suppose that the function q is of class $L_p[0, \infty)$, and let

$$\tau=-(d/dt)^2+q(t), \qquad 0\leqq t<\infty,$$

Prove that the essential spectrum of the operator $T_1(\tau, p)$ is the positive semi-axis.

F. *The Sturm-Liouville Operator*

In the following exercises the letter τ will denote the Sturm-Liouville operator

$$\tau = -\frac{d}{dt}p(t)\frac{d}{dt}+q(t)$$

where the function p is assumed to be positive throughout the interval of definition, and the function q real and continuous.

F1 Suppose that $q(t) \geqq 1$ on the real line. Prove that the essential spectrum of τ on the line is void if and only if the set of continuously differentiable functions of compact support such that

$$\int_{-\infty}^{\infty}\{p(t)|f'(t)|^2+q(t)|f(t)|^2\}dt \leq 1$$

is conditionally compact in $L_2(-\infty, \infty)$.

F2 A real number λ lies in the essential spectrum of τ if and only if there exists a *continuous* square-integrable function g on the interval of definition I such that the equation

$$(\lambda-\tau)f = g$$

has no square-integrable solution.

F3 (Hartman) Let the interval of definition of τ be $[0, \infty)$, and let $N(\lambda, t)$ be the number of zeros in the interval $[0, t)$ of a solution of the equation $(\lambda-\tau)f = 0$. Prove that a point λ_0 on the real axis belongs to the essential spectrum of τ if and only if whenever $\lambda < \lambda_0 < \mu$ then

$$\liminf_{t\to\infty}\,[N(\mu, t)-N(\lambda, t)] = \infty.$$

F4 Let the operator τ and the function Q be as in Theorem 7.66. Assume that

$$\int_a^b p(t)^{-1}dt = \infty,$$

Then:

(a) If $Q(t) \to -\infty$ and

$$\int_A^b \left|\left[\frac{q'(t)}{|q(t)|^{3/2}}\right]' + \frac{1}{4}\frac{[q'(t)]^2}{|q(t)|^{5/2}}\right| dt < \infty$$

for large A, and

$$\int_A^\infty Q(t)^{-1/2}\,dt < \infty,$$

then the essential spectrum of τ is void.

(b) If $Q(t) \to -\infty$, Q is monotone decreasing for sufficiently large t,

$$\int_A^\infty \left| \left[\frac{q'(t)}{|q(t)|^{3/2}} \right]' + \frac{1}{4} \frac{[q'(t)]^2}{|q(t)|^{5/2}} \right| dt < \infty$$

for large A, and

$$\int_A^\infty Q(t)^{-1/2}\,dt = \infty$$

for all A, then the essential spectrum of τ is the entire real axis.

F5 Using the device of Theorem 7.55, derive from each of the criteria given in Section G a corresponding criterion for the determination of the essential spectrum or of the number of boundary values of a Sturm-Liouville operator.

G. *The Sturm-Liouville Operator* $-(d/dt)^2+q(t)$

The following set of exercises deals with the operator $\tau = -(d/dt)^2+q(t)$, where the function q is assumed to be real and continuous. The interval of definition will be $[0, \infty)$. The symbol λ will denote a real number.

G1 Suppose that for sufficiently large t,

$$q(t) \leq -Kt^{-2}, \qquad K > 2.$$

Prove that the equation $\tau f = 0$ has a square-integrable solution.

G2 (Wintner) Suppose that the operator τ has the property that whenever f is a square-integrable solution of the equation $(\lambda-\tau)f = 0$, then f' is also square-integrable.

(a) Prove that τ has no boundary values at infinity.

(b) Prove that τ has this property if the function q is bounded below.

(c) If it is also assumed that q is bounded, then any solution which is linearly independent of a square-integrable solution is unbounded.

G3 Suppose that the operator τ has the property that for some λ the derivative of every square-integrable solution of the equation $(\lambda-\tau)f = 0$ is bounded. Prove that τ has no boundary values at infinity.

G4 (Wintner) Suppose that the function q satisfies the Lipschitz condition

$$|q(t)-q(s)| \leqq K|t-s|$$

for large s and t on $[0, \infty)$. Prove that the operator τ has the property of Exercise G3.

G5 (Hartman and Wintner). Suppose that q is bounded below on the interval $[0, \infty)$, and let f be a bounded solution of the equation $(\lambda-\tau)f = 0$. Prove:

(a) either f is square-integrable, or the point λ belongs to the essential spectrum of τ;

(b) if all solutions of the equation $(\lambda-\tau)f = 0$ are bounded, then λ belongs to the essential spectrum of τ.

G6 Assume that τ has no boundary values at infinity, and that

(i) if f is a bounded solution of the equation $(\lambda-\tau)f = 0$, then f' is bounded;

(ii) if g is a square-integrable solution of the same equation, then there exists a sequence t_n tending to infinity such that

$$f(t_n)^2+f'(t_n)^2 \to 0$$

as $n \to \infty$.

Prove that if f is a bounded solution of the equation $(\lambda-\tau)f = 0$, then either f is square-integrable or λ belongs to the essential spectrum of τ.

G7 In the equation

$$\text{(A)} \qquad (P(t)F'(t))'-Q(t)F(t) = 0, \qquad 0 \leqq t < \infty,$$

suppose that P is positive and Q is non-negative. Prove that the equation has a bounded solution.

G8 Suppose that $q_1(t) \leqq q_2(t)$ $(0 \leqq t < \infty)$ and also that the equation

$$f''(t)-q_1(t)f(t) = 0$$

has a solution with a finite number of zeros and has a square-integrable solution. Prove that the equation

$$g''(t) \quad q_2(t)g(t) \quad 0$$

has a square-integrable solution.

(Hint: Write $g(t) = h(t)f(t)$ and obtain, by variation of parameters, an equation of type (A) as in G7 for h. Infer that $g(t) = O(f(t))$.)

G9 Let $N(t)$ be the number of zeros in the interval $[0, t)$ of a solution of the equation $\tau f = 0$. Prove the following:

(a) If $N(t) > 1$, then

$$\int_0^t |q(s)|ds > 4t^{-1}.$$

(b) Suppose q is non-positive, and let t_n be the n-th zero of a solution of the equation $\tau f = 0$. Then

$$-\int_0^{t_n} q(t)dt > 4(n \quad 1)^2 t_n^{-1}.$$

(c) $$N(t) \quad O\left(\left(t\int_0^t \max(0, \quad q(s))ds\right)^{1/2}\right) + O(1).$$

(d) If

$$\int_0^t \max(0, \quad q(s))ds \quad O(t^2),$$

then the operator τ has no boundary values at infinity.

(Hints: *Ad* (a): $\int_0^t |f''(s)(f(s))^{-1}|ds > (\max_{0\leq s\leq t}|f(s)|)^{-1}\int_0^t |f''(s)|ds > (\max_{0<s<t}|f(s)|)^{-1}\max_{0<s,u\leq t}|f'(u) \quad f'(s)|$, and apply Rolle's theorem. *Ad* (c): reduce to the case where q is non-positive and apply (b). *Ad* (d): Apply Exercise G14 via (c).)

G10 Suppose that q is negative and non-increasing, and let f be a solution of the equation $\tau f = 0$. Let $\{s_n\}$ be the increasing sequence of zeros of f.

(a) Prove that $s_n - s_{n-1} \geqq s_{n+1} - s_n$.

(b) Let m_n be the point between s_{n-1} and s_n at which $f'(s)$ vanishes. Suppose that the function f is square-integrable. Prove that

$$\sum_{n=1}^{\infty} f(m_n)^2(m_{n+1} - m_n) < \infty.$$

(c) Prove that

$$[f(s)^2 - f'(s)^2 q(s)^{-1}]' \leqq 0$$

assuming that the function q is differentiable.

(d) Infer that, if the function f is square-integrable, then

$$\int_0^\infty [f(s)^2 - f'(s)^2 q(s)^{-1}] ds < \infty.$$

(e) (Hartman and Wintner). Suppose in addition that

$$\int_0^\infty |q(t)|^{-1/2} dt = \infty$$

and prove that the operator τ has no boundary values at infinity.

G11 (Hartman and Wintner) Suppose that the function q is negative and non-increasing, and that

$$\int_0^\infty |q(t)|^{-1} dt = \infty.$$

Prove that the origin lies in the continuous spectrum of every self adjoint extension of τ.

(Hint: Use the inequality

$$[f'(s)^2 + q(s)^2]' \geqq 0,$$

valid for a solution of $\tau f = 0$.)

G12 Suppose that q is non-positive, and that the equation $\tau f = 0$ has a square-integrable solution. Prove that f has an infinite number of zeros.

G13 Let f and g be solutions of the equation $\tau f = 0$ such that $fg' - f'g = 1$. Show that they can be expressed in the form

$$f(t) = r(t) \cos \theta(t)$$
$$g(t) = r(t) \sin \theta(t)$$

where $r(t) > 0$, $0 \leqq t < \infty$. Show that the operator τ has two boundary values at infinity if and only if

$$\int_0^\infty \theta'(t)^{-1} dt < \infty.$$

What is the relationship between $\theta(t)$ and the number of zeros of a solution of the above equation?

G14 Use the result of the preceding exercise to show that if the operator τ has two boundary values at infinity, then

$$\lim_{t\to\infty} \frac{N(t)}{t^2} = \infty,$$

where $N(t)$ is the number of zeros of a solution of the equation

$$\tau f = 0.$$

G15 Let the functions f and g be chosen as in Exercise G13. Suppose that the function $(f(t)^2+g(t)^2)^{-1}$ is integrable on $[0, \infty)$. Prove that every solution of the equation $\tau f = 0$ has a finite number of zeros.

(Hint: it follows that $\int_0^\infty \theta'(t)dt < \infty$. Express in terms of $N(t)$ and integrate.)

G16 (Hartman) Suppose that the equation $\tau f = 0$ has a solution with a finite number of zeros. Prove that there exists a solution g of the same equation such that $g(t)^{-1}$ is square-integrable on a semi-axis sufficiently far removed from the origin.

(Hint: Let f be a solution, say positive. Then $g(t) = f(t)\int_0^t f(s)^{-2}ds$ is also a positive solution. Hence for some constant K, $f(t)g(t)^{-1} = K - \int_0^t g(s)^{-2}ds$, from which the assertion follows easily.)

G17 (Hartman and Wintner). Let f and g be solutions of the equation $\tau f = 0$ such that $fg' - f'g = 1$. Then not both f' and g' are square-integrable.

(Hint: Otherwise $(f^2+g^2)^{-1}$ is integrable; hence for some solution h, h^{-1} is square-integrable (cf. G15, G16). But h' is square-integrable, hence $h'h^{-1}$ is integrable, but this is easily seen to be impossible.)

G18 (Hartman and Wintner). If the equation $\tau f = 0$ has a solution with a square-integrable derivative, then the operator τ has no boundary values at infinity.

(Hint: Let f and g be as in G17, f' and g square-integrable. Then $(gf)' - 1$ is integrable, and $fg - t$ tends to a limit. Hence f^{-1} is eventually square-integrable, and $f^{-1}g$ tends to a limit which is easily shown to be zero. Thus $f^{-1}g = -\int_0^t f(s)^{-2}ds$, whence a contradiction).

G19 (Hartman and Wintner). Suppose that the equation

$(\lambda-\tau)f = 0$ has a solution which is not square-integrable but has a square-integrable derivative. Prove that the point λ belongs to the essential spectrum of τ.

G20 (Wintner). Suppose that q is bounded below, and suppose that λ does not belong to the essential spectrum of τ. Let f be a square-integrable solution of the equation $(\lambda-\tau)f = 0$, and let g be a second solution of the same equation such that $fg'-f'g = 1$.

(a) Let h be any square-integrable function, and let r be a square-integrable solution of the equation $(\lambda-\tau)r = h$. Prove that r' is square-integrable.

(b) Prove that

$$f(t)r'(t)-r(t)f'(t) = \int_0^\infty f(t)r(t)dt.$$

(c) Prove that

$$\int_0^\infty tf(t)^2dt < \infty.$$

(d) Infer that

$$\int_0^\infty tf'(t)^2dt < \infty.$$

(e) By an inductive repetition of the same argument, prove that

$$\int_0^\infty t^kf(t)^2dt < \infty \quad \text{and} \quad \int_0^\infty t^kf'(t)^2dt < \infty.$$

(Hint: *Ad* (a): Use the identity $ff'' = (ff')'-f'^2$: if f' is not square-integrable, then $(ff')'$ tends to infinity, and hence so does $(f^2)'$, and thus also f^2, contradiction. *Ad* (d): derive the intermediary estimates

$$\int_0^t sf(s)f''(s)ds + \int_0^s s\min(-q(t), 0)f(s)^2ds = o(1)$$

and

$$tf(t)f'(t) - \int_0^t sf'(s)ds + \int_0^t s\min(-q(t), 0)f(s)^2ds = O(1)$$

from which the desired result easily follows. *Ad* (e): start by proving that $t^k(f(t)r'(t)-r(t)f'(t))$ is integrable.)

G21 Let $q(t) = -e^{kt}$, where k is a positive constant. Prove that the operator τ has two boundary values at infinity.

G22 Let $q(t) = -t^2\log^4 t$. Prove that the operator τ has two boundary values at infinity.

G23 (Hartman) Let $Q(t)$ be a positive continuous non-decreasing function, and let $\theta(t)$ be defined as in Exercise G13.

(a) Prove that for $u < t$,

$$\int_u^t \theta'(s)^{-1} ds \geq 2\int_u^t Q(s)^{-1} ds - \int_u^t Q(s)^{-2}\theta'(s) ds.$$

(b) Using the relationship between θ and the number of zeros $N(t)$ of a solution in the interval $[0, t)$ deduced in Exercise G13, show that the right-hand side of the above inequality exceeds the product of π and the quantity

$$N(u)(Q(u))^{-2} - N(t)(Q(t))^{-2} + \int_u^t N(s) d\big((Q(s))^{-2}\big) - Q(t)^{-2}$$
$$Q(u)^{-2} + \int_u^t d\big((Q(s))^{-2}\big).$$

(c) Infer that

$$\int_u^t \theta'(s)^{-1} ds \geq \int_u^t Q(s)^{-1} ds - \pi\int_u^t Q(s)^{-2} dN(s)$$
$$\pi Q(t)^{-2} - \pi Q(u)^{-2} + \pi\int_u^t d\big((Q(s))^{-2}\big).$$

(d) Prove that, if $u = u(t)$ may be chosen in such a way that

$$\limsup_{t\to\infty}\Big[2\int_u^t Q(s)^{-1} ds - \pi\int_u^t Q(s)^{-2} dN(s) - \pi Q(t)^{-2} - \pi Q(u)$$
$$+ \pi\int_u^t d\big((Q(s))^{-2}\big)\Big] > 0,$$

then the operator τ has no boundary values at infinity.

G24 Suppose that

$$\liminf_{t\to\infty}\left[\left(N(t) - N\left(\frac{t}{2}\right)\right)t^{-2}\right] < \infty$$

and prove that the operator τ has no boundary values at infinity. (Hint: Set $u = t/2$ and $Q(t) = t/\varepsilon$ in the preceding exercise.)

G25 Suppose that

$$\liminf_{t\to\infty}\,[N(t) - N(t-1)] < \infty$$

and prove that the operator τ has no boundary values at infinity. (Hint: Set $u = t - 1$ and $Q(t) = 1/\varepsilon$ in Exercise G23.)

G26 Suppose that there is an infinite sequence of disjoint

intervals of length at least one, such that the function q is uniformly bounded below on the union of all these intervals. Prove that the operator τ has no boundary values at infinity.

(Hint: Apply the preceding exercise.)

G27 (Hartman) Let Q be a positive continuous function which is non-decreasing and of bounded variation on every finite interval, and such that

$$\int_0^\infty Q(s)^{-1}ds = \infty.$$

As usual, let $N(t)$ be the number of zeros in the interval $[0, t)$ of a solution of the equation $\tau f = 0$, and suppose that

$$N(t) \leqq \int_0^t Q(s)ds + KQ(t)^{2-\varepsilon}$$

where $0 < \varepsilon < 2$.

(a) Show that

$$\int_u^t Q(s)^{-2}dN(s) \leqq \int_u^t Q(s)^{-1}ds + K\int_1^t Q^{-1-\varepsilon}dQ(s).$$

(b) Using the result of Exercise G23, show that the operator τ has no boundary values at infinity.

G28 Suppose that the function Q is positive, continuous, of bounded variation on every finite interval, non-increasing, and that

$$\int_0^\infty Q(s)^{-1}ds = \infty.$$

Suppose that $N(t)$ satisfies the condition of the preceding exercise, Prove that the operator τ has no boundary values at infinity.

(Hint: This follows immediately from G27.)

G29 Suppose that Q is a positive continuous monotone function satisfying the equation

$$\int_0^\infty Q(s)^{-1}ds = \infty,$$

and suppose that

$$\int_0^t \max\,(-q(s), 0)ds \leqq t^{-1}\left(\int_0^t Q(s)ds\right)^2.$$

Prove that the operator τ has no boundary values at infinity.

(Hint: Apply Exercise G19.)

G30 Suppose that

$$N(t) = O(t^2 \log t),$$

and prove that the operator τ has no boundary values at infinity.

G31 Prove the following refinement of the result of Exercise G19(d): if

$$\int_0^t \max\big(-q(s), 0\big)ds = O(t^2 \log^2 t),$$

then the operator τ has no boundary values at infinity.

G32 (Hartman and Wintner) Suppose q is negative and differentiable. Let $N(t)$ be the number of zeros of the solution f of the equation $\tau f = 0$ such that $f(0) = 0$ in the interval $[0, t)$, and let

$$\theta(t) = \arc\tan\big([-q(t)]^{1/2} f(t)(f'(t)^{-1})\big).$$

(a) Prove that

$$|\theta(t) - \pi N(t)| \leq \pi.$$

(b) Prove that

$$\theta'(t) = \big(-q(t)\big)^{1/2} - [(f'(t))^2 - q(t)(f(t))^2][(-q(t))^{1/2} f(t) f'(t)]\frac{q'(t)}{2q(t)}.$$

(c) Prove that

$$N(t) = \frac{1}{\pi}\int_0^t (-q(s))^{1/2} ds + \int_0^t k(s)\big(\log(-q(s))\big)' ds + r(t),$$

where k and r are measurable functions such that

$$-\tfrac{1}{2} < k(t) < \tfrac{1}{2} \qquad \text{and} \qquad -\pi < r(t) < \pi.$$

G33 Suppose that the function Q is differentiable and

$$Q(t) \geq K > 0, \qquad 0 \leq t < \infty$$
$$|Q'(t)|Q(t)^{-2} \leq M$$

for some constants M and K, and suppose that

$$\left|\int_0^t q(s)ds\right| \leq Q(t).$$

Let f be a square-integrable solution of the equation $(\lambda - \tau)f = 0$. Prove the following assertions in order:

(a) $$\int_0^t [f'(s)Q(s)^{-1}]^2 ds = \int_0^t [f(s)f'(s)]'Q(s)^{-2}ds$$
$$+\lambda\int_0^t [f(s)Q(s)^{-1}]^2 ds - \int_0^t q(s)f(s)^2Q(s)^{-2}ds = J_1+J_2+J_3.$$

(b) $$J_1 = O(|f(t)f'(t)|) + O\left(\left(\int_0^t [f'(s)Q(s)^{-1}]^2 ds\right)^{1/2}\right) + O(1),$$
$$J_2 = O(1),$$
$$J_3 = O(f(t)^2) + O\left(\left(\int_0^t f'(s)Q(s)^{-1} ds\right)^{1/2}\right) + O(1).$$

(c) $$\int_0^t [f'(s)Q(s)^{-1}]^2 ds = O(|f(t)f'(t)|+f(t)^2) + O\left(\left(\int_0^t f'(s)Q(s)^{-1} ds\right)^{1/2}\right).$$

(d) The function $f'(t)Q(t)^{-1}$ is square-integrable. (Hartman).

G34 (Borg) Suppose that

$$\limsup\,(\log t)^{-1}\int_0^t |q(s)|ds = A < \infty.$$

Prove that the continuous spectrum of every self adjoint extension of the operator τ contains the half-line $[A^2, \infty)$.

(Hint: Set $Q(t) = \log t$ in the preceding exercise and let $(\lambda-\tau)f = 0$, where $\lambda > A^2$. If f is square-integrable, deduce a contradiction from the inequality

$$f(t)^2 + \lambda^{-1}f'(t)^2 \geq K\exp\left(-\frac{1}{\sqrt{\lambda}}\int_0^t |q(s)|ds\right).$$

G35 (Hartman) Using Exercises F3 and G33, prove that if the function q is negative, non-increasing and $|q(t)| = O(t^2)$ then the essential spectrum of the operator τ is the entire real axis.

G36 (Hartman) Assume that q is monotone, negative, tends to $-\infty$ and that for some $k > 1$,

$$\int_0^\infty |q(s)|^{-k/2}ds = \infty.$$

Prove:

(a) Given $t > 0$, let $s(t)$ be the largest real number such that $q(s) = 2q(t)$. Then

$$\limsup_{t\to\infty}\int_t^{s(t)} |q(s)|^{k/2}ds = \infty.$$

(b) The essential spectrum of the operator τ is the entire real axis.

(Hint: F3, G8.)

G37 Let f be the solution of the equation $(\lambda-\tau)f = 0$ such that $f(0) = 0$, and set

$$\theta(t, \lambda) = \arctan \sqrt{\lambda} f(t) f'(t)^{-1}, \qquad \theta(0, \lambda) = 0, \qquad \lambda > 0.$$

This defines a unique continuous function on the positive semi-axis for each $\lambda \geqq 0$.

(a) Prove that

$$|\theta'(t, \lambda)-\sqrt{\lambda}| \leqq \lambda^{-1/2}|q(t)|.$$

(b) Infer that

$$|\pi N(t, \lambda)-\lambda^{1/2}t| \leqq O(1)+\lambda^{-1/2}\int_0^t |q(s)|ds,$$

where $N(t, \lambda)$ is the number of zeros of f in the interval $[0, t)$.

G38 (Hartman) Suppose that

$$\liminf_{t\to\infty} t^{-1}\int_0^t |q(s)|ds = 0$$

and infer that the essential spectrum of the operator τ contains the positive semi-axis.

(Hint: Use the previous exercise and F8.)

G39 (Hartman) Suppose that

$$\liminf_{t\to\infty} t^{-1}\int_0^t |q(s)|ds = A$$

and prove that every interval $[\lambda, \lambda+4A+4A^2\lambda^{-1}] (\lambda > 0)$ meets the essential spectrum of the operator τ.

(Hint: Use the method of the preceding exercise.)

G40 (Šnol) Suppose that there exists a sequence $\{(a_n, b_n)\}$ of intervals on the positive real axis such that $b_n-a_n \to \infty$ and

$$(b_n-a_n)^{-1}\int_{a_n}^{b_n} q(t)^2 dt \to 0.$$

(a) For large enough n, let h_n be an infinitely differentiable function which vanishes outside (a_n, b_n) and which is identically equal to one on the interval (a_n+1, b_n-1), and let

$$f_n(t) = h_n(t) \sin t\sqrt{\lambda}, \qquad \lambda > 0.$$

Prove that

$$|(\lambda - \tau)f_n| = O(\sqrt{(b_n - a_n)}).$$

(b) Prove that the essential spectrum of τ contains the positive semi-axis.

(Hint: Apply Theorem 7.1.)

G41 Suppose that the function q is bounded below, Suppose that the origin belongs to the essential spectrum of τ.

(a) Let $\{f_n\}$ be a sequence in $\mathfrak{D}(T_0(\tau))$ such that $|f_n| = 1$, $|\tau f_n| \to 0$, and such that f_n vanishes in the interval $[0, n)$. Set

$$g_n(t) = f_n(t) \sin t\sqrt{\lambda}, \qquad \lambda > 0,$$

and show that $|g_n| > 1/8$.

(b) Show that $|(\lambda - \tau)g_n| = O(\sqrt{\lambda})$.

(c) (Šnol) Prove that for large λ every interval of the form $[\lambda - \sqrt{\lambda}, \lambda + \sqrt{\lambda}]$ meets the essential spectrum of the operator τ.

(Hint: Exercise A6.)

G42 Suppose that q is bounded below, Prove that the essential spectrum of the operator τ is either empty or unbounded above.

G43 (Molčanov) Suppose that the function q is bounded below. Prove that the essential spectrum of the operator τ is empty if and only if for every $u > 0$,

$$\lim_{t \to \infty} \int_t^{t+u} q(s)ds = \infty.$$

(Hint: F2.)

G44 (Putnam) Suppose that the function q is square-integrable, Prove that the positive real axis belongs to the essential spectrum of the operator τ.

(Hint: Say there is a square-integrable f such that $(\lambda - \tau)f = 0$, where $\lambda > 0$, and let $g_1(t)$ and $g_2(t)$ be linearly independent solutions of the equation $g''(t) - \lambda g(t) = 0$. If λ does not belong to the essential spectrum, then there exist square-integrable f_1 and f_2 such that

$$f_i'' + (\lambda - q(t))f_i g = g_i g.$$

Derive a contradiction.)

G45 Suppose that q is square-integrable, and let f be a square-integrable solution of $(\lambda-\tau)f = 0$. Prove that f' and f'' are square-integrable and $o(1)$. Prove that if $\lambda \neq 0$, then

$$\lim_{t\to\infty}\int_0^t f(s)ds$$

exists and is finite.

H. *The Operator* $-(d/dt)^2+q(t)$ *with* q *integrable.*

The following set of exercises deals with the operator $\tau = -(d/dt)^2+q(t)$ on an interval which will, unless otherwise specified, always be $[0, \infty)$, where the function q is assumed to be real, continuous and integrable. The parameter λ will be tacitly assumed to be real, unless otherwise stated.

H1 Prove that the operator τ has no boundary values at infinity and that its essential spectrum is the positive semi-axis.

H2 (Auxiliary lemma). Let f be a continuous function, g an integrable function on the interval $[0, A]$. Suppose that

$$f(t) \leqq O(1) + \int_0^t f(s)g(s)ds, \qquad 0 < t \leqq A.$$

Prove that

$$f(t) \leqq k\exp\left(\int_0^t g(s)ds\right).$$

H3 Let $f(t, \lambda)$ and $g(t, \lambda)$ be solutions of the equation $(\lambda-\tau)f = 0$ which satisfy the boundary conditions

$$f(0, \lambda) = \sin\theta, \qquad f'(0, \lambda) = -\cos\theta$$
$$g(0, \lambda) = \cos\theta, \qquad g'(0, \lambda) = \sin\theta.$$

Prove that the function f satisfies the integral equation

$$f(t, \lambda) = \sin\theta\cos t\sqrt{\lambda}-(\sqrt{\lambda})^{-1}\cos\theta\sin t\sqrt{\lambda}$$
$$+(\sqrt{\lambda})^{-1}\int_0^t \sin\left((t-s)\sqrt{\lambda}\right)q(s)f(s, \lambda)ds, \qquad \lambda > 0.$$

H4 From Exercises H3 and H2 infer that for $\lambda > 0$,

$$f(t, \lambda) = O\left(\left(1+(\sqrt{\lambda})^{-1}\right)\exp\left((\sqrt{\lambda})^{-1}\int_0^t |q(s)|ds\right)\right)$$

and that a similar estimate holds for the solution g.

H5 Obtain the following estimate for the solution f:

$$\begin{aligned} f(t,\lambda) &= m(\lambda)\cos t\sqrt{\lambda} + n(\lambda)\sin t\sqrt{\lambda} + o(1) \\ &= \sqrt{m(\lambda)^2 + n(\lambda)^2}\sin\big(\sqrt{\lambda}(t-\alpha(\lambda))\big) + o(1) \end{aligned}$$

as $t\to\infty$, where

$$m(\lambda) = \sin\theta - (\sqrt{\lambda})^{-1}\int_0^\infty \sin(s\sqrt{\lambda})q(s)f(s,\lambda)ds$$

$$n(\lambda) = -(\sqrt{\lambda})^{-1}\cos\theta + (\sqrt{\lambda})^{-1}\int_0^\infty \cos(s\sqrt{\lambda})q(s)f(s,\lambda)ds.$$

Show that a similar asymptotic estimate holds for g. Show that the estimate can be differentiated.

H6 Let T be the self adjoint extension of $T_0(\tau)$ obtained by the imposition of the boundary condition

$$B(f) = f(0)\cos\theta + f'(0)\sin\theta = 0.$$

Taking as determining set (cf. Theorem 5.28) the solution $f(t,\lambda)$ of the equation $(\tau-\lambda)f = 0$ such that $f(0,\lambda) = \sin\theta$ and $f'(0,\lambda) = -\cos\theta$, prove that the measure μ associated with this determining set (cf. Theorem 5.23) is given by the expression

$$\mu(I) = \frac{1}{\pi}\int_I (\sqrt{\lambda})^{-1}[m(\lambda)^2 + n(\lambda)^2]^{-1}d\lambda, \qquad I \subseteq (0,\infty).$$

Show that this result may be formulated as follows: For each $\mu>0$, let $\sigma(t,\mu)$ be that unique solution of the equation $\tau\sigma = \mu^2\sigma$ which has the asymptotic form $\sin(\mu t - \alpha(\mu))\big(1+o(1)\big)$ as $t\to\infty$, and which satisfies the given boundary condition at $t=0$. Then the formula

$$(Uf)(\mu) = \sqrt{\frac{2}{\pi}}\ \underset{A\to\infty}{\mathrm{l.i.m.}}\int_0^A f(t)\sigma(t,\mu)dt, \qquad \mu>0,$$

defines a bounded operator mapping $L_2[0,\infty)$ into itself; and for each bounded Borel function F,

$$\big(E((0,\infty);T)F(T)f\big)(t) = \sqrt{\frac{2}{\pi}}\ \underset{A\to\infty}{\mathrm{l.i.m.}}\int_0^A (Uf)(\mu)F(\mu^2)\sigma(t,\mu)d\mu,$$

and

$$|E((0, \infty); T)f|^2 = \int_0^\infty |(Uf)(\mu)|^2 d\mu.$$

H7 Let T be any self adjoint extension of τ. Prove that any measure obtained by Theorem 5.23 is absolutely continuous with respect to Lebesgue measure.

H8 Consider an operator $\tau = -(d/dt)^2 + q(t)$ on the interval (a, ∞), where $\infty \leqq a < \infty$. Suppose that $a < b < \infty$, and that $\int_b^\infty q(t)dt < \infty$. Let τ_1 be the restriction of τ to the interval (a, b), and let Λ be a subinterval of $(0, \infty)$ not intersecting $\sigma_e(\tau_1)$. Let T be a self adjoint extension of $T_0(\tau)$.

(a) Show that either $T = T_1(\tau)$, or T is determined by a simple boundary condition $B(f) = 0$ at a; and that the latter is the case if and only if every solution of an equation $\tau f = \lambda f$ is square-integrable in the vicinity of a.

(b) Generalize the final conclusion of Exercise 6 to obtain the following statement: For each $\mu > 0$ such that $\mu^2 \in \Lambda$, let $\sigma(t, \mu)$ be that unique solution of the equation $\tau\sigma = \mu^2\sigma$ which has the asymptotic form $\sin\,(\mu t - \alpha(\mu))(1 + o(1))$, which is square-integrable in the vicinity of $t = a$, and which satisfies the boundary condition $B(f) = 0$ at a defining T, if $T \neq T_1(\tau)$. Then the formula

$$(Uf)(\mu) = \sqrt{\frac{2}{\pi}} \operatorname*{Lim.}_{A \to \infty} \int_0^A f(t)\sigma(t, \mu)dt, \qquad \mu > 0, \qquad \mu^2 \in \Lambda,$$

defines a bounded operator mapping $L_2(0, \infty)$ into $L_2(\Lambda_0)$, where $\Lambda_0 = \{\mu > 0 | \mu^2 \in \Lambda\}$. For each bounded Borel function F,

$$\big(E(\Lambda_0, T)F(T)f\big)(t) = \sqrt{\frac{2}{\pi}} \int_{\Lambda_0} (Uf)(\mu)F(\mu^2)\sigma(t, \mu)d\mu,$$

the integral on the right existing in the mean-square sense; and

$$|E(\Lambda_0, T)F(T)f|^2 = \int_{\Lambda_0} |Uf(\mu)|^2 |F(\mu^2)|^2 d\mu.$$

H9 Use Exercise H8 to establish the properties of
(a) the Fourier sine and cosine transforms;
(b) the Hankel transform.

H10 Let q be integrable on $(-\infty, +\infty)$, and let

$$\tau = \left(\frac{d}{dt}\right)^2 + q(t).$$

For each $\lambda = \mu^2$, where $\mu > 0$, let $\sigma_+(t, \lambda)$ and $\sigma_-(t, \lambda)$ be solutions of the equation $\tau\sigma = \lambda\sigma$ which have the respective asymptotic forms $e^{it\mu}$ and $e^{-it\mu}$ as $t \to -\infty$ (cf. Lemma 6.18), and let $\Sigma_+(t, \lambda)$ and $\Sigma_-(t, \lambda)$ be solutions of this same equation which have the same respective asymptotic forms as $t \to +\infty$.

(a) Show that $\sigma_+, \sigma_-, \Sigma_+$, and Σ_- are uniquely determined by their asymptotic forms, continuously dependent upon λ, and that σ_+ is linearly independent of σ_- and Σ_+ is linearly independent of Σ_-.

(b) The uniquely determined coefficients $\rho(\lambda)$ and $\tau(\lambda)$ in the equation

$$\tau(\lambda)\Sigma_+(t, \lambda) = \sigma_+(t, \lambda) + \rho(\lambda)\sigma_-(t, \lambda)$$

are called, by physicists, the *reflection and transmission coefficients for the potential* q. Show that $|\tau(\lambda)|^2 + |\rho(\lambda)|^2 = 1$, and that if we take $\sigma_-(t, \lambda)$ and $\sigma_+(t, \lambda)$ as a determining set for τ on the interval $(0, \infty)$, the corresponding positive matrix measure (cf. Theorem 5.23) is given by

$$\begin{pmatrix} \mu_{11}(I) & \mu_{12}(I) \\ \mu_{21}(I) & \mu_{22}(I) \end{pmatrix} = \frac{1}{\pi}\int_I \begin{pmatrix} 1 & \rho(\lambda) \\ \overline{\rho(\lambda)} & 1 \end{pmatrix} \frac{d\lambda}{\sqrt{\lambda}}.$$

(c) From this, derive the Plancherel theorem by considering the special case $q = 0$

(d) Using Exercise H8, make the corresponding calculations for the operator

$$\left(\frac{d}{dt}\right)^2 + ae^{-t}$$

on the interval $(-\infty, +\infty)$.

H11 Suppose that the function q satisfies the condition

$$\int_0^\infty (1 + t^k)|q(t)|dt < \infty$$

for some positive integer k, and let T and μ be as in Exercise H6. Prove that the measure μ is of the form

$$\mu(I) = \int_I g(\lambda)d\lambda, \qquad I \subseteqq (0, \infty),$$

where the function g is k times differentiable.

H12. (Jost and Pais.) Suppose that

$$\int_0^\infty (1+t)|q(t)|dt < \infty.$$

Prove that a self adjoint extension of the operator has a negative eigenvalue only if

$$\int_0^\infty t|q(t)|dt \geqq 1.$$

H13 Suppose that

$$\int_0^\infty (1+t)|q(t)|dt < \infty.$$

Prove that the origin lies in the continuous spectrum of every self adjoint extension of the operator $T_0(\tau)$.

(Hint: There are two linearly independent solutions f and g of the equation $\tau f = 0$ such that $f(t) \sim 1$ and $g(t) \sim t$ as $t \to \infty$.)

H14 The operator $T_1(\tau, 1)$ is defined in $L_1[0, \infty)$ as follows: $\mathfrak{D}(T_1(\tau, 1))$ is the set of functions f in $A^2[0, \infty)$ such that f and τf are integrable. and for f in $\mathfrak{D}(T_1(\tau, 1))$

$$(T(\tau, 1)f)(t) = -f''(t) + q(t)f(t), \qquad 0 \leq t < \infty.$$

Let $T_0(\tau, 1)$ be the restriction of $T_1(\tau, 1)$ to the class of functions in its domain vanishing outside a variable compact subset of the positive semi-axis.

Prove that the operator $T_1(\tau, 1)$ is closed in $L_1(0, \infty)$.

H15 Prove that the essential spectrum of the operator $T_1(\tau, 1)$ in $L_1[0, \infty)$ is the positive semi-axis.

(Hint: Use the method of Exercise G44.)

H16 Formulate a definition of boundary value for the operator $T_1(\tau, 1)$ in $L_1[0, \infty)$ analogous to Definition 2.17. Prove the result analogous to Theorem 2.19, and show that the operator T_1 in $L_1[0, \infty)$ has no boundary values at infinity.

H17 Suppose that $\mathscr{I}\lambda \neq 0$, and let f be a square-integrable solution of the equation $(\lambda - \tau)f = 0$. Prove that the function f is integrable.

(Hint: This is an immediate consequence of the preceding exercise.)

I. *Special Functions*

The following set of exercises is concerned with the calculation of the specific formulae which arise out of the spectral analysis of a number of particular formal differential operators. The calculations are in each case to be performed by the methods of Section 8.

I1 Given the formally symmetric formal differential operator

$$\tau = -\left(\frac{d}{dt}\right)t\left(\frac{d}{dt}\right) + \frac{k^2}{t}, \qquad k > 1/2$$

on the interval $(0, \infty)$, show that:

(a) The operator $T_0(\tau)$ has no boundary values, and $T_1(\tau)$ is a self adjoint operator.

(b) The operator $T_1(\tau)$ has no point spectrum, and has a continuous spectrum covering the entire positive real axis.

(c) Let U be the isometry of $L_2(0, \infty)$ into itself given by

$$(Uf)(t) = (2t)^{1/2}f(t^2)$$

and let H_{2k} denote the Hankel transform

$$(H_{2k}f)(\lambda) = \underset{N\to\infty}{\text{l.i.m.}} \int_0^N (\lambda s)^{1/2}J_{2k}(\lambda s)f(s)ds,$$

where $J_{2k}(t)$ is a solution of Bessel's equation

$$\left(\frac{d}{dt}\right)^2 f + \left(\frac{1}{t}\right)\left(\frac{d}{dt}\right) f + (1-(2k)^{1/2}t^{-2}) = 0,$$

$$J_{2k}(t) = \frac{1}{2^{4k}\Gamma(2k+1)} e^{-it}t^{2k}\Phi(2k+\tfrac{1}{2}, 1+4k; 2it).$$

Show that the isomorphism $U^{-1}H_{2k}U$ yields the spectral resolution of the operator $T_1(\tau)$.

I2 Given the formally self adjoint formal differential operator

$$\tau = -\frac{d}{dt}t\frac{d}{dt} + t + \frac{b^2}{t}, \qquad b^2 > \tfrac{1}{4},$$

on the interval $(0, \infty)$, show that:

(a) The operator has no boundary values and $T_1(\tau)$ is a self adjoint extension of $T_0(\tau)$.

(b) $T_1(\tau)$ has a pure point spectrum with eigenvalues at the points $1+2n+2b$ $(n = 1, 2, \ldots)$.

(c) Letting

$$\psi_n(t) = \frac{2^b}{2b\Gamma(2b)}\left(\frac{\pi\Gamma(1+n+2b)}{n!}\right)^{1/2} e^{-t}t^b\Phi(-n, 1+2b; 2t), \qquad n = 1, 2, \ldots,$$

the set $\{\psi_n\}$ forms a complete orthonormal system in $L_2(0, \infty)$.

(d) If a function $f \in \mathfrak{D}(T_1(\tau))$ is expressed in the form

$$f(t) = \sum_{n=1}^{\infty} \psi_n(t) \int_0^{\infty} \overline{\psi_n(s)} f(s)ds,$$

then

$$(T_1(\tau)f)(t) = \sum_{n=1}^{\infty} \psi_n(t)(1+2n+2b) \int_0^{\infty} \overline{\psi_n(s)} f(s)ds.$$

I8 (The radial equation of the hydrogen atom.) Analyze the formally self adjoint differential operator

$$L = -\left(\frac{d}{dt}\right)^2 - \frac{1}{t} + \frac{k^2-\frac{1}{4}}{t^2}, \qquad k \geqq 1,$$

and show that:

(a) The operator $T_0(\tau)$ has no boundary values.

(b) The self adjoint operator $T_1(\tau)$ has a point spectrum on the negative real axis with eigenvalues located at the points

$$\lambda_n = \frac{-1}{(1+2k+2n)^2}$$

and with corresponding normalized eigenfunctions

$$\psi_n(t) = \frac{2^{k+1}}{(1+2k+2n)^{k+2}\Gamma(1+2k)}\left(\frac{\Gamma(2+2k+n)}{n!}\right)^{1/2} t^{1/2+k}$$
$$\times \exp\left(-\frac{t}{1+2k+n}\right)\Phi\left(-n, 1+2k; \left(\frac{2t}{1+2k+2n}\right)\right),$$

and a continuous spectrum covering the positive real axis.

(c) Show that the function $\sigma(t, \lambda)$ of Theorem 5.23 is given by

$$\sigma(t, \mu) = \frac{\mu^{-1}}{2\sqrt{\pi}} \frac{|\Gamma(\frac{1}{2}+k+i\mu)|}{\Gamma(1+2k)} e^{\pi\mu/2} \left(\frac{t}{\mu}\right)^{1/2+k} e^{-(it/2\mu)}$$
$$\Phi\left(\tfrac{1}{2}+k+i\mu, 1+2k; \frac{it}{\mu}\right),$$

where $\mu = 2^{-1}\lambda^{-1/2}$.

14 Given the formally symmetric formal differential operator

$$\tau = -\left(\frac{d}{dt}\right) t \left(\frac{d}{dt}\right) - t + \frac{b^2}{t}, \qquad b \geq \tfrac{1}{2},$$

on the interval $(0, \infty)$, show that:

(a) The operator τ has no boundary values and therefore the operator $T_1(\tau)$ is a self adjoint operator.

(b) The continuous spectrum of the operator $T_1(\tau)$ covers the entire real axis.

(c) Let

$$\psi(t, \lambda) = (\sqrt{2\pi})^{-1/2b}\Gamma(1+2b)e^{-(1/4)\lambda}|\Gamma(\tfrac{1}{2}+b+\tfrac{1}{2}i\lambda)|t^b$$
$$e^{-it}\Phi(\tfrac{1}{2}+b-\tfrac{1}{2}i\lambda, 1+2b; 2it),$$

and let

$$(Uf)(\lambda) = \operatorname*{l.i.m.}_{N\to\infty} \int_0^N \psi(t, \lambda)f(t)dt, \qquad f \in L_2(0, \infty).$$

Show that U is an isometric isomorphism of $L_2(0, \infty)$ onto $L_2(-\infty, \infty)$ whose inverse is given by the equation

$$(U^{-1}g)(t) = \operatorname*{l.i.m.}_{N\to\infty} \int_{-N}^{N} \psi(t, \lambda)g(\lambda)d\lambda.$$

(c) Show that the isomorphism U yields the spectral resolution of the operator $T_1(\tau)$.

15 Given the formally symmetric formal differential operator

$$\tau = -\left(\frac{d}{dt}\right) t^2 \left(\frac{d}{dt}\right) - t^2 - \tfrac{1}{4}$$

on the interval $(0, \infty)$, prove that:

(a) The operator τ has no boundary values at zero and two boundary values at infinity.

(b) Two linearly independent boundary values at infinity are given by the equations

$$B_1(f) = \lim_{t\to\infty} e^{2it}(tf(t)e^{-it})',$$

$$B_2(f) = \lim_{t\to\infty} e^{-2it}(tf(t)e^{it})'.$$

(c) Prove that the extension T_θ of the operator $T_0(\tau)$ obtained by imposing the boundary condition

$$B_\theta(f) = e^{1/2(\pi i)(1/2-\theta)} B_1(f) - e^{-1/2(\pi i)(1/2-\theta)} B_2(f) = 0, \qquad 0 < \theta < 2,$$

is a self adjoint operator.

(d) Prove that the operator T_θ has a continuous spectrum covering the positive semi-axis and eigenvalues at the points

$$-(2n+\theta)^2$$

with normalized eigenfunctions given by

$$(4n+2\theta)^{1/2}t^{-1/2}J_{2n+\theta}(t),$$

where $J_\mu(t)$ denotes the Bessel function of order μ:

$$J_\mu(t) = \frac{1}{2^\mu\Gamma(1+\mu)} e^{-it}t^\mu\Phi(\mu+\tfrac{1}{2};\, 2\mu+1;\, 2it).$$

(e) Let

$$\psi(t,\mu) = t^{-1/2}(2\pi)^{-1}|\Gamma(1+i\mu)|\left[\sinh\left[\left(\frac{\pi}{2}\right)(\mu+i\theta)\right]J_{i\mu}(t)\right.$$
$$\left.+\sinh\left[\left(\frac{\pi}{2}\right)(\mu-i\theta)\right]J_{-i\mu}(t)\right],$$

and let E be the resolution of the identity for the operator T_θ. Then the operator

$$(Uf)(\lambda) = \underset{N\to\infty}{\mathrm{l.i.m.}} \int_0^N \psi(t,\mu)f(t)\,dt$$

is an isometric isomorphism of $E([0,\infty))L_2(0,\infty)$ onto $L_2(0,\infty)$ whose inverse is given by

$$(U^{-1}g)(t) = \underset{N\to\infty}{\mathrm{l.i.m.}} \int_0^N \psi(t,\mu)g(\mu)\,d\mu.$$

(f) Show that the operator U gives a spectral representation of the Hilbert space $E([0, \infty))L_2(0, \infty)$ relative to the restriction of the operator T_θ to this space.

I6 Given the formally symmetric formal differential operator

$$\tau = -\left(\frac{d}{dt}\right) t^2 \left(\frac{d}{dt}\right) + t^2 + bt, \qquad b \geqq 0,$$

on the interval $(0, \infty)$,

(a) show that the operator τ has no boundary values, and hence that the closure of the operator $T_0(\tau)$ defines a unique self adjoint operator $T_1(\tau)$;

(b) show that the spectrum of the operator $T_1(\tau)$ consists of a continuous spectrum on the half-line $[\frac{1}{4}, \infty)$;

(c) putting $\nu = \sqrt{4\lambda - 1}$, and taking as determining set the functions

$$f_1(t) = t^{(-1+i\nu)/2} e^{-t} \Phi\left(\frac{1+b+i\nu}{2}, 1+i\nu; 2t\right),$$

$$f_2(t) = t^{(-1-i\nu)/2} e^{-t} \Phi\left(\frac{1+b-i\nu}{2}, 1-i\nu; 2t\right),$$

show that the spectral density matrix of Theorem 5.23 is given by

$$h_{ij}(d\lambda) = \frac{1}{2\pi\nu}\begin{bmatrix} 1 & c \\ c^{-1} & 1 \end{bmatrix} d\lambda \quad \text{where} \quad c = -2^{i\nu} \frac{\Gamma\left(\frac{1+b-i\nu}{2}\right)\Gamma(1+i\nu)}{\Gamma\left(\frac{1+b+i\nu}{2}\right)\Gamma(1-i\nu)}.$$

I7 Given the formally symmetric formal differential operator

$$\tau = -(t^2+1)^{-1/2}\left(\frac{d}{dt}\right) t^2 \left(\frac{d}{dt}\right) (t^2+1)^{-1/2}$$

on the interval $(0, \infty)$, show that:

(a) τ is a positive operator with no boundary values.

(b) The spectrum of the operator $T_1(\tau)$ consists of a continuous spectrum covering the positive semiaxis.

(c) The multiplicity of the continuous spectrum of $T_1(\tau)$ is one for $0 \leqq \lambda < \frac{1}{4}$ and two for $\frac{1}{4} < \lambda < \infty$.

(d) Let E be the resolution of the identity for the operator $T_1(\tau)$, and let

$$\psi(t, k) = \sqrt{\frac{k}{\pi}}\,\frac{\Gamma(\frac{1}{2}+k)}{\Gamma(1+2k)}\,2^{2k}(t+t^{-1})^{1/2}J_k(t\sqrt{\tfrac{1}{4}-k^2}),$$

where

$$k = \sqrt{\tfrac{1}{4}-\lambda}.$$

Show that the operator

$$(Uf)(\lambda) = \operatorname*{l.i.m.}_{N\to\infty}\int_0^N \psi(t, k)f(t)dt$$

is an isometric isomorphism of $E(0, \frac{1}{4})L_2(0, \infty)$ onto $L_2(0, \frac{1}{2})$ whose inverse is given by

$$(U^{-1}g)(t) = \operatorname*{l.i.m.}_{N\to 1/2}\int_0^N \psi(t, k)g(k)dk.$$

(e) Taking as determining set the functions

$$f_1(t) = (t^2+1)^{1/2}t^{-1/2+ir}e^{-i\sqrt{\lambda}t}\Phi(\tfrac{1}{2}-ir, 1-2ir; 2it\mu),$$
$$f_2(t) = (t^2+1)^{1/2}t^{-1/2-ir}e^{it\sqrt{\lambda}}\Phi(\tfrac{1}{2}-ir, 1\ \ 2ir; -2it\mu),$$

where

$$r = \sqrt{\lambda-\tfrac{1}{4}},$$

show that the spectral density matrix of Theorem 5.23 is given by

$$h_{ij}(d\lambda) = \frac{1}{2\pi}\begin{bmatrix} 1 & \left(\sqrt{\frac{\lambda}{2}}\right)^{2ir}\frac{\Gamma(1-ir)}{\Gamma(1+ir)}e^{-\pi r} \\ \left(\sqrt{\frac{\lambda}{2}}\right)^{-2ir}\frac{\Gamma(1+ir)}{\Gamma(1-ir)}e^{-\pi r} & 1 \end{bmatrix} dr.$$

18 Given the formally symmetric formal differential operator

$$\tau = -(1+t^2)^{-1/2}\left(\frac{d}{dt}\right)t^2\left(\frac{d}{dt}\right)(1+t^2)^{-1/2}-(\tfrac{1}{4})(1+t^2)^{-1},$$

on the interval $(0, \infty)$:

(a) Show that τ has no boundary values.

(b) Show that the spectrum of the self adjoint operator $T_1(\tau)$

consists of a continuous spectrum extending over the positive semi-axis.

(c) Putting $\mu = \lambda^{1/2}$, and taking as determining set the functions

$$f_1(t) = (1+t^2)^{1/2} t^{-1/2+i\mu} e^{-it\mu} \Phi(\tfrac{1}{2}+i\mu, 1+2i\mu; 2it\mu),$$
$$f_2(t) = (1+t^2)^{1/2} t^{-1/2-i\mu} e^{it\mu} \Phi(\tfrac{1}{2}-i\mu, 1-2i\mu; -2it\mu),$$

show that the density matrix of Theorem 5.23 is given by

$$\{h_{ij}(d\lambda)\} = \begin{bmatrix} 1 & \dfrac{\Gamma(\frac{1}{2}-i\mu)\Gamma(1+2i\mu)}{\Gamma(1-2i\mu)\Gamma(\frac{1}{2}+i\mu)} (2\mu)^{-2i\mu} e^{-\pi\mu} \\ (2\mu)^{2i\mu} e^{-\pi\mu} \dfrac{\Gamma(\frac{1}{2}+i\mu)\Gamma(1-2i\mu)}{\Gamma(1+2i\mu)\Gamma(\frac{1}{2}-i\mu)} & 1 \end{bmatrix} d\mu.$$

I9 Given the formally symmetric formal differential operator

$$\tau = -\left(\frac{d}{dt}\right) t^2 \left(\frac{d}{dt}\right) + bt$$

on the interval $(0, \infty)$:

(a) Show that the operator τ has no boundary values.

(b) Show that the spectrum of the operator $T_1(\tau)$ consists of a continuous spectrum on the semiaxis $[\frac{1}{4}, \infty)$.

(c) Putting $c = b^{1/2}$, $s = t^{1/2}$, $\mu = \sqrt{4\lambda - 1}$, and taking as determining set the functions

$$f_1(t) = e^{-2cs} s^{(-1+i\mu)} \Phi\big((\tfrac{1}{2}) + i\mu, 1+2i\mu; 4cs\big)$$
$$f_2(t) = e^{-2cs} s^{(-1-i\mu)} \Phi\big((\tfrac{1}{2}) - i\mu, 1-2i\mu; 4cs\big)$$

calculate the density matrix as in Theorem 5.23.

I10 Given the formally symmetric formal differential operator

$$\tau = \left(\frac{d}{dt}\right)^2 + e^{kt}, \qquad k > 0,$$

(a) determine specific expressions for the boundary values at infinity,

(b) show that τ has no boundary values at $-\infty$,

(c) give a complete spectral analysis of the operator in the interval $[0, \infty)$.

(d) give a complete spectral analysis of the operator in the interval $(-\infty, 0]$.

(e) give a complete spectral analysis of the operator in the interval $(-\infty, +\infty)$.

(Hint: The solutions can be expressed in terms of the hypergeometric function.)

J. *Miscellaneous Problems*

J1 Consider the formal differential operators

$$\tau_0 = (-1)^n \left(\frac{d}{dt}\right)^{2n}$$

$$\tau = \tau_0 + q_{2n-2}(t)\left(\frac{d}{dt}\right)^{2n-2} + q_{2n-3}(t)\left(\frac{d}{dt}\right)^{2n-3} + \dots + q_0(t)$$

on a compact interval $I = [a, b]$. Show that:

(a) The equation $\tau\sigma + \mu^{2n}\sigma = 0$ has a set $\sigma_1, \sigma_2, \dots \sigma_{2n}$ of solutions obeying the asymptotic relations.

$$\sigma_j(t, \mu) = e^{i\omega_j\mu t}\big(1 + O(\mu^{-1})\big) \text{ as } \mu \to \infty,$$

for $j = 1, 2, \dots, 2n$, uniformly for t in I, where ω_j are the various roots of the equation $\omega^{2n} = -1$.

(b) If

$$\tau\sigma(t, \mu) + \mu^{2n}\sigma(t, \mu) = 0, \qquad t \in I, \qquad \mu > 0,$$

and $|\sigma(\cdot, \lambda)|_2$ remains bounded as $\mu \to \infty$, then $|\sigma(t, \mu)| \to 0$ as $\mu \to \infty$, uniformly for t in any closed subinterval J of the interior of I.

(c) For a square-integrable function h the equations

$$\tau_0 h_0 + \mu^{2n} h_0 = h,$$

and

$$\tau h + \mu^{2n} h = h,$$

respectively, have solutions $h_0(t, \mu)$ and $h(t, \mu)$ satisfying

$$\lim_{\mu\to\infty} |h_0(t,\mu)-h(t,\mu)| = 0$$

uniformly for t in I, while

$$|h_0(\cdot,\mu)|_2 + |h(\cdot,\mu)|_2 = O(\mu^{-2n})$$

as $\mu \to \infty$.

(Hint: Let $\omega_1, \omega_2, \ldots, \omega_{2n}$ be enumerated in such a way that $\mathscr{R}i\omega_j < 0$ for $j \leq n$, and $\mathscr{R}i\omega_j > 0$ for $j > n$. Then the inhomogeneous differential equation

$$(\tau + \mu^{2n})f = g$$

may be solved by the formula

$$f(t) = \sigma(t) - \frac{1}{2n}\mu^{1-2n}\left[\sum_{j=1}^{n}\int_a^t i\omega_j e^{i\omega_j\mu(t-s)}g(s)ds \right.$$
$$\left. \sum_{j=n+1}^{2n}\int_t^b i\omega_j e^{i\omega_j\mu(t-s)}g(s)ds\right],$$

where σ denotes any solution of the homogeneous equation

$$(\tau + \mu^{2n})\sigma = 0.$$

Using this fact, construct σ_j, h, and h_0 as the solutions of appropriate integral equations.

J2 (*General Summability Principle*). Let the hypotheses of Theorem 5.28 be satisfied, and suppose in addition that τ is bounded below, that $\Omega \subseteq \sigma(T)$, and that f is a square-integrable function. Show that:

(a) For λ_0 sufficiently large

$$\int_{-\infty}^{\infty}\left[(\lambda_0+\lambda)^{-1}\sum_{i,j=1}^{k}(Vf)_i(\lambda)\sigma_j(t,\lambda)h_{ij}(d\lambda)\right]$$
$$= \lim_{a\to\infty}\int_{-\infty}^{\infty}\left[(\lambda_0+\lambda)^{-1}\sum_{i,j=1}^{k}(Vf)_i(\lambda)\sigma_j(t,\lambda)h_{ij}(d\lambda)\right]$$

exists uniformly for t in any compact subinterval of I.

(b) We have

$$\lim_{\lambda_0\to\infty}\int_{-\infty}^{\infty}\left[\lambda_0(\lambda_0+\lambda)^{-1}\sum_{i,j=1}^{k}(Vf)_i(\lambda)\sigma_j(t,\lambda)h_{ij}(d\lambda)\right]$$
$$= \lim_{\lambda_0\to\infty}\big(-\lambda_0 R(-\lambda_0; T)f\big)(t) = f(t)$$

at all points t interior to I for which

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{t-\varepsilon}^{t+\varepsilon}|f(s)-f(t)|ds = 0.$$

(c) If $\hat{\tau}$ is any operator defined on a subinterval $\hat{I}$ of I, if $\hat{\tau}$ is bounded below, and if $\hat{T}$ is a self adjoint extension of $T_0(\hat{\tau})$, then

$$\lim_{\lambda_0\to\infty}\left[(-\lambda_0 R(-\lambda_0; \hat{T})(f|\hat{I}))(t)+(\lambda_0 R(-\lambda_0; T)f)(t)\right] = 0$$

uniformly for t in any compact subinterval of the interior of I.

(Hint: Use the preceding exercise to establish (c), and deduce (b) from (c)).

J3 Let τ be a formally self adjoint formal differential operator on an interval I, and let T be a self adjoint extension of $T_0(\tau)$. Let $\{f_n\}$ be an orthonormal family of eigenfunctions of T, i.e., $Tf_n = \lambda_n f_n$, and suppose that the set λ_n is bounded. Then, for each square-integrable function f, the series

$$\sum_{n=1}^{\infty}(f, f_n)f_n(t)$$

converges absolutely and uniformly on every bounded closed subinterval I, and may be differentiated arbitarily often under the sign of summation, the differentiated series also converging absolutely and uniformly.

J4 Suppose that the differential operator τ on the interval $[a, b)$ satisfies the conditions given in Theorem 7.16(d). Let h be any measure obtained in the inversion formulas of Theorem 5.23. Show that h is absolutely continuous with respect to Lebesgue measure.

J5 Consider the differential operator

$$\tau = t\left(\frac{d}{dt}\right)$$

on the interval $[1, \infty)$. Show that the essential spectrum of τ is the strip $0 < \mathscr{R}(\lambda) \leq 1$.

(Hint: Theorem 7.5 and the fact that $\sigma_e(\tau^*) = \overline{\sigma_e(\tau)}$.)

J6 Show that the formally symmetric formal differential operator

$$\tau = (1+t)\left(\frac{d}{dt}\right)(t+1)\left(\left(\frac{d}{dt}\right)^2 - 1\right)(t+1)\left(\frac{d}{dt}\right)(1+t)$$

on the interval $[0, \infty)$ has two boundary values at infinity.

J7 In various problems of applied mathematics one is led to consider generalized "eigenvalue problems" of the form

$$\tau f = \lambda r f,$$

on an interval I, where τ is a formally symmetric formal differential operator of order n defined on I, and where r is a positive function which is infinitely often differentiable on I.

(a) Let $L_2(I, r)$ denote the set of measurable functions defined on I for which

$$|f|^2 = \int_I r(t)|f(t)|^2 dt < \infty,$$

and for f, g in $L_2(I, r)$ and set

$$(f, g)_r = \int_I r(t)f(t)\overline{g(t)}dt.$$

Let $T_1(r^{-1}\tau, r)$ denote the operator defined by

$$\mathfrak{D}(T_1(r^{-1}\tau, r)) = \{f \in L_2(I, r) | f \in A^{(n)}(I), r^{-1}\tau f \in L_2(I, r)\}$$
$$(T_1(r^{-1}\tau, r)f)(t) = r^{-1}(t)(\tau f)(t).$$

Show that the operator

$$T_0(r^{-1}\tau): L_2(I, r) \to L_2(I, r)$$

is symmetric, and that its adjoint is $T_1(r^{-1}\tau, r)$.

(b) Let T be a self adjoint restriction of $T(r^{-1}\tau, r)$. Show that, if l denotes the mapping $(lf)(t) = r^{1/2}(t)f(t)$ of $L_2(I)$ into $L_2(I, r)$, then lTl^{-1} is a self adjoint restriction of $T_1(r^{-1/2}\tau r^{-1/2})$. Show that the converse of this statement is also valid.

(c) Assume the hypotheses of (b), and let $E(\cdot\,; T)$ and $E(\cdot\,; lTl^{-1})$ denote the spectral resolutions of T and lTl^{-1} respectively. Then $lE(\cdot\,; T)l^{-1} = E(\cdot\,; lTl^{-1})$.

(d) The operator $T_0(r^{-1/2}\tau r^{-1/2})$ is bounded below as an operator in $L_2(I, r)$ if and only if $T_0(r^{-1/2}\tau r^{-1/2})$ is bounded below as an operator in $L_2(I)$. Suppose that $T_0(r^{-1}\tau)$ is bounded below, and let T be its Friedrichs extension (in the sense of Theorem XII.5.2 and Corollary

XII.5.8). Let $\lambda_n(T)$ be the numbers of Exercise D2 as defined for the operator T. Then

$$\lambda_n(T) = \sup_{\mathfrak{H}_0 \in \mathcal{G}^{(n-1)}} \inf_{\substack{(f, \mathfrak{H}_0) = 0 \\ 0 \neq f \in \mathfrak{D}(T_0(\tau))}} \frac{(T_0(\tau)f, f)}{\int_I r(t)|f(t)|^2 dt}.$$

(e) Let τ be a differential operator of the second order, of the form

$$\tau = \left(\frac{d}{dt}\right) p(t) \left(\frac{d}{dt}\right) + q(t)$$

where p is positive. Then $T_0(r^{-1}\tau)$ is bounded below (as an operator in $L_2(I, r)$) if and only if there exists a λ sufficiently large so that the solutions of the equation $(\lambda r - \tau)f = 0$ have only a finite number k of zeros in I. Suppose that such a λ exists, and let T be a self adjoint extension of $T_0(r^{-1}\tau)$. Then T has at least $k-1$ eigenvalues (counted according to multiplicity) in the interval $(-\infty, -\lambda)$.

(f) (J. Berkowitz). Let τ be of the form described in (e), and suppose that $(r(t))^{-1}q(t)$ is bounded below. Suppose that the interval I is of the form $[a, b)$. Then the operator $T_0(r^{-1/2}\tau r^{-1/2})$ (considered as an operator in $L_2(I)$) is bounded below, and its essential spectrum is a subset of the interval $[c, \infty)$, where

$$c = \liminf_{t \to b} (r(t))^{-1} q(t).$$

J8 Let $\tau = -(d/dt)^2 + q(t)$ on the interval $[0, \infty)$, and let $q(t) \to \infty$ as $t \to \infty$. Let T be the restriction of the operator $T_1(\tau)$ defined by the boundary condition $f(0) = 0$.

(a) Show that T is self adjoint, and has a spectrum consisting of an infinite sequence of simple eigenvalues $\lambda_n(q)$ converging to infinity.

(b) Let $N(\mu, q)$ be the number of eigenvalues $\lambda_n(q)$ in the interval $(-\infty, \mu]$. Show that

$$\mu^{1/2} = o(N(\mu, q)) \qquad \text{as } \mu \to \infty.$$

(c) Suppose that

$$t^{1/\varepsilon} = o(q(t)) \quad \text{as } t \to \infty.$$

Show that

$$N(\mu, q) = O(\mu^{1/2+\varepsilon}) \qquad \text{as } \mu \to \infty.$$

J9 Let τ be a formally self adjoint formal differential operator of order n defined on an interval I. Suppose that τ is bounded below, and $\sigma_1(\tau)$ is void. Let T be a self adjoint extension of $T_0(\tau)$, and let $N(\mu, T)$ be the number of eigenvalues of T in the interval $(-\infty, \mu)$. Show that

$$\mu^{1/n} = O(N(\mu, T)) \qquad \text{as } \mu \to \infty.$$

If the interval I is infinite, and the leading coefficient of τ is bounded above, or if the leading coefficient of τ approaches zero at one of the free endpoints of I, show that the stronger assertion

$$\mu^{1/n} = o(N(\mu, T))$$

is valid.

10. Notes and Remarks

A. *Historical remarks.* The spectral theory of self adjoint operators which has been developed in Chapters X and XII reveals much of its power in its application to the study of linear differential operators. Such concepts as that of an expansion into the eigenfunctions of a differential operator of second order, a boundary value for a singular differential operator, or the generalized "Fourier Integral" of Section 5 have been so fully understood since their connection with the geometry of Hilbert space was discovered that it may seem paradoxical that the eigenvalue theory of a Sturm-Liouville operator should have been chronologically the first approach to spectral theory, even before the dawn of matrix theory.

The idea of decomposing a function into a linear combination of simpler functions dates back to the eighteenth century. We owe to Daniel Bernoulli the first glimmerings of the possibility of expanding a continuous function defined on a finite interval into a series of eigenfunctions. Indeed, his treatment of the problem of the vibrating string led him to the first formulation of what was later to be called the principle of superposition (1753), which became the inspiration for much of the nineteenth century work on differential operators of the second order and on harmonic analysis. To Daniel Bernoulli we owe a reawakening of interest in the problem of the vibrating string,

which had lain dormant since it was first attacked by B. Taylor in 1713.

Spurred on by the work of Bernoulli and by related investigations of Euler and d'Alembert, Lagrange, in 1779, was the first to apply to the equation of the string the idea of approximating the solution of a linear differential equation by the solutions of a finite system of linear equations, thus using an idea that would later become one of the basic principles of functional analysis. Lagrange replaced a continuous distribution of mass in the string by a finite number of equally spaced point masses.

A number of expansions in terms of what is now known as a system of orthogonal functions, such as spherical functions and Legendre polynomials, were known in the eighteenth century, but a systematic approach to the problem of expansions in eigenfunctions of an arbitrary self adjoint second order operator was not made until 1830, when, almost simultaneously, J. C. F. Sturm [1, 2] and J. Liouville [1, 2] developed, in a series of four memoirs in Liouville's Journal, a systematic and elegant theory of the differential operator

$$\left(\frac{d}{dt}\right) p(t) \left(\frac{d}{dt}\right) + q(t),$$

where q is a real and p a positive twice differentiable function on a finite closed interval. Their work, while it may now appear to be limited by the lack of a proper theory of integration and by the use of some arguments of heuristic rather than strictly logical character (which were not made rigorous until 1905), nevertheless already contains practically all the characteristic features of the theory of expansion in eigenfunctions of a differential operator on a closed interval, such as the existence of an infinite sequence of eigenvalues with no finite cluster point, the orthogonality property of the eigenfunctions and the Parseval equality.

To Sturm we also owe the theory of oscillation of the solutions of a differential operator of second order; his renowned "oscillation" and "comparison" theorems have revealed their usefulness in the analysis of the most disparate problems, not the least of which is the problem of the location of spectra of singular operators, discussed in the latter part of Section 7 and later in the present section of notes and remarks.

The researches of Sturm and Liouville were followed by a great deal of work in the theory of special functions, which we shall briefly touch upon later. However, the "spectral" theory which they had discovered was to wait until the first decade of this century before it was actively taken up again. It was Dini [1] who, in his studies on expansions in Fourier, Bessel and spherical series, inaugurated, in 1880 and carried forward until 1910, the study of convergence along modern lines. Dini gave the first criterion of equiconvergence of Fourier series and Sturm-Liouville series. In 1905, A. C. Dixon [1] gave a first rigorous proof of the existence of an infinite discrete set of eigenvalues. Dixon's work was almost contemporary with that of Kneser [1, 2, 3, 4], which relaxed the requirements of differentiability of the coefficients. Shortly thereafter, with the advent of the Lebesgue integral, E. W. Hobson (1908, [2]) and W. Stekloff [1] studied the convergence properties of the expansion of an arbitrary integrable function in terms of the eigenvalues of a Sturm-Liouville operator. Hobson succeeded in proving a "principle of localization" similar to the corresponding principle for trigonometrical series. A. Haar (1910–11, [8]) continued the investigation, and concluded with a general principle permitting the comparison of a Sturm-Liouville series with a trigonometrical series. In the same year M. Picone [1] published a particularly elementary proof of Sturm's comparison theorems.

In the meanwhile the problem of extending the Sturmian theory to differential operators of higher order or of more general type had been receiving attention. Already in the dissertation of Westfall (1905, [1]) we find an extension of the expansion theory to real self adjoint differential operators of even order on a closed interval, but the most comprehensive theory of these operators is due to G. D. Birkhoff [1 through 7], who, in an exhaustive series of papers starting in 1908, constructs a theory of expansion in terms of biorthogonal systems obtained from a (not necessarily self adjoint) differential operator of arbitrary order on a closed interval. Some of the work of E. Hilb [2, 3], M. Bôcher [3, 4, 5] and J. Tamarkin [2, 3] in this period also deals with particular cases of the problems dealt with by G. D. Birkhoff. A modern version of Birkhoff's theory will be given in Chapter XIX.

After the thorough investigations of these authors the interest in differential operators on finite closed intervals slowly wanes. The contemporary research of Hellinger [1], E. Schmidt [1, 2] and Hilbert [1] on integral equations naturally led to an interest in a similar approach to differential operators whose coefficients are singular at one or both of the endpoints of the interval of definition.

The exploratory work of Hilb [2] in 1909 deals with differential operators of the form

$$(\tau f)(t) = \left(\frac{d}{dt}\right) t \left(\frac{d}{dt}\right) f(t) + [g(t) + \lambda h(t)]t^{-1} f(t)$$

on the interval (0, 1], where g and h are assumed to be analytic. The reduction to an integral equation via the Green's kernel is used, and then the theory of integral equations is applied. To Hilb we also owe the first remark on the fundamental dichotomy arising in the boundary value theory of a second order differential operator with real coefficients, exhibited by the definite or indefinite character of the eigenvalue problem obtained by imposing one boundary condition at the non-singular endpoint.

It was Hermann Weyl, however, who brought together these loose threads and developed a unified and far-ranging theory of the most general singular formally self adjoint differential operator. Weyl's two papers in the *Göttinger Nachrichten* [7, 8], and particularly his memoir in the 68th volume of the *Mathematische Annalen* [5] brought to light an exhaustive and systematic theory which, while embracing all the preceding discoveries, would remain for years to come one of the stepping stones of linear analysis.

Weyl's memoir is remarkable in several respects. With the "Fourier integral" representation of a square-integrable function in terms of the "eigendifferentials" of singular differential operators it opens a new horizon for the newly discovered spectral theorem of Hilbert, which, moreover, is applied here to a first example of what is now known as an unbounded operator. In fact, many of the structural properties of the closed self adjoint unbounded operator, such as the existence of "deficiency indices" and their invariance (Theorem XII.4.19 in this book), as well as the corresponding extension theory, make their first appearance in this memoir, in Weyl's use of the

ingenious geometric method of the contracting circles. In Weyl's work we also find the first examples of criteria permitting the determination of the continuous spectrum and the number of boundary values of a differential operator by direct inspection of the singularities of the coefficients, which thirty years later became the starting point for the work of Hartman, Wintner and their school on these same questions.

Remarkably enough, the discoveries of Weyl did not lead immediately to more extensive research. While abstract linear analysis became a fashion in the twenties, the difficulties of generalizing the limit point limit-circle method to operators of higher order obstructed any attempts to obtain a spectral theory for differential operators of higher order. The isolated attack of Windau [1] in 1921 seems to be the only one in the decade.

In the years following Weyl's work, up to the late forties, three distinct schools carry the work forward. The year 1920 marks the appearance of the wave mechanics of Schrödinger, which assigns a fundamental role to a certain singular equation. The following two decades witness the development of quantum mechanics, in which the spectral theory of differential operators of second order assumes a physically significant role. Thus the physicists of the time were naturally led to interest themselves in the theory of Weyl. But, while many of the useful insights of the new physical theories later afforded inspiration for further mathematical research, unfamiliarity with some of the analytic tools of the new analysis enveloped several concepts with an aura of mystery and awe. Square-integrability, spectral measures and continuous spectra were looked upon with a suspicion which disappeared only after the work of mathematicians such as J. von Neumann and K. Friedrichs in the thirties. A typical instance of this situation is found in a misunderstanding of the extension of the continuous spectrum: a heuristic "principle of infection" led to the belief that the continuous spectrum would always include a half-line (cf. the book of Dirac [1]).

Among mathematicians we find two lines of investigation which lead us to the final systematization which occurred in the last eight years. As early as 1915, Hilb [4] gave a proof of the fundamental inversion formula of Weyl which dispensed with the machinery of

singular integral equations in favor of a wide use of function theory. The formula was obtained by an argument on the residues of Green's kernel. The same idea was taken up by Titchmarsh [5] in 1938. Reducing his use of functional analysis to a minimum, Titchmarsh undertook in a long series of papers [5 through 16] the task of calculating the spectral measure of a singular differential operator by an exclusive use of residue arguments. His work culminated with the derivation of a version of the fundamental formula 5.18 for a differential operator of the second order.

In the first thirty years following the publication of Weyl's memoirs, the development of the spectral theory of differential operators in the spirit of functional analysis proceeded rather slowly. A paper of M. H. Stone [17] in 1926 studies the case of the operator

$$\left(\frac{d}{dt}\right)^2 + q(t)$$

with q integrable, and derives in particular the formulas for the Hankel transform. Several other papers of Stone in this period [18, 19, 20] treat the more delicate questions of convergence of the series expansions in the theory of Birkhoff. A later book by the same author [3] contains a sketch of the application of the spectral theorem in Hilbert space to differential operators. A few years later, in 1937, Halperin [5] published a first study of the closure and adjoint properties of unbounded operators in Hilbert space obtained from ordinary differential operators. Almost simultaneously, Calkin [1, 3] proposed the notion of an abstract boundary value, whose usefulness we have tried to make apparent in the development of this chapter. In the same period K. Friedrichs [8, 5, 7] carried forward Weyl's program.

In recent years, many mathematicians have worked on various aspects of the spectral theory of self adjoint differential operators. Following a suggestion of H. Weyl, K. Kodaira in 1949 undertook to unify the results of Weyl and Titchmarsh by the methods of functional analysis. His work [1, 4, 5] completes the theory of operators of even order, and presents several points of contact with the development given in Section 5 of this chapter, the main divergence being Kodaira's use of a method of contracting hypersurfaces generalizing

Weyl's method of contracting circles. K. Yosida [10] in 1950 threw further light upon the formulas of Titchmarsh and Kodaira. Levinson [5, 6] produced a simplified proof of the expansion theorem. Coddington (1954, [2]) proved the expansion theorem by a different method and proved the uniqueness of the spectral matrix. An exposition of the development of the theory along the lines developed by them can be found in the recent book of these authors [1].

Starting in 1947 and continuing to the present day, numerous contributions to the analysis of various spectral problems connected with the operator of the second order have been given by P. Hartman [1 through 16], A. Wintner [1 through 20], Hartman and Wintner. [1 through 21] and their school (Putnam [1 through 18], Hartman and Putnam [1, 2], Wallach [1, 2], Wolfson [1, 2]). The problem of determining the essential spectrum and the number of boundary values for a second order operator was studied in a long series of notes in the *American Journal of Mathematics*. Some of their results are generalized in Sections 6 and 7, others are summarized in paragraphs C and D of this section.

In Russia, interest in the theory started with a paper of M. G. Krein [12] in 1948, where what he calls the "method of directing functionals" is used to obtain transparent proofs of the fundamental theorems. This method seems to have been the first to do without the use of procedures of approximation from finite to infinite intervals, and is strong enough to be applicable to operators of any order. Another proof of the expansion theorems—which, however, omits the key fact of the uniqueness of the matrix measure—is due to B. M. Levitan [4]. In the dissertation of I. M. Glazman in 1949 the problems connected with boundary values of singular operators are finally settled in a particularly simple way, without appealing to the geometric method of contracting hypersurfaces.

In recent years the Russian school has prospered. It suffices to refer the reader to the papers of Berezanskiĭ [1, 2], Dorodnicyn [1], Fage [1, 2], Gelfand and Kostyučenko [1] Gelfand and Levitan [1, 2], Karaseva [1] Krein [10 through 18], Krein, Krasnosel'skiĭ and Milman [1], Levitan [1 through 7], Lidskiĭ [1], Livšic [5], Marčenko [1, 2], Molčanov [1], Naĭmark [4, 9, 10, 11, 12], Nemyčkiĭ [2], Povzner [5, 6, 7, 8], Rapoport [1, 2], Šnol [1] and

Staševskaya [1], as well as to the books of Ahiezer and Glazman [1], Levitan [2], and Naĭmark [5].

B. *Comments on the text. Section 1.* The literature on the solution of linear differential equations by the method of successive approximations is very extensive, and runs beyond the scope of these notes. We refer the reader to any standard treatise on differential equations for further references.

The idea used in Theorem 5 is due to Peano ([1], 1888), who developed a matrix calculus for this express purpose. See also the paper of Baker [1]. The method is called by the names of these two authors.

Section 2. The first explicit calculation of the formal adjoint of a linear differential operator was published by Lagrange [2] in 1765 in the *Miscellanea Tauriniensia*. The term "adjoint" was first used by L. Fuchs [1] in 1873.

It is difficult to establish the origin of the various formal expressions connected with adjoint operators and Green's formolas. The interested reader will find more detailed treatments in the paper of E. Borel [1], in the book of M. Bôcher [5], or in the third volume of Darboux's *Théorie des Surfaces*. The same may be said of the analytical machinery connected with the calculation of the adjoint of differential operators in Hilbert space. Von Neumann's paper in 1929 [7] takes the first steps in the case of a Sturm-Liouville operator. Stone's book [3] in 1932 contains a slightly expanded account. A more systematic treatment is to be found in Halperin's paper [5] in 1937. The analytical device of Lemma 3.16 is to be found in a recent paper of J. T. Schwartz [2].

After the publication of Weyl's memoir and until the publication of the thesis of Glazman, all attempts at generalizations of the algebra of boundary values (which extensions are required in order to apply to formally symmetric differential operators the theory of von Neumann) were concerned with an extension of Weyl's geometric formalism to surfaces of higher order. Along this line we notice the paper of Windau [1] and a series of works by D. Šin [1, 2, 3]. Insistence on carrying out this program even led to the belief that a differential operator of order $2n$ would always have either zero or n

boundary values at each singular endpoint, a conjecture which was first disproved by Glazman in 1949.

The present definition of boundary value (Definition 17) was first expounded in the thesis of Calkin [1]. Many results concerning the deficiency indices, notably Corollary 22 and Theorem 27, are due to Glazman [1, 2], whose methods of proof are, however, dissimilar from the ones adopted here. The two corollaries attributed to Kodaira were first proved [1, 2, 4] by a somewhat more involved method of contracting hypersurfaces. The terminology explained in Definition 29 seems to have a physical origin. The first satisfactory discussion of the boundary values of the singular Sturm-Liouville operator was given by Weyl [5] in 1910. For an explanation of various problems connected with the determination of boundary values, the reader may refer to works of Fage [1, 2] and Feller [4–7], as well as to the standard works of Ahiezer and Glazman [1] and Naĭmark [5].

A comprehensive exposition of the theory of boundary values for linear and non-linear differential operators is contained in the paper of W. M. Whyburn [1]. The algebra of boundary value problems for operators on a finite closed interval is treated in the well-known paper of Birkhoff [3]; see also Jackson [1] and Latshaw [1].

Section 3. It is beyond the scope of this brief survey to give a history of the theory of Green's kernel for differential operators on a finite interval. It suffices to say that the first published account of Green's kernel for an ordinary differential operator appeared in the paper of H. Burkhardt [1] in 1894 for the differential operator $(d/dt)^2$ on a finite interval. The generalization to differential operators of arbitrary order was carried out by M. Bôcher [2] in 1901. Three years later Hilbert [1] gave the first reduction of a differential problem to an integral problem, thus establishing the integral representation of the resolvent. Hilbert's method was later exploited by Hilb [2] and Weyl [5]. The method followed by these authors for the determination of the integral representation of the resolvent is based on an approximation of the Green's kernel for the singular operator by a sequence of Green's kernels obtained by restricting the operator to finite intervals. The method followed in this chapter seems to be new. As an alternate approach, we mention the method of Glazman,

a detailed exposition of which can be found in the book of Ahiezer and Glazman [1], as well as the approximation method of Coddington and Levinson [1].

For a detailed exposition of the problems connected with the calculation of the Green's kernel for a differential operator on a finite interval, the recent paper of E. Mohr [1] may be found valuable.

Section 4. The work of Hilbert [1] in 1904 already contains a remark to the effect that Green's kernel for an operator of the second order on a compact interval is a kernel of the Hilbert-Schmidt type. As soon as the results of Hilbert and E. Schmidt on such an integral kernel became available, the idea of obtaining the expansion theory for a differential operator on a closed interval by inverting the differential operator was used: see the two papers of Kneser [3, 4]. Later, Weyl established the more general fact that Green's kernel for a formally self adjoint differential operator of second order with four boundary values is a Hilbert-Schmidt kernel

Section 5. The historical development of the main theorems in this section has been sketched in the first section of these notes. We recall that alternate proofs of these results are due to Weyl [5] (for the operator of second order on a semi-axis), Hilb [4] (the inversion formula for the operators of Weyl), Stone [17] (operator $-(d/dt)^2+q(t)$ with q integrable and general discussion of Sturm-Liouville operator in [3]), Titchmarsh [5] (derivation of Theorem 5.18 for the operator $(d/dt)^2+q(t)$), Kodaira [1, 4, 5] (for the operators of even order), Levinson [5, 6], Coddington [2], Krein [12], Levitan [4]. Sears [6] has sketched an approach to the operators of the second order which proceeds from the L_1-theory to the L_2-theory in analogy with a classical approach to Plancherel's theorem.

The codification of the terminology connected with matrix measures seems to be due to Kodaira [4, 5]. The proof of Theorem 5.10 seems to have been first obtained by I. Kaz [1]. The first general proofs of the uniqueness Corollary 21 are due to Coddington [3] and Marčenko [1, 2].

Section 6. Most of the results given in this section have been slowly evolved from the first results of Weyl in papers appearing in the last fifteen years, notably by Titchmarsh [5 through 16], Naĭmark [4, 9, 10, 12], Hartman and Wintner, Levinson [2],

Friedrichs [3, 5, 6, 7, 10, 18] and Sears [1, 2, 5, 7]. For other papers of interest see Glazman [1 through 4], Krein [13 through 16], Lidskiĭ [1], Molčanov [1] and Šnol [1], and the recent book of Naĭmark [5].

The term "essential spectrum" was first used by Hartman and Wintner in their papers in the *American Journal of Mathematics* in 1948. The defining property used by them coincides with the property we have given in Theorem 4. The development followed in this section and the next, which makes extensive use of Definition 1, has also been used by Šnol [1] and Naĭmark [5]. A version of Lemma 7 is due to Glazman [1, 2]. A result similar to Lemma 8 can be found in the joint paper of Krein, Krasnosel'skiĭ and Milman [1]. Theorem 11 was obtained by Weyl for operators of the second order, and extended by Glazman [1]. The trick used in Theorem 14 has been utilized by Wintner in some of his notes. Theorem 15 is due to Levinson [2].

The asymptotic result given in Theorem 18 is well known: it was used by Stone in 1928 [17]. A version of Theorem 20 can be found in the work of Coddington and Levinson [1]. Theorem 23 appears in the paper of Friedrichs [1]. It was improved by Sears [1] and Berkowitz [1]. Theorem 28 is new.

A first result in the spirit of Lemma 32 is due to Esclangon [1] and Landau [2]: they assumed that the function f in the statement of the Lemma is bounded. Lemma 33 is well-known: see the book of Hardy, Littlewood and Pólya [1]. Lemma 34 is due to von Neumann: its first published proof can be found in the paper of Halperin [5]. Theorem 35 was known for operators of the second order: the general result has been obtained by Naĭmark [5].

Section 7. Most of the results given in this section are found in the literature only in the special case of operators of second order. Some of the theorems have been proved by Naĭmark [5]. In all cases the present method of proof is different.

The characterization of the essential spectrum (for self adjoint operators) given in Theorem 1 is due to Weyl [5]. Theorems 8, 9, 10 and 18 were previously known for operators of the second order (cf. the book of Titchmarsh [16]). The proof of Corollary 14 was finally achieved by Hartman after a series of successive weakenings of the assumptions on the coefficient function q. For condition (c) of

Theorem 16, see the book of Coddington and Levinson [1]. For Theorem 17 see the papers of Friedrichs [10].

For a different proof of Lemma 22 see the paper of Rellich [6]. Semi-bounded differential operators were studied by Friedrichs [3] and Rellich [6]. Lemma 23 is a special case of the fact that a gap in the essential spectrum implies equality of the deficiency indices.

Lemma 85 is the renowned "Sturm comparison theorem." The proof given here is similar to that which has been given by Picone [1]. Corollary 87 goes back to Kneser [1]. The study of the connection between the essential spectrum and the oscillatory behavior of the solutions was initiated by Hartman [8] and Putnam (Hartman and Putnam [1, 2]) (Theorems 50 to 55). The paper of Rellich [6] studies the special case of operators bounded below.

There is a vast literature on the asymptotic distribution of the eigenvalues of the Sturm-Liouville operator on a compact interval. Let it suffice to mention here the works of Birkhoff [2], Brillouin [1], Dunham [1], Kramers [1], Kemble [1], Langer [1, 2, 3], Milne [3] and Titchmarsh [16].

The study of "periodic potentials" for the operators of second order began in the physical laboratory (see, e.g., Kramers [2]). A more rigorous study was first carried out by Titchmarsh [10] and Wallach [2]. The reader interested in the physical aspects of the theory and its relations with the theory of crystal structures may profitably consult the book of Seitz [1].

Corollary 68 was given by Friedrichs [10]. The dissertation of J. Berkowitz [1] will be found useful for further exploration of this type of theorem.

Various related results on the topics treated in Sections 6 and 7 will be found in the following sections of the present collection of notes and comments.

Section 8. Most of the classical formulas connected with special functions which are used in this section can be found in the treatise of Whittaker and Watson [1]. The formalism used for the study of the hypergeometric function goes back to Riemann. The extension to the confluent hypergeometric function is new. Other studies of the spectral theory of the operators in this section are scattered through the physical literature. Some similar examples are worked out in the book of Titchmarsh [16], and in that of Ahiezer and Glazman [1].

C. *The Essential Spectrum of a Differential Operator of the Second Order.* Under this heading we shall give a relatively complete summary of the known results on the relationship between the asymptotic behavior of the coefficients of the operator

$$\tau = -\left(\frac{d}{dt}\right) p(t) \left(\frac{d}{dt}\right) + q(t)$$

and the essential spectrum of τ. The interval of definition will be $[a, b)$ or $(a, b]$, with $a > -\infty$ or $a = -\infty$ and $b < \infty$ or $b = \infty$ and this interval will be specified with each case. It is understood that when the interval of definition of τ is not specified, the statement is true in any finite or infinite interval. The functions p and q are assumed to be real and continuous, and p is assumed to be positive.

The following statements describe situations in which the essential spectrum is void:

(1) For some (real or complex) λ, the equation $(\lambda - \tau)f = 0$ has two linearly independent square-integrable solutions (6.6).

(2) For every real λ, the solutions of the equation $(\lambda - \tau)f = 0$ have only a finite number of zeros (7.89).

(3) The function p is bounded away from zero, the function q is bounded below, and the set of square-integrable functions f for which

$$\int_a^b [p(t)|f'(t)|^2 + q(t)|f(t)|^2]dt \leqq 1$$

is conditionally compact. When p is bounded away from zero and q is bounded below this condition is also necessary (Rellich [6], Naĭmark [5], Exercise 9.F1).

(4) More generally, the condition given in (3) is necessary and sufficient whenever the operator τ is bounded below (Rellich [6]).

(5) In $[a, b)$, assume:

(a) the functions p and q are positive in a neighborhood af b,

(b) $$\int_a^b \left| \left[\frac{(q(t)p(t))'}{q(t)^{3/2}p(t)^{1/2}}\right]' + \frac{1}{4}\frac{([q(t)p(t)]')^2}{(p(t))^{3/2}(q(t))^{5/2}} \right| dt < \infty,$$

(c) (6.21) for x in a neighborhood of b,

$$\int_a^b [p(t)q(t)]^{-1/2} dt < \infty.$$

(6) In the interval $[a, b)$ $(b \leqq \infty)$ assume

(a) the function g is non-negative and piecewise continuous in the interval $[0, \infty)$,

(b) the solutions of the differential equation

$$\frac{d^2 f(t)}{dt^2} + g(t)f(t) = 0 \qquad (0 \leq t < \infty)$$

have only a finite number of zeros,

(c) the function h, defined in $[a, b)$, is positive,

(d) $|h'(t)| = \dfrac{1}{p(t)}$,

(e) either

$$h(t) \to 0 \text{ or } h(t) \to \infty$$

as t tends toward b,

(f) the function Z is defined by

$$Z(t) = q(t) + \frac{g(h(t))}{p(t)}$$

or

$$Z(t) = q(t) + \frac{1}{p(t)}\left[\frac{1}{4(h(t))^2} + \frac{1}{(h(t))^2} g\left(\log \frac{1}{h(t)}\right)\right]$$

according as $h(t) \to \infty$ or $h(t) \to 0$ as t tends to b, and we have $\lim_{t \to b} \inf Z(t) = \infty$, (Berkowitz [1]).

(7) (7.66) In the interval $[a, b)$, let

$$Q(t) = q(t) + \tfrac{1}{4}[p''(t) - \tfrac{1}{4}(p(t))^{-1}(p'(t))^2]$$

and assume that

$$\lim_{t \to b} Q(t) = \infty.$$

(8) (7.67) In the interval $[a, b)$, let Q be defined as in (7), and assume that

$$\int_a^b p(t)^{-1/2} dt < \infty,$$

$$\limsup_{t \to b} \left[\int_t^b p(s)^{-1/2} ds\right] |Q(t)| < \tfrac{3}{4}.$$

Other criteria for the determination of the essential spectrum are the following:

(9) Under the assumptions (a) and (b) of (5), and with the further assumption that for all x

$$\int_x^b |p(t)q(t)|^{-1/2}dt = \infty,$$

the essential spectrum of τ is the entire real axis, (6.21).

(10) Assuming (a) through (e) of (6), and letting the function Z be as in (f), if

$$\liminf_{t\to b} Z(t) \geqq c > -\infty,$$

then the essential spectrum of τ is contained in the half-line $[c, \infty)$ (Berkowitz [1]).

(11) In the interval $[a, b)$, let Q be defined as in (7). Assume that

$$\int_a^b p(t)^{-1/2}dt = \infty, \qquad \lim_{t\to b} Q(t) = c.$$

Then the essential spectrum of τ is the half-line $[c, \infty)$ (7.66).

(12) In the interval $[a, b)$ let

$$Z(t) = q(t) + \left[4p(s)\left(\int_a^t p(s)^{-1}ds\right)^2\right]^{-1}$$

and assume that

$$\liminf_{t\to b} Z(t) = K.$$

Then the essential spectrum of τ is contained in the half-line $[K, \infty)$ (7.68).

(13) Suppose that in the interval $[a, b)$

$$\liminf_{t\to b} q(t) = K.$$

Then the essential spectrum of τ is contained in the half-line $[K, \infty)$ (7.19).

(14) In the interval $[a, b)$, let $N(t, \lambda)$ be the number of zeros of a solution of the equation $(\lambda-\tau)f = 0$ in the interval $[a, t)$ $(t < b)$. Then a point λ_0 on the real axis belongs to the essential spectrum of τ if and only if for every pair λ and μ such that $\lambda < \lambda_0 < \mu$,

$$\liminf_{t\to b}\,[N(t,\mu)-N(t,\lambda)] = \infty$$

(Hartman [8], Exercise 9.F 3).

(15) Suppose that the operator τ is bounded below, and that the function r is non-negative in the interval of definition of τ. Then the least point in the essential spectrum of τ is smaller than the least point in the essential spectrum of $\tau+r$. This assertion is not true if it is not assumed that τ is bounded below (cf. the exercises in Section 9.D).

Additional criteria are given below for the more special operator

$$\tau = -\left(\frac{d}{dt}\right)^2 + q(t)$$

where the function q is assumed to be real and continuous. The following conditions permit a complete determination of the essential spectrum of τ:

(16) If in the interval $[0, \infty)$ the function q is bounded below, and for every positive real number a,

$$\lim_{t\to\infty}\int_t^{t+a} q(s)ds = \infty,$$

then the essential spectrum of τ in $[0, \infty)$ is empty. When the function q is bounded below, this condition is also necessary (Šnol [1], Naĭmark [5], Exercise 9.G 43).

(17) In the interval $[a, \infty)$ assume that

$$\lim_{t\to b} q(t) = -\infty,$$

$$\int_a^\infty \left|\left[\frac{q'(t)}{q(t)^{3/2}}\right]' + \frac{1}{4}\frac{[q'(t)]^2}{q(t)^{5/2}}\right| dt < \infty.$$

Then,

(a) if for large x the integral

$$\int_x^\infty |q(t)|^{-1/2}dt$$

is convergent, the essential spectrum of τ is empty;

(b) if the above integral is divergent for all x in $[a, \infty)$, then the essential spectrum of τ is the entire real axis. (7.16).

(18) In the interval $[0, \infty)$, suppose that

(a) $$\lim_{t\to\infty} q(t) = \infty,$$

(b) $$\limsup_{t\to\infty} \frac{(q'(t))^2}{|q(t)|^3} < \infty,$$

(c) $$\int_M^\infty \frac{(q'(t))^2}{|q(t)|^{5/2}}\, dt < \infty,$$

for large M. Then the essential spectrum of τ is empty (Wintner [8]).

(19) In the interval $[a, \infty)$, suppose that

$$\lim_{t\to\infty} q(t) = c.$$

Then the essential spectrum of τ is the semi-axis $[c, \infty)$ (7.16).

(20) In the interval $[0, \infty)$, suppose that the function $q(t) - c$ is of class $L_p[0, \infty)$ for some p, $1 \leq p < \infty$. Then the essential spectrum of τ in the interval $[0, \infty)$ is the semi-axis $[c, \infty)$ (Naĭmark [5]).

(21) Suppose that q tends monotonically to $-\infty$ in the interval $[0, \infty)$ and that

$$q(t) = o(t^2).$$

Then the essential spectrum of τ is the entire real axis (Hartman [16]).

(22) In the interval $(0, a]$ suppose that q is negative and non-decreasing, and that

$$\lim_{t\to 0} q(t) = -\infty.$$

Then the essential spectrum of τ is void (6.27, Sears [1]).

(23) In the interval $[0, \infty)$ suppose that q tends monotonically to $-\infty$ and that for some $k > 1$,

$$\int_0^\infty |q(t)|^{-k/2} dt = \infty.$$

Then the essential spectrum of τ is the entire real axis (Hartman [16], Exercise 9.G 36).

(24) In the interval $[0, \infty)$ suppose that q tends monotonically to $-\infty$ as $t \to \infty$, and that for some integer n

$$\int_0^\infty |q(t)|^{-1/2} [\log |q(t)| \log_2 |q(t)| \ldots (\log_n |q(t)|)^{k/2}]^{-1} dt = \infty,$$

(where $\log_2 t = \log \log t$, $\log_{n+1} t = \log \log_n t$). Then the essential spectrum of τ is the entire real axis (Hartman [16]).

(25) In the interval $(0, b]$ suppose that

$$\liminf_{t\to 0} t^2 q(t) > -\tfrac{1}{4}.$$

Then the essential spectrum of τ is void, and τ is bounded below. (Use Theorem 7.34.)

(26) In the interval $(0, b]$, suppose that

$$\lim_{t\to 0} q(t) = \infty.$$

Then the essential spectrum of τ is void (7.17).

(27) In the interval $(0, b]$ suppose that

$$\limsup_{t\to 0} |t^2 q(t)| < \tfrac{3}{4}.$$

Then the essential spectrum of τ is void (7.17).

(28) In the interval $(0, b]$ suppose that

(a) $$\lim_{t\to 0} q(t) = -\infty,$$

(b) $$\int_0^x \left| \left[\frac{q'(t)}{q(t)^{3/2}} \right]' + \frac{1}{4} \frac{(q'(t))^2}{q(t)^{5/2}} \right| dt < \infty$$

for x in a neighborhood of 0, and

(c) $$\int_0^x |q(t)|^{-1/2} dt < \infty,$$

for x in a neighborhood of 0. Then the essential spectrum of τ is void (7.17).

(29) In the interval $(0, b]$, suppose that conditions (a) and (b) of (26) are satisfied, and that in addition

(c) the function q is monotone decreasing in a neighborhood of zero, and

(d) for all $x > 0$,

$$\int_0^x |q(t)|^{-1/2} dt = \infty.$$

Then the essential spectrum of τ is the entire real axis (7.17).

(30) In the interval $(0, b]$ assume that as $t \to 0$,

$$q(t) + \frac{1}{4t^2} + \frac{1}{4t^2 \log^2 t} \to \infty,$$

then the essential spectrum of τ is void (Berkowitz [1]).

Other conditions which allow the approximate determination of the essential spectrum are the following:

(31) Let $$K = \limsup_{t\to\infty} q(t) - \liminf_{t\to\infty} q(t)$$

in the interval $[0, \infty)$. Then on the positive real axis every interval of length K contains a point of the essential spectrum of τ (Glazman [4], Exercise 9.A 6).

(32) On the interval $[0, \infty)$, if $q(t)$ tends monotonically to $-\infty$, and if

$$-q(t) \leq Ct^2$$

for large t, then there exists a constant K (depending only on C) such that every interval of length K contains a point of the essential spectrum of τ (Hartman [16], Exercise 9.G 35).

(33) Suppose that q is bounded on the interval $[0, \infty)$. Let

$$v(t, \varepsilon, q) = \limsup |q(s) - q(t)|, \text{ for } |s-t| < \varepsilon$$

$$v(\varepsilon, q) = \limsup_{t\to\infty} \frac{1}{t} \int_0^t v(s, \varepsilon, q) ds.$$

Then there exists a constant $K = K(q)$ such that for large λ every interval $[\lambda, \lambda + K(v(\pi/\lambda^{1/2}, q) + 1/\lambda)]$ meets the essential spectrum of τ (Hartman and Putnam [2]).

(34) Under the assumptions of (33), suppose that $b = \infty$ and that the function q has a bounded derivative q'. Then there exists a constant $M = M(q)$ such that for large λ every interval $[\lambda, \lambda + M(1/\lambda^{1/2})(v(\pi/\lambda^{1/2}, q') + 1/\lambda)]$ meets the essential spectrum of τ (Hartman and Putnam [2]).

(35) Under the assumptions of (34) suppose that the derivatives q' and q'' are bounded. Then there exists a constant $N = N(q)$ such that for large λ every interval $[\lambda, \lambda + N(1/\lambda)(v(\pi/\lambda^{1/2}, q'') + 1/\lambda^{1/2})]$

meets the essential spectrum of τ (Hartman and Putnam [2]).

(36) Suppose the function q is twice differentiable, and let (λ, μ) be an open interval which does not meet the essential spectrum of τ but whose end points belong to the essential spectrum of τ. If

$$\int_0^t |q''(s)|ds = O(t),$$

then

$$|\lambda-\mu| = O\left(\frac{1}{\mu}\right)$$

(Hartman and Putnam [2]).

(37) Similarly, if q is three times differentiable and if

$$\int_0^t |q'''(s)|ds = O(t),$$

then

$$|\lambda-\mu| = O\left(\frac{1}{\mu^{3/2}}\right)$$

(Hartman and Putnam [2]).

(38) Suppose that q is bounded below in the interval $[0, \infty)$ and that the essential spectrum of τ is not void. For real positive λ let $d(\lambda)$ be the distance from λ to the essential spectrum of τ. Then

$$d(\lambda) = O(\sqrt{\lambda}).$$

(Putnam [12], Šnol [1]). It follows in particular that the essential spectrum of τ is either empty or unbounded (Exercise 9.G 41 and 9.G 42).

(39) Suppose the function q is bounded on the interval $[0, \infty)$, and that the essential spectrum of τ is not void. Then (cf. (38))

$$d(\lambda) = O\left(\frac{1}{\sqrt{\lambda}}\right)$$

(Šnol [1]).

(40) In the interval $[0, \infty)$ suppose that

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t |q(s)|ds = 0.$$

Then the essential spectrum of τ contains the positive semi-axis. The

inclusion may be proper (Hartman [14], Exercise 9.G 38).

(41) In the interval $[0, \infty)$, suppose that there is a sequence of intervals $[a_n, b_n]$ whose length increases beyond all bounds and for which

$$\lim_n \frac{1}{b_n - a_n} \int_{a_n}^{b_n} |q(t)|^2 dt = 0.$$

Then the essential spectrum of τ contains the positive semi-axis (Šnol [1], Exercise 9.G 40).

The following are criteria for deciding whether a particular point λ lies in the essential spectrum of τ.

(42) On the interval $[0, \infty)$, if q is bounded below and the equation $(\lambda - \tau)f = 0$ has two linearly independent bounded solutions, then λ belongs to the essential spectrum of τ (Hartman and Wintner [8], Exercise 9.G 5).

(43) On the interval $[0, \infty)$, if q is bounded below and the equation $(\lambda - \tau)f = 0$ has a bounded solution which is not square-integrable, then λ belongs to the essential spectrum of τ (Hartman and Wintner [8], Exercise 9.G 5).

(44) In $[0, \infty)$, if q is monotone and negative, and if

$$\int_0^\infty |q(t)|^{-1} dt = \infty,$$

then the origin lies in the essential spectrum of τ (Hartman and Wintner [4]).

(45) In $[0, \infty)$, if q is negative and monotone, and if there exists a sequence $\{t_n\}$ of real numbers tending to infinity for which

$$\limsup_{n\to\infty} -\frac{q(t_n)}{t_n} < \infty,$$

then the origin belongs to the essential spectrum of τ (Hartman and Wintner [4]).

(46) In $[0, \infty)$, suppose that there is a solution of the equation $(\lambda - \tau)f = 0$ which is not square-integrable and which satisfies

$$\int_0^t |f(s)| ds = O(t^2).$$

Then the point λ belongs to the essential spectrum of τ (Hartman and Wintner [14]).

(47) In $[0, \infty)$, suppose that the equation $(\lambda-\tau)f = 0$ has two linearly independent solutions f and g such that

$$\int_0^t |f'(s)|^2 ds = O(t^2) \qquad \text{and} \qquad \int_0^t |g'(s)|^2 ds = O(t^2).$$

Then the point λ belongs to the essential spectrum of τ (Hartman and Wintner [14]).

(48) Suppose that the function q is bounded below, and let f be a real solution of the equation $(\lambda-\tau)f = 0$ on $[0, \infty)$ which is not square-integrable but which satisfies

$$\int_0^t |f(s)|^2 ds = O(t^k)$$

for some $k > 0$. Then the point λ belongs to the essential spectrum of τ (Wintner [17]).

(49) Suppose that the function q is bounded below, and that for some constant $k > 0$ every solution of the equation $(\lambda-\tau)f = 0$ satisfies

$$\int_0^t |f(s)|^2 ds = O(t^k).$$

Then the point λ belongs to the essential spectrum of τ (Wintner [17]).

(50) Suppose that the function q is bounded, and that a real solution f of the equation $(\lambda-\tau)f = 0$ on $[0, \infty)$ is not square-integrable but satisfies

$$\int_0^t |f(s)|^2 ds = O(e^{kt})$$

for every $k > 0$. Then the point λ belongs to the essential spectrum of τ (Hartman [10]).

(51) Suppose that the function q is bounded, and that every solution of the equation $(\lambda-\tau)f = 0$ satisfies

$$\int_0^t |f(s)|^2 ds = O(e^{kt})$$

for every $k > 0$. Then the point λ belongs to the essential spectrum of τ on $[0, \infty)$ (Hartman [10]).

(52) In $[0, \infty)$, suppose that the function $Q \geqq 0$ is non-decreasing, that $q(t) \geqq -Q(t)$, and that

$$\int_0^\infty (Q(t))^{-\frac{1}{2}} dt = \infty.$$

If every solution of the equation $\tau f = 0$ is bounded, then the origin lies in the essential spectrum of τ (Hartman [15]).

(53) In $[0, \infty)$, if

$$\int_0^t \max(0, -q(s))ds = O(t^2)$$

and the equation $\tau f = 0$ has two linearly independent bounded solutions, then the origin lies in the essential spectrum of the operator τ (Hartman [15]).

(54) In $[0, \infty)$, if the function q is bounded below, and if there exists a square-integrable solution of the equation $(\lambda - \tau)f = 0$ such that for some positive N

$$f(t) = O(t^{-N}) \qquad \text{as} \qquad t \to \infty,$$

then the point λ belongs to the essential spectrum of τ (Wintner [17], Exercise 9.G20).

(55) In $[0, \infty)$, suppose that the function q is bounded below. Suppose that there exists a square-integrable solution of the equation $(\lambda - \tau)f = 0$ such that for all $k > 0$

$$\limsup_{t\to\infty} [f(t)^2 + f'(t)^2]e^{kt} = \infty.$$

Then the point λ belongs to the essential spectrum of τ (Hartman [10]).

(56) Suppose that in the interval $[0, \infty)$ the function q satisfies the inequality

$$|q(t)| \leq \frac{p+1}{t^2p^2} + \frac{p+2}{t^2p^2}\sum_{i=1}^{n}\frac{1}{\log t \log_2 t \ldots \log_i t} + g(t)$$

($\log_2 t = \log\log t$, $\log_{n+1} t = \log \log_n t$, etc.) for some $p > 1$ and for some non negative n, where the function g satisfies the conditions

$$\int_0^\infty t|g(t)|dt < \infty, \qquad \int_0^\infty t^{2q-1}|g(t)|^q dt < \infty, \qquad (p^{-1}+q^{-1} = 1).$$

Then the equation $\tau f = 0$ has no solutions of class $L_p[0, \infty)$. In particular, if $p = 2$, the origin lies in the essential spectrum of τ in the interval $[0, \infty)$.

If $p = 1$ the condition is replaced by

$$|q(t)| \leq 2t^{-2} + 3t^{-2} \sum_{i=1}^{n} \frac{1}{\log t \log_2 t \ldots \log_i t} + g(t),$$

where the function g satisfies the conditions

$$\int_0^\infty t|g(t)|dt < \infty, \qquad g(t) = O(t^{-2}) \qquad \text{as} \qquad t \to \infty.$$

All constants in these inequalities are the best possible (Sears [5]).

Remark. In many cases in which the essential spectrum of a Sturm-Liouville operator is bounded below it can be established that the operator is bounded below: cf. 7.31, 7.32, 7.33, Exercises 9.C1 – 7, Berkowitz [1], Rellich [6], Friedrichs [3, 10, 13]. We shall not discuss this point in the present section except for giving these few references.

D. *The number of boundary values of a singular differential operator of the second order.* We shall give a relatively complete summary of published results concerning the relationship between the behavior of the coefficients of the Sturm-Liouville operator

$$\tau = -\left(\frac{d}{dt}\right) p(t) \left(\frac{d}{dt}\right) + q(t)$$

and the number of boundary values at the singular endpoint(s) of τ. The functions p and q will be subjected to the conditions specified in Section 2. However, in general, the function q will only be assumed to be continuous. The interval of definition of the operator will be $[a, b)$ or $(a, b]$, and $a = -\infty$, $a > -\infty$, $b = \infty$, $b < \infty$ will be specified in each case.

Every criterion in the preceding paragraphs which implies that the essential spectrum is not empty implies that there are no boundary values at the free endpoint. We shall not repeat these criteria but shall confine the following list to conditions which are specific for the existence or non-existence of boundary values.

In the same way, every criterion in this paragraph which implies that the operator τ has two boundary values at a singular endpoint implies that the essential spectrum of the operator at that endpoint is empty.

(1) If q is bounded below in the interval $[0, \infty)$, then τ has no boundary values at infinity.

(2) In the interval $[0, \infty)$, suppose that there exists a positive continuously differentiable function M such that

(a) $\quad p(t)^{1/2}M'(t)M(t)^{-3/2}$ is bounded above,

(b) $\quad \int_0^\infty p(t)M(t)^{-1/2}dt = \infty,$

(c) $\quad q(t)M(t)^{-1}$ is bounded below.

Then τ has no boundary values at b (6.14).

(3) In the interval $[0, b)$, suppose that:

(a) q is positive in a neighborhood of b,

(b) $\displaystyle\int_0^b \left|\left[\frac{q(t)p'(t)}{q(t)^{3/2}p(t)^{1/2}}\right]' + \frac{1}{4}\frac{([q(t)p(t)]')^{1/2}}{p(t)^{3/2}q(t)^{5/2}}\right| dt < \infty,$

(c) $\int_0^x |p(t)q(t)|^{-1/2}dt < \infty$ for x in a neighborhood of b.

Then τ has two boundary values at b (6.20).

(4) In the interval $[0, \infty)$, if the equation $\tau f = 0$ has a solution such that

$$\int_0^t |f'(s)|^2 ds = O\left(\int_0^t p(s)^{-1}ds\right),$$

and if the integral on the right is not bounded, then τ has no boundary values at infinity (Hartman and Wintner [12]).

(5) In the interval $[a, b)$ let

$$Z(t) = q(t) + \left[4p(t)\left(\int_a^t p(s)^{-1}ds\right)^2\right]^{-1}.$$

If

(a) Z is eventually bounded below

(b) $$\int_a^b p(t)^{-1}dt = \infty,$$

then τ has no boundary values at b (7.69).

The following criteria apply to the operator

$$\tau = -\left(\frac{d}{dt}\right)^2 + q(t).$$

(6) If, in the interval $[0, \infty)$, the function $(1+t^2)^{-1}q(t)$ is bounded below, then τ has no boundary values at infinity (6.17).

(7) In the interval $(0, \infty]$:

(a) If $\liminf_{t\to 0} t^2 q(t) > \frac{3}{4}$, then τ has no boundary values at zero.

(b) If $\limsup_{t\to 0} |t^2 q(t)| < \frac{3}{4}$, then τ has two boundary values at zero (6.23).

(8) If q is monotone increasing in the interval $(0, b]$, then τ has two boundary values at zero (6.24).

(9) If $[a, b) = [0, \infty)$ and for some real λ the equation $(\lambda-\tau)f = 0$ has a solution whose derivatives are square-integrable, then τ has no boundary values at the singular endpoint (Hartman and Wintner [12]).

Each of the following conditions on the function q implies that the operator τ on the interval $[0, \infty)$ has no boundary values at infinity:

(10) $\int_0^t \max\big(0, -q(s)\big)ds = O(t^3)$ as $t \to \infty$ (Hartman and Wintner [12]).

(11) $q(t) \geqq -Q(t)$, where the function Q is positive and non-decreasing,

$$\int_0^\infty \big(Q(t)\big)^{-1}dt = \infty,$$

and the equation $\tau f = 0$ has a bounded solution (Hartman [15]).

(12) There exists a constant K such that for large s and t

$$q(t)-q(s) \geqq K(s-t)$$

(Wintner [4]).

(13) q is monotonic and

$$\int_0^\infty |q(t)|^{-1/2}dt = \infty$$

(Hartman and Wintner [4])

(14) q is differentiable and

(a) $$\int_0^\infty |q(t)|^{-1/2}dt = \infty,$$

(b) $$\limsup_{t\to\infty} \left|\frac{q'(t)}{q(t)^{3/2}}\right| < \infty,$$

(Hartman and Wintner [7]).

(15) $q(t) \geqq -Q(t)$, where the function Q is continuous and monotonic, $Q(t) \geqq K > 0$, and

$$\int_0^\infty |Q(t)|^{-1/2} dt = \infty$$

(Sears [2]).

(16) $q(t) \geqq -Q(t)$, where the function Q is differentiable, $Q(t) \geqq K > 0$, and

$$\limsup_{t \to \infty} \left| \frac{Q'(t)}{Q(t)^{3/2}} \right| < \infty,$$

(Sears [2]).

(17) Let Q be a positive continuous function which is of bounded variation on every finite interval. Let $N(t, \lambda)$ be the number of zeros in the interval $[0, t)$ of a solution of the equation $(\tau-\lambda)f = 0$. For some real λ

$$\limsup_{t \to \infty} \left[2 \int_0^t Q(s)^{-1} ds - \int_0^t (Q(s))^{-2} dN(s, \lambda) \right.$$
$$\left. (Q(t))^{-2} - \int_0^t dQ^2(s)] = \infty \right]$$

(Hartman [9]).

Note. This criterion seems to be the most general obtained thus far. The reader may verify that it implies several of the preceding ones.

E. *Essential spectrum and deficiency indices for differential operators of arbitrary order.* We summarize below the few scattered results which relate the asymptotic behavior of the coefficients of a regular formal differential operator of order higher than 2 to the essential spectrum and to the deficiency indices of the operator. Many of these results are due to Naĭmark [5].

Unless otherwise specified, the letter τ will denote a regular formal differential operator of the form

$$[*] \qquad \tau = \sum_{k=0}^{n} a_k(t) \left(\frac{d}{dt}\right)^k$$

on an interval which will be specified in each case. The coefficients a_k will be subjected to the assumptions specified in the following subsections I and II.

I. *Essential spectrum.* The essential spectrum of τ is empty if:

(1) τ is self adjoint and the deficiency indices of τ are (n, n) (6.12).

(2) The operator τ is of the form

$$\tau = (-1)^n \left(\frac{d}{dt}\right)^{2n} + \sum^{2n-1} p_k(t) \left(\frac{d}{dt}\right)^k + p_0(t)$$

on an interval $[a, b)$, where $p_1, p_2, \ldots, p_{2n-1}$ are bounded functions and $\mathscr{R}p_0(t) \to \infty$ as $t \to b$ (7.9).

(3) On an interval $[a, b)$ the operator τ is of the form

$$\tau = (-1)^n \left(\frac{d}{dt}\right)^{2n} + \sum_{k=1}^{2n-1} (-1)^k \left(\frac{d}{dt}\right)^k p_k(t) \left(\frac{d}{dt}\right)^k + p_0(t),$$

where $\mathscr{R}p_k(t)$ is bounded below for $1 \leqq k \leqq n-1$ and $\mathscr{R}p_0(t) \to \infty$ as $t \to b$ (7.10).

(4) On the interval $[0, \infty)$ the operator τ has the form

$$[**] \qquad \tau = \sum_{k=0}^{n} (-1)^k \left(\frac{d}{dt}\right)^k p_k(t) \left(\frac{d}{dt}\right)^k$$

where all the values of the coefficients p_k lie in the right half-plane and $\mathscr{R}p_0(t) \to \infty$ as $t \to \infty$ (7.8).

(5) Let τ have the form [**] on the interval $[0, \infty)$, where the coefficients p_k are real and let

(a) $p_0(t) \to \infty$ as $t \to \infty$,

(b) p_0' and p_0'' are eventually of constant sign,

(c) $p_0'(t) = O(p_0(t)^k)$ for some $0 < k < 1 + \frac{1}{2}n$, as $t \to \infty$,

(d) the functions

$$\frac{p_n'}{p_n},\ p_{n-1}p_n^{-1/2n},\ p_{n-2}p_n^{-3/2n}, \ldots, p_{n-1}p_n^{-(2n-3)/2n}$$

are summable in the interval $[0, \infty)$,

(e) $\lim_{t \to \infty} p_n(t) = 1$,

(Naimark [5]).

(6) Let τ have the form [**] on the interval [0, ∞), let the coefficients p_j be real, and assume the following conditions:

(a) $p_0(t) \to \infty$ as $t \to \infty$,

Conditions (b), (c), (d) and (e) of (5), and

(f)
$$\int_{-\infty}^{\infty} |p_0(t)|^{-1+1/2n} dt < \infty.$$

Then $\sigma_e(\tau)$ is void (Naĭmark [5]).

Other conditions which allow determination of the essential spectrum are the following:

(7) If there exists a constant M such that

$$|f^{(k)}|_2^2 \leqq M(\tau|f|_2^2 + |f|_2^2),$$

if

$$\tau_1 = \sum_{j=0}^{k} b_j(t) \left(\frac{d}{dt}\right)^j,$$

where

$$\limsup_{t \to b} b_j(t) = 0, \qquad (0 \leqq j \leqq k),$$

and if the operator $\tau + \tau_1$ on the interval $[a, b)$ has a non-vanishing leading coefficient, then the essential spectrum of τ coincides with the essential spectrum of $\tau + \tau_1$ (7.11).

(8) If the coefficients $|a_1(t)|, |a_2(t)|, \ldots, |a_{n-1}(t)|$ of τ are bounded and $|a_n(t)|$ is bounded below in the interval $[a, b)$ and if τ_1 has the property stated in (7) with $k = n$, then the essential spectrum of τ coincides with the essential spectrum of $\tau + \tau_1$ (7.12).

(9) In the interval [0, ∞), if

$$\lim_{t \to \infty} a_k(t) = q_k, \qquad 0 \leqq k \leqq n,$$

then the essential spectrum of τ is the set

$$\{\lambda | \lambda = \sum_{j=0}^{n} q_j(it)^j, -\infty < t < \infty\}$$

(7.13).

(10) If τ has the form [**] on the interval [0, ∞), where $\mathscr{R}p_j$ is bounded below for $0 \leqq j < n$ and $\mathscr{R}p_n$ is bounded below by a positive constant and if τ_1 is of the form given in (7) with $k = n$, then the essential spectrum of τ coincides with the essential spectrum of $\tau + \tau_1$ (7.15).

(11) If τ has the form [$**$], where all the coefficients are real and $(1/p_n)'$, $p_{n-1}, \ldots, p_0$ are summable in $[0, \infty)$, then the essential spectrum of τ in the interval $[0, \infty)$ is the positive semi-axis (Naĭmark [5]).

(12) Let $[a, b) = [0, \infty)$. If assumptions (a), (b), (c), (d), (e) of (6) are satisfied and if the integral in (f) diverges, then the essential spectrum of τ is the entire real axis (Naĭmark [5]).

Other conditions which allow approximate determination of the essential spectrum are the following:

(14) Suppose that τ has the form given in (4) and that all coefficients are real and eventually non-negative. Then the essential spectrum is contained in the positive real axis (7.7).

(15) If, under the assumptions of (14) in the interval $[0, \infty)$

$$\lim_{t\to\infty} p_n(t) = P,$$

then the essential spectrum of τ is contained in $[-P, \infty)$ (Naimark [5]).

(16) Suppose that $[a, b) = [0, \infty)$, that the deficiency indices of τ are equal and that there exists a sequence $\{f_n\}$ of square-integrable functions such that f_n vanishes in the interval $[0, n]$, $|f_n| = 1$, and

$$|(\lambda - \tau)f_n| \leqq K.$$

Then the interval $[\lambda - K, \lambda + K]$ contains a point in the essential spectrum of τ (Šnol [1], Exercise 9.A6).

(17) Let

$$K = \limsup_{t\to\infty} q(t).$$

Then the distance of any point in the essential spectrum of an operator of the form [$*$] on the interval $[0, b)$ from the essential spectrum of the operator

$$\tau_1 = \tau + q$$

is smaller than K (Putnam [9]).

II. *Deficiency indices and boundary values.* Since every estimation of the number of deficiency indices immediately yields an estimation of the number of boundary values, but not vice versa, we shall state the results in terms of the deficiency indices whenever possible,

leaving it to the reader to interpret them in terms of boundary values at the singular endpoint(s). We assume, unless it is otherwise stated, that τ is a formally symmetric formal differential operator of the form [*] on an interval which will be specified in each case, and with the understanding that when no interval is mentioned, the result is valid in an arbitrary interval of definition.

(1) If the essential spectrum of τ is not the entire real axis, the deficiency indices of τ are equal (6.6).

(2) In particular, the deficiency indices are equal if τ is bounded below.

(3) If for some real or complex λ all solutions of the equation $(\lambda-\tau)f = 0$ are square-integrable, then the deficiency indices of τ are (n, n) (6.11).

(4) If q is a bounded function, then τ and $\tau+q$ have the same deficiency indices (6.30).

(5) If $[a, b) = [a, \infty)$, if $a_n(t)$ is bounded away from zero, and $a_k(t)$ is bounded, $0 \leq k < n$, then the sum of the deficiency indices of τ in the interval $[0, \infty)$ is n (6.35).

In the following cases the operator τ will be assumed to be of the form [**] on the interval $[0, \infty)$. All coefficients will be assumed to be real. All the following theorems are due to Naĭmark [5].

(6) If conditions (a), (b), (c), (d) of I.(6) are satisfied, and if

$$\lim_{t\to\infty} p_n(t) > 0,$$

then:

(a) if $p_0(t) \to \infty$ as $t \to \infty$, then the deficiency indices are (n, n),

(b) if $p_0(t) \to -\infty$ as $t \to \infty$, and the integral

$$\int_0^\infty |p_0(t)|^{-1+1/2n}\, dt$$

converges, then the deficiency indices are $(n+1, n+1)$,

(c) if $p_0(t) \to -\infty$ as $t \to \infty$ and the integral in (b) diverges, then the deficiency indices are (n, n).

(7) If for some constants $c_0, c_1, \ldots, c_n$ the functions

$$c_n - \left(\frac{1}{p_n}\right)', \qquad c_{n-1}-p_{n-1}, \ldots, c_0-p_0$$

are integrable in $[0, \infty)$, then the deficiency indices of τ are (n, n).

(8) If the functions $(1/p_n)'$, $p_{n-1}, \ldots, p_0$ are integrable, if

$$\lim_{t\to\infty} p_n(t) > 0$$

and if q is a function of bounded variation, then the deficiency indices of $\tau + q$ are (n, n).

(9) If the limits

$$\lim_{t\to\infty} p_k(t), \qquad 0 \leqq k \leqq n$$

exist, then the deficiency indices of τ are (n, n).

F. *Differential operators which are not formally self adjoint.* The later chapters of this work will be largely concerned with the study of linear operators which are nonselfadjoint, and hence operators to which the spectral theorem of Chapters X and XII does not apply. The more general theory of spectral operators, to be developed in Chapters XV, XVI, XVII and XVIII will be applied in Chapters XIX and XX to differential operators. In this way, we shall succeed in showing that a large class of nonselfadjoint differential operators possess a spectral resolution in a suitably generalized sense. However, the results to be given in these chapters will be obtained by the application of perturbation methods to self adjoint operators, and thus will, in the last analysis, lean upon the results of this chapter.

Little has been done in the study of non-formally self adjoint differential operators independently of perturbation methods. A generalization of the theory of boundary values and extensions can be found in the dissertation of Rota [1], whose results are as follows.

A closed operator $T_1(\tau, \mathfrak{X})$ in a Banach space $\mathfrak{X}$, which can be one of the spaces $L_p(I)$ $(1 \leqq p \leqq \infty)$ or $C(I)$, can be defined in terms of a formal differential operator τ by a method which is similar to that of Section 2. While the proof of closure and the calculation of the adjoint operator presents no particular difficulty in the spaces $L_p(I)$ $(1 < p < \infty)$, the derivation of similar results in the spaces $L_1(I)$ and $C(I)$ requires the use of several of the deeper notions of the theory of Banach spaces, such as the notions of the bounded $\mathfrak{X}$-topology of $\mathfrak{X}^*$ given in Chapter V.

The space of boundary values can be defined much as in Section

2, and its dimension can be proved to be finite. The essential spectrum is to be defined as in Section 6, and is a closed subset of the complex plane which coincides with the essential spectrum of the formal adjoint operator in the conjugate space. The essential spectrum of a formal differential operator enjoys the following "spectral mapping" property: If p is a polynomial with constant coefficients, and if $p(\tau)$ is the corresponding "polynomial" in τ, then the essential spectrum of the formal differential operator $p(\tau)$ is the set

$$\{p(\lambda)|\lambda \in \sigma_e(\tau)\}.$$

The complement of the essential spectrum of τ in $\mathfrak{X}$ may be decomposed into a countable or finite number of connected components. A remarkable fact is that as λ ranges over any one such component the dimension of the space of solutions of the equation $(\lambda-\tau)f = 0$ which belong to the space $\mathfrak{X}$ remains constant. Furthermore, if λ is in the complement of the essential spectrum of τ in $\mathfrak{X}$, the sum of the dimensions of the space of solutions of the equation $(\lambda-\tau)f = 0$ which lie in the space $\mathfrak{X}$ and of the space of solutions of the equation $(\lambda-\tau^*)g = 0$ which lie in the conjugate space $\mathfrak{X}^*$ is a constant which equals the dimension of the space of boundary values of τ in $\mathfrak{X}$. This result permits the assignment, to each connected component of the complement of the essential spectrum of τ, of two "deficiency indices" on the basis of which an extension theory can be constructed.

A restriction of the operator $T_1(\tau, \mathfrak{X})$ is obtained by restricting the domain of definition to all functions which satisfy a prescribed set of boundary conditions, just as in the Hilbert space theory. While the essential spectrum of τ in $\mathfrak{X}$ is invariant under restriction of $T_1(\tau, \mathfrak{X})$, the remaining part of the spectrum depends on the restriction chosen, and may lie in the residual spectrum and/or the point spectrum or in the resolvent set of the restricted operator. The main problem consists in finding characterizations of those restrictions whose spectra are minimal. A general answer to this problem is given by the following statement: if all the deficiency indices of τ in $\mathfrak{X}$ are equal, then a restriction of $T_1(\tau, \mathfrak{X})$, whose spectrum consists of the essential spectrum of τ in $\mathfrak{X}$, taken together with a finite or infinite sequence of eigenvalues all of whose limit points lie in the essential spectrum of τ, can be found.

G. *Spectral asymptotics.* (a) A refinement of the result given in Theorem 7.58 has been obtained by Titchmarsh [16] by means of the W.K.B. (Wentzel-Kramers-Brillouin) method used by physicists. The method is applied by Titchmarsh to the case when q is a polynomial t^k, but it can be generalized. A sequence of increasingly accurate asymptotic expressions can be obtained for the eigenvalues. A second order approximation gives

$$N(\lambda) = \frac{1}{\pi}\int_0^{t(\lambda)} (\lambda - s^k)^{1/2} ds - \tfrac{1}{2} + O(\lambda^{-1/2-1/k}).$$

A further approximation gives

$$N(\lambda) = \frac{1}{\pi}\int_0^{t(\lambda)} (\lambda - s^k)^{1/2} ds - \tfrac{1}{2} - \frac{1}{8\pi}\int_0^{t(\lambda)} \frac{(q'(s))^2}{(\lambda - q(s))^{5/2}} ds + O(\lambda^{-1-2/k})$$

and so forth, where $t(\lambda) = \lambda^{1/k}$. We refer the reader to the book of Titchmarsh for the details.

(b) Hartman [11] has shown that the formula given in Theorem 7.58 is valid under less restrictive assumptions on the coefficient q. He assumes only that q is an increasing function satisfying

$$\lim_{t\to\infty} \inf_{t\leqq u, v\leqq\infty} [q(v) - q(u)] \Big/ \left(\int_u^v s^{-3} ds\right) = \infty.$$

Subsequent work on related problems was done by Hartman and N. S. Rosenfeld. We give under the next four subheadings a summary of the literature on results in the spirit of Theorem 7.58 for the asymptotic distribution of the eigenvalues of $(d/dt)^2 + q(t)$, $0 \leqq t < \infty$, with appropriate boundary conditions. We denote the corresponding operator in $L_2(0, \infty)$ by T.

(a) $\lim_{t\to\infty} q(t) = \infty$ and $q(t)$ is convex. References: Milne [2], Titchmarsh [16], Hartman and Wintner, On the Asymptotic Problems of the Zeros in Wave Mechanics, *Amer. J. Math* 70, 461–480 (1948) and Hartman, On the Zeros of Solutions of Second Order Linear Differential Equations, *J. London Math. Soc.* 27, 492–496 (1952). The best result is that of Hartman, who proves (in the notation of Theorem 7.58) the following.

THEOREM. *Let $q(t)$ be continuous, increasing and convex. Then*

[*] $$\pi N(\lambda) = \int_0^{t(\lambda)} [\lambda - q(t)]^{1/2} dt + O(1).$$

(a′) $\lim_{t\to\infty} q(t) = \infty$ and $q(t)$ is not necessarily convex. References: Atkinson [5], Hartman [11].

(b) $\lim_{t\to\infty} q(t) = 0$. Reference: Rosenfeld, N. S., The Eigenvalues of a Class of Singular Differential Operators, *Comm. Pure Appl. Math.* 13, 395–405 (1960). He proves the following theorem.

THEOREM. *Let* $q(t) < 0$ *be twice continuously differentiable,* $\lim_{t\to\infty} q(t) = 0$, $q'(t) > 0$, $q''(t) \leqq 0$ *and* $\lim_{t\to\infty} q''(t)[q'(t)]^{-4/3} = 0$. *Then* T *has an infinite, negative sequence of eigenvalues converging to zero. Let* $N(\lambda)$, $\lambda < 0$, *denote the number of eigenvalues not exceeding* λ. *Then, as* λ *increases to zero,*

$$\pi N(\lambda) \sim \int_0^{t(\lambda)} [\lambda - q(t)]^{1/2} dt.$$

If, furthermore, $q''(t)$ *is of bounded variation on compact intervals and* $[q''(t)]^2[q'(t)]^{-7/3}$ *is integrable, then* [∗] *holds.*

(c) The distribution of the eigenvalues of the operator

$$\tau = -\left(\frac{d}{dt}\right)^2 + q(t)$$

on the interval $[0, \infty)$ in some cases in which τ has two boundary values at infinity has recently been investigated in the dissertation of P. Heywood [1]. Specifically, it is assumed that

(i) $q(0) = 0$,
(ii) $q'(t) < 0$ for $t > 0$,
(iii) $\int_0^\infty q(t)^{-1/2} dt < \infty$,
(iv) $q''(t)$ is eventually of one sign,
(v) $q''(t) = O(q'(t))^k$ for some $1 < k < \frac{4}{3}$,
(vi) for some $r < \frac{3}{2}$, the function

$$\frac{q''(t)}{(-q(t))^r}$$

is eventually monotonic decreasing.

For any self adjoint extension of τ, let $N_1(\lambda)$ be the number of non-negative eigenvalues not exceeding λ, and let $N_2(\lambda)$ be the number of negative eigenvalues not exceeding λ in absolute value. Let $t(\lambda)$ be as in Theorem 7.58. Then

$$N_1(\lambda) \sim \frac{1}{\pi}\int_0^{\infty} [(\lambda \quad q(t))^{1/2} \ -q(t)^{1/2}]\, dt;$$

$$N_2(\lambda) \sim \frac{1}{\pi}\int_0^{t(\lambda)} |q(t)|^{1/2} dt + \frac{1}{\pi}\int_{t(\lambda)}^{\infty} [|q(t)|^{1/2} \quad (\quad q(t) - \lambda)^{1/2}]\, dt.$$

(d) A qualitative result on the distribution of the eigenvalues of the Sturm-Liouville operator has been obtained by Hartman and Wintner [11]. Suppose that an interval $[a, b)$ on the real axis does not meet the essential spectrum. Let $\lambda_1, \lambda_2, \ldots$ be an enumeration in increasing order of the eigenvalues of a self adjoint extension of a Sturm-Liouville operator. Then any other self adjoint extension has one and only one eigenvalue between λ_k and λ_{k+1}.

(e) Little is known about the asymptotic properties of the measures and matrix measures obtained from the spectral resolutions of differential operators with non-void essential spectrum. It is worth noting in this connection that the results of Gelfand and Levitan on the converse problem, given in paragraph I, can be interpreted as asymptotic results for the continuous spectrum.

H. *Convergence and summability.* Several questions of convergence and summability analogous to those which have formed part of the classical theory of Fourier series and Fourier integrals have been studied, mostly by the British school, for the eigenfunction expansions arising from a self adjoint extension of a Sturm-Liouville operator. These problems were studied for a finite closed interval of definition in the first decade of this century by Haar, and later by Walsh (Haar [3], Walsh [1]). A corresponding study of the summability properties of the generalized Fourier integral representation arising from the spectral analysis of the operator

$$[*] \qquad \tau = -\left(\frac{d}{dt}\right)^2 + q(t)$$

on the real line, where the function q is assumed to be integrable, can be found in the paper of Stone [17] in 1928. The work of Titchmarsh [16] deals with singular operators on the interval $[0, \infty)$ which have no essential spectrum. Sears [4] extends the results of Titchmarsh and Stone. A recent paper of Rutovitz [1] takes the first steps

in the study of convergence in L_p. The main results of these papers are the following:

(1) Haar [8] proved that for the Sturm-Liouville eigenfunctions obtained by the impositions of separated boundary conditions

(a) there exist continuous functions whose Sturm-Liouville series is divergent at a given point;

(b) the convergence or divergence of the Sturm-Liouville series $\sum_{n=1}^{\infty} a_n f_n(t)$ is determined by the convergence or divergence of the cosine series $\sum_{n=1}^{\infty} a_n \cos nt$ or the sine series $\sum_{n=1}^{\infty} a_n \sin nt$;

(c) the Sturm-Liouville series of a continuous function is uniformly Cesàro summable to the function.

(2) Walsh [2] proved the following theorem.

Let $\{f_n\}$ *be the orthonormal eigenfunctions of a Sturm-Liouville operator obtained by the imposition of separated boundary conditions, and let* $\{g_n\}$ *be the orthonormal eigenfunctions obtained from the operator* $-(d/dt)^2$ *by the imposition of the same boundary conditions on an interval* $[a, b]$. *Let* f *be a square-integrable function, and let*

$$\sum_{k=1}^{\infty} a_k f_k(t) \qquad \text{and} \qquad \sum_{k=1}^{\infty} b_k g_k(t)$$

be the two corresponding expansions of f. *Then the series*

$$\sum_{k=1}^{\infty} [a_k f_k(t) - b_k g_k(t)]$$

is absolutely convergent and converges uniformly to zero.

A summary of later work on the convergence properties of eigenfunction expansions on a finite closed interval is given in the section of notes and remarks appended to Chapter XIX.

(3) If q is a continuous function tending monotonically to infinity on the positive real axis, then the inversion formula for the spectral representation of a self adjoint extension T of the operator τ of [*] takes the familiar form of a series expansion (for a square-integrable function f)

$$f(t) \sim \sum_{n=1}^{\infty} c_n f_n(t), \qquad c_n = \int_0^{\infty} f(t)\overline{f_n(t)}\,dt,$$

where f_n is the nth normalized eigenfunction. Titchmarsh [16] found that, as in the case of Fourier series, the series on the right converges at any point t in a neighborhood of which the function f is of bounded variation.

(4) Some results on the summability of the eigenfunction expansion under the same conditions as in (3) have been obtained by Titchmarsh [16]. The natural method of summation to apply is

$$\lim_{\lambda\to\infty} -\lambda\big(R(-\lambda;T)f\big)(t) = \lim_{\lambda\to\infty} \sum_{n=1}^{\infty} \frac{\lambda}{\lambda+\lambda_n} c_n f_n(t)$$

where λ_n is the nth eigenvalue. The limit exists for every square-integrable function f at all points t at which

$$\int_0^{\varepsilon} |f(t+s)-f(t)|\,ds = O(\varepsilon),$$

and in particular, equals $f(t)$ at all points of continuity of f. (Cf. paragraph 8 below, where a general form of this principle is given.)

(5) (Titchmarsh [15]) If it is further assumed that the function q behaves like a polynomial, that is, that

$$q(t) \sim At^k, \qquad q'(t) \sim t^{k-1}, \qquad q''(t) = O(t^{k-2}),$$
$$q'''(t) = O(t^{k-3}),$$

then the result can be shown to apply to a class of functions wider than the class of square-integrable functions. A summability method "(R, p)" is defined by the operation

$$\lim_{\lambda\to\infty} \sum_{\lambda_n<\lambda} \left(\frac{\lambda-\lambda_n}{\lambda}\right)^p c_n f_n(t).$$

It is then shown that if the function f is integrable over any finite interval, and if for some sufficiently large positive number a depending on p

$$\int_0^N |f(t)|\,dt = O(N^a),$$

then the series expansion of f exists and is (R, p)-summable at all points t at which

$$\int_0^{\varepsilon} |f(t+s)-f(t)|\,ds = O(\varepsilon).$$

(6) Assume that the function q, in addition to being continuous, is integrable in the interval $[0, \infty)$. Then the spectrum of a self adjoint extension of the operator τ on the interval $[0, \infty)$ obtained by the imposition of a boundary condition at zero will consist of a continuous spectrum covering the positive real semi-axis, and a sequence of eigenvalues with simple normalized eigenfunctions $\{f_n\}$. Let $f(t, \lambda)$ be a solution of the equation $(\lambda - \tau)f = 0$ which satisfies the boundary condition at zero, and let μ be the measure obtained from the spectral representation (cf. Theorem 5.13). Then it can be shown (cf., e.g., Stone [17]) that as λ ranges over a compact set the functions $f(t, \lambda)$ are uniformly bounded in t and λ. Thus, the integral

$$\int_0^\infty g(t)f(t, \lambda)dt$$

is defined for every integrable function g. The most general result obtained by Sears [4] is the following: the series

$$\sum_{n=1}^\infty c_n f_n(t), \qquad c_n = \int_0^\infty g(t)\overline{f_n(t)}dt,$$

is convergent for $0 < t < \infty$, and for every $R > 0$ we have

$$\lim_{\varepsilon \to 0} \int_\varepsilon^R f(t, \lambda)\left[\int_0^\infty g(t)f(t, \lambda)dt\right]\mu(d\lambda) + \sum_{n=1}^\infty c_n f_n(t)$$
$$= \frac{1}{\pi}\int_0^\infty g(s)\frac{\sin\left((t-s)\sqrt{R}\right)}{s-t}\,ds + w(R, t),$$

where $w(R, t)$ tends to zero as R tends to infinity, uniformly over any finite closed interval not containing the origin.

This result shows that the generalized Fourier integral converges in the same way as the ordinary Fourier integral. For example, if g is of bounded variation in a neighborhood of the point $t > 0$,

$$\lim_{R \to \infty} \lim_{\varepsilon \to \infty} \int_\varepsilon^R f(t, \lambda)\left[\int_0^\infty g(t)f(t, \lambda)dt\right]\mu(d\lambda) + \sum_{n=1}^\infty c_n f_n(t)$$
$$= (\tfrac{1}{2})[g(t+0)+g(t-0)].$$

(7) Sears [4] also investigates the Cesàro summability of the generalized Fourier integral for integrable and square-integrable

functions. For any square-integrable function f the integral

$$\int_0^\infty f(t, \lambda)\hat{f}(\lambda)\mu(d\lambda)$$

where

$$\hat{f}(\lambda) - \underset{N\to\infty}{\text{l.i.m.}} \int_0^N f(t, \lambda)f(t)dt$$

is shown to be summable by any pth Cesàro mean ($p > 0$) at any point at which

$$\int_0^\varepsilon |f(t+s) \quad f(t \quad s)|ds \qquad O(\varepsilon)$$

as $\varepsilon \to 0$, in particular, almost everywhere and at all points of continuity of f. The same result holds for integrable functions.

(8) (cf. Exercise 9.J2). With the hypotheses and in the notation of Theorem 5.23 we can enunciate a general summability principle as follows: suppose that the differential operator τ is bounded below, and that f is a square-integrable function on the interval of definition of τ. Then

(a) for λ_0 sufficiently large

$$\int_{-\infty}^{\infty} (\lambda_0+\lambda)^{-1} \sum_{i,j=1}^{k} (Vf)_i(\lambda)\sigma_j(t, \lambda)\rho_{ij}(d\lambda)$$
$$= \lim_{a\to\infty} \int_{-\infty}^{a} (\lambda_0+\lambda)^{-1} \sum_{i,j=1}^{k} (Vf)_i(\lambda)\sigma_j(t, \lambda)\rho_{ij}(d\lambda)$$

exists uniformly for t in any compact subinterval of I;

(b)

$$\lim_{\lambda_0\to\infty} \int_{-\infty}^{\infty} \lambda_0(\lambda_0+\lambda)^{-1} \sum_{i,j=1}^{k} (Vf)_i(\lambda)\sigma_j(t, \lambda)\rho_{ij}(d\lambda)$$
$$= \lim_{\lambda_0\to\infty} (\quad \lambda_0 R(-\lambda_0; T)f)(t) - f(t)$$

for each t interior to I for which

$$\int_{t-\varepsilon}^{t+\varepsilon} |f(s)-f(t)|ds \qquad O(\varepsilon)$$

as $\varepsilon \to 0$;

(c) if $\hat{\tau}$ is any operator defined on a subinterval $\hat{I}$ of I, if $\hat{\tau}$ is bounded below and if $\hat{T}$ is a self adjoint extension of $T_0(\hat{\tau})$, then

$$\lim_{\lambda_0 \to \infty} \left[\left(-\lambda_0 R(-\lambda_0; \hat{T})(f|\hat{I}) \right)(t) + \lambda_0 R(-\lambda_0; T)f(t) \right] = 0$$

uniformly for t in any compact subinterval of the interior of I.

(9) (cf. Exercise 9 J1–J8). Let τ be a formally self adjoint formal differential operator on an interval I, and let T be a self adjoint extension of $T_0(\tau)$. Let $\{f_n\}$ be an orthonormal family of eigenfunctions of T: $(\lambda_n - T)f_n = 0$, and suppose that the set $\{\lambda_n\}$ is bounded. Then, for each f in $L_2(I)$, the series

$$\sum_{n=1}^{\infty} (f, f_n) f_n(t)$$

converges absolutely and uniformly in each bounded closed subinterval of I, and may be differentiated arbitrarily often under the sign of integration, the differentiated series also converging absolutely and uniformly.

(10) The problem of the validity of the inversion formulas for the generalized Fourier integrals arising from singular operators of the form

$$\tau = -\left(\frac{d}{dt}\right)^2 + q(t)$$

on the interval $[0, \infty)$ for functions in $L_p[0, \infty)$ $(1 \leq p \leq 2)$ has been recently attacked by Rutovitz [1]. It is assumed that

(a) $\int_0^\infty |(1+t)q(t)| dt < \infty$,

(b) the function $(1+t)(q(t))$ is of bounded variation.

Let a self adjoint extension T of τ be defined by a boundary condition at zero and let f be a function of class $L_p[0, \infty)$ $(1 \leq p < 2)$. Let $f(t, \lambda)$ be a solution of the equation $(\lambda - \tau)g = 0$ which satisfies the given boundary condition at zero. Let μ be the corresponding measure (cf. Theorem 5.18) obtained from the spectral resolution of T. Then the limit

$$\operatorname*{l.i.m.}_{N \to \infty} \int_0^N f(t) f(t, \lambda) dt = \hat{f}(\lambda)$$

exists in the mean of order p. Furthermore, there is a constant $K(p, q)$ such that

$$\left| \int_{-A}^{A} \hat{f}(\lambda) f(t, \lambda) \mu(d\lambda) \right|_p \leq K(p, q) |f|_p$$

and we have

$$\lim_{\Lambda\to\infty}\left|\int_{-\Lambda}^{\Lambda} f(\lambda)\psi(t,\lambda)\mu(d\lambda) - f\right|_p = 0.$$

I. *The converse problem.* The wide application of the spectral theory of differential operators to a variety of problems in modern physics not only has shown a physical meaning for many of the mathematical notions, but has also raised some problems which have not been completely answered by the mathematician. Outstanding among these is the so-called "converse problem", namely, the problem of determining necessary and sufficient conditions on a given matrix measure in order that it be the measure obtained from the spectral resolution of a self adjoint extension of some differential operator, and of calculating this differential operator explicitly, if it exists.

The first approach to this problem is to be found in the paper of G. Borg [2] in 1945, where the problem is studied in the case in which the second order operator is given on a compact interval. Borg showed that, given the distribution of eigenvalues for *two* self adjoint extensions of the operator

$$\tau = -\left(\frac{d}{dt}\right)^2 + q(t), \qquad 0 \leq t \leq \pi,$$

then the function q is uniquely determined. Levinson [4] simplified Borg's arguments, and proved that the distribution of eigenvalues of one self adjoint extension determines the function q uniquely, provided it is further assumed that $q(\pi - t) = q(t)$.

The singular problem for the second order operator defined on the positive semiaxis was first studied by Krein [13] [14], who made use of his theory of extension of positive definite functions. The exhaustive paper of Gelfand and Levitan [1] gives a complete account of what is known so far on the problem. Recently Kay and Moses [1] [2] have considerably clarified the ideas involved. We give here a brief summary of the results of these papers.

The main result is the following:

Let μ be the measure which is obtained from the spectral representation of the self adjoint extension of the operator

$$\tau_0 = -\left(\frac{d}{dt}\right)^2$$

on the interval $[0, \infty)$ (cf. the end of Section 5) obtained by imposing the boundary condition $B(f) = f'(0) = 0$, namely

$$\mu([a, b]) = \frac{1}{\pi}\int_a^b \frac{d\lambda}{\sqrt{\lambda}}.$$

Let ρ be a positive Borel measure on the positive real axis, and let $\sigma = \rho - \mu$. Suppose that ρ satisfies the following conditions:

(1) $$\int_{-\infty}^{0} e^{\sqrt{|\lambda|}\,t}\rho(d\lambda) < \infty, \qquad t > 0.$$

(2) The function

$$a(t) = \int_1^\infty \frac{\cos t\sqrt{\lambda}}{\lambda}\,\sigma(d\lambda)$$

is four times differentiable.

Under these conditions we can find a differential operator

$$\tau = -\left(\frac{d}{dt}\right)^2 + q(t) \qquad 0 \leqq t$$

such that ρ is the measure associated with the spectral representation of the self adjoint extension T of τ (cf. Theorem 5.13 and Corollary 21) obtained by imposition of the boundary condition $B(f) = f'(0) = 0$.

Conversely, if the function q is assumed to have a continuous derivative, then the measure obtained from the spectral representation (cf. Theorem 5.13) of any self adjoint extension T of τ satisfies conditions (1) and (2).

The idea used by Gelfand and Levitan is intuitively simple. Suppose that the operator τ is given, and let $f(t, \lambda)$ be a solution of the differential equation $(\lambda - \tau)f = 0$ satisfying the boundary condition $B(f) = 0$. Let ρ be the measure described in Theorem 5.18 for the self adjoint operator T. Then the functions $f(t, \lambda)$ enjoy the orthogonality property

$$\int_{-\infty}^{\infty} f(s, \lambda)f(t, \lambda)\rho(d\lambda) = 0, \qquad 0 < s \neq t < \infty.$$

In the same way, the functions $\cos t\sqrt{\lambda}$ obtained from the operator τ_0 enjoy the property

$$\int_{-\infty}^{\infty} \cos s\sqrt{\lambda} \cos t\sqrt{\lambda}\,\mu(d\lambda) = 0, \qquad s \neq t.$$

Making use of these relations, Gelfand and Levitan attempt to reconstruct the functions $f(t, \lambda)$ by "orthogonalizing" the functions $\cos t\sqrt{\lambda}$ with respect to the measure ρ. Their first step consists in obtaining an expression for $f(t, \lambda)$ as a "linear combination" of the functions $\cos t\sqrt{\lambda}$ in terms of an integral operator with a kernel of Volterra type:

$$(*) \qquad f(t, \lambda) = \cos t\sqrt{\lambda} + \int_0^t K_1(t, s) \cos s\sqrt{\lambda}\, ds.$$

Let us indicate briefly how the kernel K_1 is obtained once the functions $f(t, \lambda)$ are known. A formal differentiation gives the following partial differential equation for K_1:

$$\frac{\partial^2 K_1}{\partial t^2} - \frac{\partial^2 K_1}{\partial s^2} = q(t)K_1(s, t)$$

with the boundary conditions

$$K_1(t, 0) = 0,$$

$$K_1(t, t) = (\tfrac{1}{2}) \int_0^t q(t)dt.$$

The general theory of hyperbolic equations informs us that a solution exists and is unique. Thus, once the kernel K_1 is known, the functions $f(t, \lambda)$ are known, and the coefficient function q can be uniquely determined from the relation

$$q(t) = 2K_1'(t, t).$$

Similarly, the functions $\cos t\sqrt{\lambda}$ can be expressed as "linear combinations" of the functions $f(t, \lambda)$ in the form

$$\cos t\sqrt{\lambda} = f(t, \lambda) + \int_0^t K_2(t, s) f(s, \lambda)\, ds,$$

where the kernel K_2 can be obtained analogously.

The next step consists in obtaining an integral equation for the kernel K_1 which will not involve a previous knowledge of the functions

$f(t, \lambda)$. To this end, it is first shown that the fact that $f(t, \lambda)$ is a "linear combination" of the functions $\cos t\sqrt{\lambda}$ implies the identity

$$\int_{-\infty}^{\infty} f(t, \lambda) \cos s\sqrt{\lambda}\, d\lambda = 0, \qquad s < t.$$

By formally substituting the expression for f in terms of the kernel K_1 into the above identity we obtain

$$\int_{-\infty}^{\infty} \cos t\sqrt{\lambda} \cos s\sqrt{\lambda}\, \rho(d\lambda)$$
$$+ \int_{-\infty}^{\infty} \cos s\sqrt{\lambda} \left[\int_0^t K_1(t, s) \cos s\sqrt{\lambda}\right] \rho(d\lambda) = 0, \qquad s < t.$$

Clearly the integrals in this equation will, in general, diverge. However, using the previously defined set function σ, they can be recast in a form which, in view of the restrictions imposed on ρ, will render them convergent, namely

$$F(t, s) + \int_0^t F(t, u) K_1(t, u) du + K_1(t, s) = 0,$$

where

$$F(t, s) = \int_{-\infty}^{\infty} \cos t\sqrt{\lambda} \cos s\sqrt{\lambda}\, \sigma(d\lambda).$$

It is then shown that the homogeneous form of this Fredholm equation for K_1 (for t fixed, $s < t$) has no solutions. Hence a solution of the inhomogeneous equation above will determine the kernel K_1 and consequently the function q.

The power of the method is apparent, since it only involves the solution of an integral equation, followed by a quadrature. Furthermore, the above theorem of Gelfand and Levitan, once proved, tells us that the measures of Theorem 5.18 may exhibit several pathological properties; that, for example, they may be singular with respect to Lebesgue measure and at the same time vanish on all points.

When the operator τ is defined on a compact interval $[0, a]$ the theorem of Gelfand and Levitan assumes a particularly simple form. Given two infinite sets of real numbers $\{\lambda_n\}$ and $\{c_n\}$ satisfying the asymptotic relations

$$\lambda_n = \frac{\pi}{a} n + O(n^{-1}),$$

$$c_n = \frac{a}{2} + O(n^{-1}),$$

and given real boundary values $B_1(f) = f'(0) + hf(0)$ and $B_2(f) = f'(a) + Hf(a)$, then there exists a continuous function q such that the spectrum of the self adjoint extension of the operator obtained by the imposition of the boundary conditions $B_1(f) = 0 = B_2(f)$ is the set $\{\lambda_n\}$.

If f_n is the eigenfunction corresponding to the eigenvalue λ and subjected to the condition $f_n(0) = 1$, then

$$c_n = \int_0^a f_n(t)^2 dt.$$

Furthermore, the function q can be explicitly calculated by the method outlined above.

An investigation of the pathology of the spectrum along radically different lines has been undertaken by Hartman [18], who used the characterization of the spectrum of the second order operator which is provided by the oscillation theory (cf. Exercise 9.F3). Hartman proved that given any closed subset of the real axis, there exists a function q such that the essential spectrum of the operator

$$\left(\frac{d}{dt}\right)^2 + q(t)$$

on the interval $[0, \infty)$ is the given set. The function q can be chosen to be infinitely differentiable.

Among other recent contributions to the converse problem, the following papers may be found illuminating: Berezanskii [1], Kay [1], Krein [14] [17] [18], Newton and Jost [1], Staševskaya [1].

J. ***Related theories.*** A number of other applications of linear analysis to the study of differential operators besides the spectral theory presented in the present chapter have recently received attention. Two of these theories seem particularly interesting.

The Russian school has pursued the theory of generalized displacement operators, founded by B. M. Levitan and A. Povzner on an idea which originated with Delsarte in 1938. Starting with the formal differential operator

$$\tau = \left(\frac{d}{dt}\right)^2 + q(t)$$

on an interval I which can be the positive semiaxis or the entire real

axis, where the function q is assumed to be bounded, an operator in $L_p(I)$ $(1 \leq p < \infty)$ is defined in a manner similar to that which has been used for Hilbert space in Section 2. Let us denote this operator by S, leaving the particular space under consideration unspecified, and maintaining the understanding that the domain $\mathfrak{D}(S)$ will consist of sufficiently differentiable functions. Suppose now for the moment that the function q vanishes identically, so that the operator S reduces to ordinary differentiation. Writing

$$\big(T(s)f\big)(t) = f(t+s), \qquad s \in I$$

we find that the family of operators $T(s)$ has the following properties:

(a) for fixed s the operator $T(s)$ maps $\mathfrak{D}(S)$ into itself,

(b) for fixed t and for f in $\mathfrak{D}(S)$ the function

$$F_t(\cdot) = \big(T(\cdot)f\big)(t)$$

belongs to $\mathfrak{D}(S)$,

(c) for f in $\mathfrak{D}(S)$,

$$\big(S(T(s)f)\big)(t) = \big(S(T(t)f)\big)(s),$$

(d) (associativity)

$$(T(u)F_t)(s) = \big(T(u)(T(s)f)\big)(t),$$

(e) (commutativity)

$$T(t)\big(T(s)f\big) = T(s)\big(T(t)f\big),$$

(f) $T(0)f = f$.

If we now relax the condition $q(t) = 0$, and take properties (a), . . ., (f) as basic axioms, we arrive at the following definition: A family of bounded operators $T(s)$, $s \in I$, is called a family of *generalized displacement operators* associated with the operator S if it satisfies conditions (a) through (f). The notion of the convolution of two functions can then be extended by the following definition of a generalized convolution:

$$(f * g)(s) = \int_I \big(T(s)f\big)(t)g(t)dt.$$

By continuing in this vein, interesting generalizations of several of the notions of harmonic analysis: positive-definite functions, convolution algebras, almost periodic functions, etc., can be found, and

one more proof of the spectral theorem for self adjoint differential operators of the second order can be given.

To construct the operators $T(s)$, construct two solutions u and v of the partial differential equation

$$\frac{\partial^2 w}{\partial s^2} - q(s)w = \frac{\partial^2 w}{\partial t^2} - q(t)w$$

which satisfy the boundary conditions

$$u(s, 0) = f(s), \qquad \left.\frac{\partial u}{\partial t}\right|_{t=0} = 0$$

and

$$v(s, 0) = 0, \qquad \left.\frac{\partial v}{\partial t}\right|_{t=0} = f(s)$$

respectively, and let

$$\big(T(s)f\big)(t) = u(t, s) + av(t, s)$$

for any real number a. The operator $T(t)$ will then satisfy conditions (a) through (f). A detailed exposition of this theory can be found in Levitan [1].

An approach to the study of differential operators in a Banach space of a more complex character than Hilbert space has been carried out by Feller [1] through [7]) in his investigations of the structure of semigroups in the space $C(I)$. In particular, Feller has given a theory of boundary values which presents some points of contact with the one developed in this book for Hilbert space. Feller's work is related to the studies of Hille [5] and Yosida [8] [9] on semigroups generated by differential operators. A detailed exposition of these theories would lead us too far afield into the theory of semigroups: we therefore refer the reader to the original works.

CHAPTER XIV

Linear Partial Differential Equations and Operators

1. Introduction

The Cauchy Problem, Local Dependence

In this chapter, we shall discuss a variety of theorems having to do with linear partial differential operators. Since the theory of linear partial differential operators is vast and highly ramified, we shall only touch upon a number of its aspects, with the intention of displaying a bouquet of applications of functional analysis, rather than treat any part of the theory of partial differential operators for its own sake.

To illustrate the way in which functional analysis can be applied to yield results about differential equations, let us begin with an elementary example. By a *formal (linear) partial differential operator L of order m defined in a domain D of Euclidean n-space*, we mean a formal expression

$$(1) \qquad L = \sum_{j=0}^{m} \sum_{i_1, i_2, \ldots, i_j=1}^{n} a_{i_1 \cdots i_j}(t_1, \ldots, t_n) \frac{\partial^j}{\partial t_{i_1} \cdots \partial t_{i_j}},$$

the coefficients $a_{i_1 \cdots i_j}(t_1, \ldots, t_n)$ being assumed to be defined for $t = [t_1, \ldots, t_n]$ in D, to be symmetric in the indices $i_1, i_2, \ldots, i_j$, and, unless the contrary is explicitly specified, to be infinitely differentiable in D. The term in this formal differential operator corresponding to $j = 0$ is, by definition, the operation of multiplication by a function defined on D.

Let S be a smooth surface in D. Then the *Cauchy Problem* for L along S is the problem of finding a solution f of the partial differential equation $Lf = 0$ having arbitrarily prescribed values and first $m-1$ normal derivatives at each point of the surface S. To be specific, let us assume that D is all of Euclidean n-dimensional space, and that S is a hyperplane of $n-1$ dimensions in D. Making a suitable rotation

of Euclidean n-space, we may assume for simplicity and without loss of generality that $S = \{[t_1, \ldots, t_n] | t_n = 0\}$. Then we may say that the Cauchy problem for L consists in showing that for each set $g_0, g_1, \ldots, g_{m-1}$ of infinitely differentiable functions defined on S, there exists a unique function f in $C^\infty(D)$, satisfying the equation $Lf = 0$, and such that

$$f(t) = g_0(t), \qquad \frac{\partial f}{\partial t_n}(t) = g_1(t), \ldots, \frac{\partial^{m-1} f}{\partial t_n^{m-1}}(t) = g_{m-1}(t), \qquad t \in S;$$

where here and below we write $t = [t_1, \ldots, t_n]$ for the sake of brevity. The functions $g_0, \ldots, g_{m-1}$ are called the *data* of the Cauchy problem, and the function f is called the solution of the Cauchy problem. It will be shown by an elementary functional-analytic argument that the hypothesis that the Cauchy problem for L can be solved uniquely along S has the following surprising consequence.

If A is a compact subset of D, and in particular if $A = \{p\}$ is a single point of D, then there exists a compact subset K of S, such that if the data $g_0, \ldots, g_{m-1}$ of the Cauchy problem vanish in K, the solution f vanishes on A.

This is a very general formulation of the phenomenon of finite domains of dependence. The proof is an elementary application of the closed graph theorem. Let Ξ be the F-space of all solutions of the equation of $Lf = 0$ which are in $C^\infty(D)$, taken with the norm

$$|f| = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \sum_{i_1, \ldots, i_j = 1}^{n} \frac{1}{k!\,j!} \frac{\mu_{i_1 \cdots i_j}(k, f)}{1 + \mu_{i_1 \cdots i_j}(k, f)},$$

where

$$\mu_{i_1 \cdots i_j}(k, f) = \max_{t_1^2 + \cdots + t_n^2 \leq k^2} \left| \frac{\partial^j f(t_1, \ldots, t_n)}{\partial t_{i_1} \cdots \partial t_{i_j}} \right|.$$

In this symbolism $\mu_{i_1 \cdots i_j}$ is, for $j = 0$, the function

$$\mu(k, f) = \max_{t_1^2 + \cdots + t_n^2 \leq k^2} |f(t_1, \ldots, t_n)|.$$

Let Δ be the F-space of all sets $[g_0, \ldots, g_{m-1}]$ of data, taken with the norm

$$|[g_0, \ldots, g_{m-1}]| = \sum_{p=0}^{m-1} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \sum_{i_1, \ldots, i_j = 1}^{n-1} \frac{1}{k!\,j!} \frac{\hat{\mu}_{i_1 \cdots i_j}(k, p)}{1 + \hat{\mu}_{i_1 \cdots i_j}(k, p)},$$

where

$$\hat{\mu}_{i_1\cdots i_j}(k, p) = \hat{\mu}_{i_1\cdots i_j}(k, p; [g_0, \ldots, g_{m-1}])$$
$$= \max_{0\le q\le m-1}\ \max_{t_1^2+\cdots+t_{n-1}^2\le k^2}\left|\frac{\partial^j g_q(t_1, \ldots, t_{n-1})}{\partial t_{i_1}\cdots\partial t_{i_j}}\right|.$$

Let $T : \Delta \to \Xi$ be the mapping which assigns to every set $[g_0, \ldots, g_{m-1}]$ of data the corresponding unique solution f of the equation $Lf = 0$. It is evident that T is linear, and equally evident that T is closed. Hence, by the closed graph theorem (II.2.4), T is continuous. Hence given any $\varepsilon > 0$ and any k, there is an integer l and a $\delta > 0$ such that $\mu(k, T[g_0, \ldots g_{m-1}]) < \varepsilon$ provided that

$$|\hat{\mu}_{i_1,\ldots,i_m}(l, p; [g_0, \ldots, g_{m-1}])| < \delta, \qquad i_1, \ldots, i_m \le n-1,$$
$$p = 0, \ldots, m-1.$$

Since T is linear, it follows immediately that if

$$\hat{\mu}_{i_1\cdots i_m}(l, p; [g_0, \ldots, g_{m-1}]) = 0, \qquad i_1, \ldots, i_m < n-1,$$
$$p = 0, \ldots, m-1,$$

then $\mu(k, T[g_0, \ldots, g_{m-1}]) = 0$. Thus for each integer k there is an integer l such that if the initial data for the Cauchy problem vanish in the sphere $t_1^2+\ldots+t_{n-1}^2 \le l^2$, then the solution of the Cauchy problem vanishes in the sphere $t_1^2+\ldots+t_n^2 \le k^2$. From this assertion the earlier assertion follows readily.

Consider as an instructive example the particular case in which $n = 2$, $m = 2$, and L is the Laplace operator

$$L = \nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}.$$

Does the Cauchy problem

$$\nabla^2 f(x_1, x_2) = 0, \qquad +\infty > x_i > -\infty, \qquad f(x_1, 0) = g_0(x_1),$$
$$\frac{\partial}{\partial x_2} f(x_1, 0) = g_1(x_1),$$

have a unique solution for each pair g_0, g_1 of initial data in $C^\infty(-\infty, +\infty)$? This question may be readily answered in the negative by the following consideration. As is well-known the real

and imaginary parts of a solution of the equation $\nabla^2 f = 0$ in the region $R = \{[x_1, x_2] | x_2 > 0\}$ are the real parts of a pair of analytic functions of the complex variable $z = x_1 + ix_2$ defined in that region. Hence, a solution of $\nabla^2 f = 0$ in R is an analytic function of the variables x_1, x_2 and it follows that if f vanishes in a domain of R, it must vanish identically. On the other hand, if the Cauchy problem had a unique solution for each pair g_0, g_1 of the prescribed data, then, by what has been proved above, the phenomenon of local dependence would occur, and we would be able to construct solutions of the equation $\nabla^2 f = 0$ which vanish in a domain of R but are not identically zero. This argument is evidently very general, and shows that the following two properties of a formal differential operator L tend to be mutually exclusive.

Property A: A suitably defined Cauchy problem for L has a unique solution for each set of prescribed, smooth initial data.

Property B: The solutions of $Lf = 0$ are so smooth as to be subject to the function-theoretic principle of unique continuation.

Formal partial differential operators with properties like A are commonly called *hyperbolic* operators; those with properties like B are commonly called *elliptic* operators. We have noted that Laplace's operator $\nabla^2 = \partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$ is a typical example of the second type. It will soon be seen that the operator

$$\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}$$

is hyperbolic. The elliptic-hyperbolic dichotomy is well illustrated by this pair of operators. We have already remarked that solutions of the equation

$$\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) f(x_1, x_2) = 0$$

are analytic. On the other hand, any twice differentiable function f of the form

$$f(x_1, x_2) = g(x_1 + x_2)$$

satisfies the equation

$$\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\right) f(x_1, x_2) = 0.$$

Clearly, then, the solutions of this latter equation have no tendency to be analytic, or even to have any derivatives not initially required of them.

A third category of formal partial differential operators is the *parabolic*, typified by the operator

$$\frac{\partial}{\partial x_1} - \frac{\partial^2}{\partial x_2^2}.$$

This sort of operator is closely related to the theory of semi-groups, where we have already seen general equations of the form

$$\frac{\partial}{\partial t} T(t)f = AT(t)f,$$

A being an unbounded operator which is negative in a suitable sense (cf. Corollary VIII.1.14), play a crucial role. If A is taken to be a partial differential operator which is negative in this suitable sense, we find ourselves on the common ground of semi-group theory and the theory of parabolic partial differential equations.

It should not be supposed that the three categories *elliptic*, *parabolic*, *hyperbolic* exhaust the totality of formal linear partial differential operators. Thus, for instance, the operator

$$[*] \qquad \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \frac{\partial^2}{\partial x_4^2},$$

sometimes classified as *ultrahyperbolic*, belongs to none of these categories. An operator like

$$[**] \qquad \frac{\partial^2}{\partial x_1^2} + i\frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - i\frac{\partial^2}{\partial x_4^2}$$

is difficult to classify. In addition, various authors have differed in their usage of the terms *elliptic*, *parabolic*, *hyperbolic*, so that the boundaries between these categories are not to be drawn too rigidly. The notions of elliptic, parabolic, and hyperbolic formal partial differential operators are to be regarded as landmarks in a broad field rather than as infallible labels by which particular equations are to be distinguished. It shall be seen in what follows that many important

properties of formal partial differential operators which lie in the vicinity of one of these three landmarks can be established on quite a general basis. On the other hand, the ultrahyperbolic operator [*] and the puzzling operator [**] are both well within the terra incognita of today's theory.

As the complexity of the basic elementary expression (1) makes clear, the mere notation for partial differential operators and equations tends to become overwhelming. For this reason, we shall devote the next section to the statement of a series of notational conventions and abbreviations which will be of considerable use in the sections to follow. In the body of our analysis (as in the elementary proof of local dependence given above) we shall find it to be of crucial importance to permit the analysis to proceed in *complete* spaces of differentiable functions, for which purpose it is necessary to set up suitably defined *generalized derivatives*. This task may be regarded as being solved by the definitions customary in Laurent Schwartz' theory of *distributions*, to which Section 3 below will be devoted. Once generalized derivatives in the sense of the theory of distributions have been defined, it becomes important to have information on the continuity and differentiability in the ordinary sense of functions possessing generalized derivatives in one or another generalized sense. Such information is furnished by the general theorems of Sobolev, to which Section 4 is devoted.

In the short Section 5, a number of elementary questions belonging to the geometric theory of partial differential equations are discussed.

Section 6 takes up the theory of elliptic partial differential equations and that of the associated boundary value problems. It begins with a relatively elementary proof of the principle of differentiability of weak solutions. On the basis of this a short proof of the Mautner-Gårding-Browder generalization to arbitrary singular self adjoint elliptic partial differential operators of the Weyl-Kodaira spectral representation theorem for the singular self adjoint case of ordinary differential operators is given. Next we discuss the spectral theory of elliptic operators in bounded domains, giving the theory of the general Dirichlet problem (which is based on the fundamental inequality of Gårding) and of the principle of differentiability up to

the boundary. This accomplished, it is possible to give a very short proof of the interesting nonselfadjoint completeness theorem of Browder.

In Section 7 a number of the methods developed in the earlier sections of the present chapter are applied to the study of the Cauchy problem for hyperbolic equations, giving a proof of the general theorem of K. O. Friedrichs and P. Lax on the existence and uniqueness of a solution of the Cauchy problem for the first order symmetric hyperbolic systems of partial differential equations.

In the rather brief Section 8, application of the theory of the elliptic boundary value problem developed in Section 6 is made, to give the solution of the initial boundary value problem for the "time independent" parabolic equation. This theory makes use of a number of ideas from the abstract theory of semi-groups.

2. Notational Conventions and Preliminaries

Throughout the rest of the present chapter, the symbol J will denote an *index*, i.e., a k-tuple $J = [j_1, \ldots, j_k]$ of integers. We write $|J| = k$, $\min J = \min_{1 \leq i \leq k} j_i$, $\max J = \max_{1 \leq i \leq k} j_i$. It will be convenient to allow the possibility of J being vacuous, in which case we write $|J| = 0$. The symbol E^n will denote real Euclidean n-space, and the symbol U^n will denote complex unitary n-dimensional space. An index J will be said to be an index for E^n if $\min J \geq 1$ and $\max J \leq n$. If ξ is in U^n, so that $\xi = [\xi_1, \ldots, \xi_n]$, and J is an index for E^n, so that $J = [j_1, \ldots, j_k]$, $k = |J|$, then ξ^J will denote the expression $\xi_{j_1} \xi_{j_2} \ldots \xi_{j_k}$. If $J = [j_1, \ldots, j_k]$ and $\hat{J} = [\hat{j}_1, \ldots, \hat{j}_{\hat{k}}]$ are two indices, then $J \cup \hat{J}$ denotes the index $[j_1, \ldots, j_k, \hat{j}_1, \ldots, \hat{j}_{\hat{k}}]$. If $L = [l_1, \ldots, l_n]$ is an index such that $|L| = n$, and $\xi = [\xi_1, \ldots, \xi_n]$ is in U^n, then $L \cdot \xi = \xi \cdot L$ will denote the quantity $l_1 \xi_1 + \ldots + l_n \xi_n$. If $\xi = [\xi_1, \ldots, \xi_n]$ and $\zeta = [\zeta_1, \ldots, \zeta_n]$ are two vectors in U^n, then $\xi\zeta$ will denote the vector $[\xi_1\zeta_1, \ldots, \xi_n\zeta_n]$.

The operations $\partial/\partial x_j$ and $\partial/\partial s$ of partial differentiation will sometimes be written as ∂_{x_j} or ∂_j and ∂_s respectively. If J is an index for E^n, and $|J| = k$, then the higher partial derivative

$$\frac{\partial^k}{\partial x_{j_1} \partial x_{j_2} \ldots \partial x_{j_k}}$$

will be called a partial derivative of order $k = |J|$ and will be written as ∂^J. If $|J| = 0$ the operator ∂^J is defined to be the identity operator.

Throughout the remainder of the present chapter, except where otherwise specified, n will be a fixed positive integer, and we will analyze partial differential operators, equations, etc., which apply to various classes of functions and "generalized functions" which are themselves defined in one or another subset of E^n. On occasion, however, we shall distinguish one or another coordinate variable in E^n which is to play a special role in the analysis. To do this with the smallest number of notational inconveniences, we shall often, on these occasions, pass from consideration of subsets of E^n to consideration of subsets of E^{n+1}, and regard E^{n+1} as being identical with the direct sum $E^{n+1} = E^n \oplus E^1$ of n-dimensional space and one-dimensional space. Correspondingly, each y in E^{n+1} will be written as $[x, s]$, where $y = [y_1, \ldots, y_{n+1}]$, and $s = y_{n+1}$. In contexts in which E^n and E^{n+1} are being considered together, y will then generally denote a variable point in E^{n+1}, x will generally denote a variable point in E^n, generally equal to $[y_1, \ldots, y_n]$, and $s = x_{n+1}$ will denote a real variable generally equal to the "distinguished" last coordinate of y. As is indicated in these last two sentences, the coordinates of a vector x in E^n will be written as x_j, or, in case it is desired to emphasize the fact that x_j is the jth coordinate of x and not the jth vector in a sequence of vectors, the notation $(x)_j$ will sometimes be used. Thus, if $\{x_m\}$ is a sequence of vectors in E^n, $\{(x_m)_j\}$ is the corresponding sequence of jth coordinates. In general, unless the contrary is explicitly stated, J, $\mathfrak{J}$, $\mathcal{J}$, etc., will denote indices for E^n, that is, indices whose range of variation is restricted by the condition $\min J \geq 1$, $\max J \leq n$. The symbols J_1, $\mathfrak{J}_1$, $\mathcal{J}_1$ will similarly denote indices for E^{n+1}. Thus, for instance, if ξ is in U^n, $\sum_{|J|=m} \xi^J$ is an abbreviated notation for

$$\sum_{j_1=1,\ldots,j_m=1}^{n} \xi_{j_1}\xi_{j_2}\cdots\xi_{j_m} = \Big(\sum_{j=1}^{n} \xi_j\Big)^m = \Big(\sum_{|J|=1} \xi^J\Big)^m;$$

and similarly

$$\sum_{|J_1|=m} \xi^{J_1} = \left(\sum_{j=1}^{n+1} (\xi_1)_j\right)^m.$$

In discussions of functions or generalized functions defined on

E^n or on subsets of E^n, the symbols L, L_1, $\bar{L}$, etc., will commonly be used for indices subject to the condition $|L| = n$, but otherwise unrestricted. Thus, for instance, the Fourier series of a function f defined on E^n and periodic of period 2π in each of its variables will be written either as

$$f(x) = \sum_{|L|=n} a_L e^{iL\cdot x}$$

or simply as

$$f(x) = \sum_{L} a_L e^{iL\cdot x}.$$

If m is a positive integer, an expression

$$\tau = \sum_{|J| \leq m} a_J(x)\partial^J,$$

where the coefficients a_J are infinitely differentiable functions in an open set $I \subseteq E^n$, will be called a *formal partial differential operator defined in* I, and m will be called the *order* of τ. In this expression we allow, as one of the summands, a term for which $|J| = 0$. The formal operator $a_J(x)\partial^J$ corresponding to such a J is, by definition, that of multiplication by an infinitely differentiable function defined on I. If

$$\hat{\tau} = \sum_{|J| \leq m} \hat{a}_J(x)\partial^J$$

is another formal partial differential operator defined in I, and f is a function which is infinitely often differentiable in I, then, by the use of Leibniz' rule, the expression

$$\hat{\tau}(\tau f)(x) = \sum_{|\hat{J}| \leq \hat{m}} \sum_{|J| \leq m} \hat{a}_{\hat{J}}(x)\partial^{\hat{J}}\big((a_J(x))\partial^J f\big)(x)$$

can be written in the form

$$\sum_{|J| \leq \hat{m}+m} b_J(x)(\partial^J f)(x).$$

By using Leibniz' rule, it would be easy to express the coefficients b_J algebraically in terms of the coefficients $\hat{a}_J$, a_J, and their partial derivatives. We write

$$\hat{\tau}\tau = \tilde{\tau} = \sum_{|J| \leq \hat{m}+m} b_J(x)\partial^J,$$

and call $\tilde{\tau}$ the product of $\hat{\tau}$ and τ. Similarly, the *sum* of τ and $\hat{\tau}$ is defined by

$$\sum_{|J| \leq \max(m, \hat{m})} \left(a_J(x) + \hat{a}_J(x)\right) \partial^J,$$

where we place $a_J = 0$ for $|J| > m$ and $\hat{a}_J = 0$ for $|J| > \hat{m}$.

Let I be an open set in E^n, and $\bar{I}$ be its closure. The set $C^\infty(I)$ consists of those scalar functions f defined on I which have all partial derivatives of all orders existing and continuous. Similarly, the set $C^k(I)$ consists of those scalar functions defined in I every one of whose partial derivatives of order not more than k exists and is continuous. The sets $C_0^\infty(I)$ and $C_0^k(I)$ consist of those functions in $C^\infty(I)$ and $C^k(I)$ respectively which vanish outside a compact set. The set $C^k(\bar{I})$ consists of all functions defined on I having all partial derivatives of orders at most k at each point of I and such that each partial derivative has a continuous extension to $\bar{I}$. If this is the case, $\partial^J f(x)$ is defined for x in $\bar{I}$ and $|J| \leq k$ as the extension by continuity of $\partial^J f(x)$ from I to $\bar{I}$. Then we put

$$C^\infty(\bar{I}) = \bigcap_{k \geq 1} C^k(\bar{I}), \qquad C_0^\infty(\bar{I}) = C_0^\infty(I), \qquad C_0^k(\bar{I}) = C_0^k(I).$$

On occasion, we will consider the particular case in which $\bar{I}$ is a rectangular parallelepiped in E^n of the form

$$I = \{x \in E^n | a_j \leqq x_j \leqq b_j, \qquad j = 1, \ldots, n\},$$

$a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ being two sequences of real constants. In this case, $C_\pi^k(\bar{I})$ will denote the subset of $C^k(\bar{I})$ defined by the "periodicity" conditions

$$\partial^J f(a_1, x_2, \ldots, x_n) = \partial^J f(b_1, x_2, \ldots, x_n), \quad |J| \leq k,$$
$$a_j \leqq x_j \leqq b_j, \qquad j = 2, \ldots, n.$$

$$\cdots$$

$$\cdots$$

$$\partial^J f(x_1, \ldots, x_{n-1}, a_n) = \partial^J f(x_1, \ldots, x_{n-1}, b_n), \qquad |J| \leq k,$$
$$a_j \leqq x_j \leqq b_j, \qquad j = 1, \ldots, n-1.$$

We put $C_\pi^\infty(\bar{I}) = \bigcap_{k \geq 0} C_\pi^k(\bar{I})$. Then evidently $C_\pi^k(\bar{I})$ may be regarded as being identical with the set of functions in $C^k(E^n)$ which are multiply periodic, of period $b_j - a_j$ in the variable x_j, $j = 1, \ldots, n$.

The vector spaces $C^k(I)$, etc., may be made into F spaces as follows. Let K_m be an increasing sequence of compact subsets of I or of $\bar{I}$. Suppose that K_m are such that any compact subset of I belongs to one of the sets K_m. Then for a function f in one of the spaces $C^k(I)$, $C^k(\bar{I})$, or $C^k_\pi(\bar{I})$, we place

$$\mu(f; J, m) = \sup_{x \in K_m} |\partial^J f(x)|,$$

and define the norm of f by the equation

$$|f| = \sum_{m=0}^{\infty} \sum_{j=1}^{k} \sum_{|J|=j} \frac{1}{2^m 2^j j!} \frac{\mu(f; J, m)}{1+\mu(f; J, m)}.$$

This norm makes each of the spaces listed above into a complete F-space. If $k < \infty$ and $\bar{I}$ is compact, but not otherwise, the spaces $C^k(\bar{I})$ and $C^k_\pi(\bar{I})$ are B-spaces under a norm equivalent to the norm displayed, though not under the norm displayed itself. It is in the sense of these norms that we speak of *the topology of* $C^k(I)$, $C^k(\bar{I})$, etc.

If τ is a formal partial differential operator of the form

$$\tau = \sum_{|J| \leq m} a_J(x) \partial^J,$$

then the formal partial differential operator

$$\tau^* = \sum_{|J| \leq m} (-1)^J \partial^J \overline{a_J(x)}$$

is called the *adjoint*, or *formal adjoint* of τ. If $\tau = \tau^*$, then τ is said to be *formally symmetric*, or *formally self adjoint*. The formal partial differential operator

$$\tau^+ = \sum_{|J| \leq m} (-1)^J \partial^J a_J(x)$$

is called the *real adjoint* of τ. The formal partial differential operator

$$\bar{\tau} = \sum_{|J| \leq m} \overline{a_J(x)} \partial^J$$

is called the *complex conjugate* of τ. It is readily seen by direct computation that

$$(\tau + \hat{\tau})^* = \tau^* + \hat{\tau}^*, \qquad (\tau + \hat{\tau})^+ = \tau^+ + \hat{\tau}^+,$$
$$(\tau\hat{\tau})^* = \hat{\tau}^* \tau^*, \qquad (\tau\hat{\tau})^+ = \hat{\tau}^+ \tau^+,$$

while

$$(\alpha\tau)^* = \bar{\alpha}\tau^*, \qquad (\alpha\tau)^+ = \alpha\tau^+,$$

if α is a complex number.

It is an important fact that functions f in $C^\infty(E^n)$ can be prescribed more or less arbitrarily, i.e., that they have no global "structural" properties which do not follow immediately from the definition of the class $C^\infty(E^n)$. The following lemma expresses an aspect of this general principle that will find important applications in the later sections of the present chapter.

1 LEMMA. *Let K be a compact subset of E^n, and let U be an open set containing K. Then there exists a function φ in $C^\infty(E^n)$ such that $0 \leqq \varphi(x) \leqq 1$ for all x in E^n, $\varphi(x) = 1$ for x in K, and $\varphi(x) = 0$ for x not in U.*

PROOF. We shall construct the desired function explicitly in a sequence of steps. The function f_1 defined by the equations

$$\begin{aligned} f_1(s) &= 0, && s \leqq 0, \\ &= \exp(-s^{-2}), && s > 0, \end{aligned}$$

belongs to $C^\infty(E^1)$, vanishes for $s \leq 0$, is positive for $s > 0$, and is monotone increasing. The function f_2 defined by $f_2(s) = f_1(s)f_1(1-s)$ belongs to $C^\infty(E^1)$, vanishes for $s \leq 0$ and $s \geqq 1$, and is positive for $0 < s < 1$. The function f_3 in $C^\infty(E^1)$ defined by the integral

$$f_3(s) = \int_{-\infty}^{s} f_2(t)dt,$$

vanishes for $s \leqq 0$, is monotone increasing, identically equal to one for $s \geqq 1$, and has $0 \leqq f_3(s) \leqq 1$ for all s. The function g_ε in $C^\infty(E^n)$ defined by the equation

$$g_\varepsilon(x) = f_2\left(\frac{x_1 - \frac{1}{2}}{\varepsilon}\right) \cdots f_2\left(\frac{x_n - \frac{1}{2}}{\varepsilon}\right),$$

vanishes if $|x| > \varepsilon/2$, is positive for $|x| < \varepsilon/2$, and takes on only non-negative values. Suppose that with each point p in K we associate a positive number, ε_p such that $\{q \in E^n \mid |q-p| \leqq \varepsilon_p\} \subset U$. By the Heine-Borel theorem, a finite collection $\{U_{p_1}, \ldots, U_{p_k}\}$ of the sets $U_p = \{q \in E^n \mid |q-p| < \varepsilon_p/2\}$ with p in K, cover the set K. If we place

$$g(x) = \sum_{j=1} g_{\varepsilon_{p_j}}(x-p_j),$$

it is clear that g is in $C^\infty(E^n)$, $g(x) > 0$ for x in K, $g(x) = 0$ for x not in U, and that g takes on only non-negative values. Let δ be the minimum value assumed by the function g on the set K. Then the function φ defined by the equation

$$\varphi(x) = f_3\left(\frac{g(x)}{\delta}\right),$$

is in $C^\infty(E^n)$ and has $\varphi(x) = 1$ for x in K, $\varphi(x) = 0$ for x not in U, and takes on values in the interval $[0, 1]$. Q.E.D.

Another useful property of the functions in C^∞ is their density, expressed in the following two lemmas.

2 LEMMA. *Let I be a domain in E^n, and let $1 \leq p < \infty$. Then the subset $C_0^\infty(I)$ of $L_p(I)$ is dense in $L_p(I)$.*

PROOF. Let R be a large fixed rectangle in E^n. Let A be the closed subspace of $L_p(R)$ spanned by $C_0^\infty(R)$. Let B be the set of bounded functions in A. It is evident from Lemma 1 that B contains the characteristic function of every rectangle contained in R. If f is in $C_0^\infty(R)$ and $g_n \to g$ in the norm of $L_p(R)$ then clearly $fg_n \to fg$ in $L_p(R)$. Thus, if f is in $C_0^\infty(R)$ and g is in A, fg is also in A. Repeating this argument we find that if f and g are both in A, then fg is in A. Hence, if f and g are both in B, then fg is in B. Since A contains the difference of any two of its elements, so does B. Thus, if Σ is the family of all subsets of R whose characteristic functions belong to B, Σ contains the complement of each of its members, and the intersection and union of any two of its members; that is, Σ is a field of sets. If $\{f_n\}$ is an increasing sequence of characteristic functions belonging to B, then it is clear that $f = \lim_{n\to\infty} f_n$ belongs to A, and hence belongs to B. Thus, Σ is a σ-field. Thus, Σ includes each Lebesgue measurable subset of R (cf. III.11.3), and hence A includes all of $L_p(R)$ (cf. III.8.8).

Let G be the closed subspace of $L_p(E^n)$ spanned by $C_0^\infty(E^n)$. If χ_j denotes the characteristic function of the cube

$$\{x \in E^n \mid |x_i| < j, i = 1, \ldots, n\},$$

then it is evident that $\chi_j f \to f$ as $j \to \infty$ for each f in $L_p(E^n)$.

Hence, by the conclusion of the first paragraph of the present proof, $L_p(E^n) \subseteq G$.

Finally, let D be the closed subspace of $L_p(I)$ spanned by $C_0^\infty(I)$. If f is in $L_p(I)$, then by what has been proved above, there is a sequence $\{f_n\}$ of elements of $C_0^\infty(E^n)$ approaching f in $L_p(I)$. Let χ be the characteristic function of I. Then $\chi f_m \to \chi f = f$ as $m \to \infty$. Hence, if it is shown that χf_m is in D, it will follow that f is in D, and the lemma will be proved. Let I_0 be the bounded open subset of I where $f_m(x) \neq 0$. Since I_0 is an open subset of E^n, there exists an increasing sequence $\{K_p\}$ of compact subsets of I_0 such that $\bigcup_{p=1}^\infty K_p = I_0$. Using Lemma 1, let $\{\varphi_q\}$ be a sequence of functions in $C_0^\infty(I_0)$ with $0 \leq \varphi_q(x) \leqq 1$ for all x and such that $\varphi_q(x) = 1$ for x in K_q. Then it is evident from the Lebesgue dominated convergence theorem that $f_m \varphi_q \to \chi f_m$ as $q \to \infty$, so that χf_m is in D. Q.E.D.

3 LEMMA. *Let I be an open set in E^n and let p be a non negative integer. Then the subset $C_0^\infty(I)$ of $C_0^p(I)$ is dense in $C_0^p(I)$.*

PROOF. Setting all functions in $C_0^p(I)$ equal to zero outside their domains of definition, we can conveniently suppose them to belong to $C_0^p(E^n)$. It is clear from the principle of uniform continuity that $f(x+y) \to f(x)$ uniformly in x as $|y| \to 0$ for each f in $C_0^p(E^n)$.

Hence, it is evident that if f is in $C_0^p(E^n)$, $f(\cdot+y) \to f(\cdot)$ in the topology of $C^p(E^n)$ as $|y| \to 0$. Given a function f in $C_0^p(I)$, let K be a compact subset of I outside of which f vanishes. Let $\delta > 0$ be given, and choose $\varepsilon = \varepsilon(\delta) > 0$ such that

(i) no point of $E^n - I$ lies within a distance 2ε of a point of K,

(ii) $|\partial^J f(x+y_0) - \partial^J f(x)| < \delta, \quad x \in E^n, \quad |y_0| < \varepsilon, \quad |J| \leq p.$

Using Lemma 1, let φ be a non-negative function in $C_0^\infty(E^n)$ such that

(iii) $\varphi(x) = 0, \quad |x| \geq \varepsilon,$

(iv) $\int_{E^n} \varphi(x)dx = 1.$

Let

$$(f * \varphi)(x) = \int_{E^n} f(x-y)\varphi(y)dy = \int_{E^n} \varphi(x-y)f(y)dy.$$

From the second expression for $f * \varphi$ and from (i) and (iii) it follows immediately that $f * \varphi$ is in $C_0^\infty(I)$. From the first expression for $f * \varphi$ and from (ii), (iii), and (iv), we have

$$|\partial^J(f*\varphi)(x) - \partial^J f(x)| \leq \delta \int_{E^n} \varphi(x)dx = \delta, \qquad |J| \leq p, \qquad x \in E^n.$$

Thus f is closely approximated in the norm of $C^p(E^n)$ by the function $\varphi * f$ in $C_0^\infty(I)$, proving the present lemma. Q.E.D.

The following elementary lemma states the "principle of the partition of unity" which will be very useful in what follows.

4 Lemma. *Let K be a compact subset of E^n, and let $\{U_\alpha\}$ be a covering of K by open sets. Then there exists a finite set $f_1, \ldots, f_q$ of non-negative elements of $C_0^\infty(E^n)$ with the properties*

(i) $\sum_{j=1}^{q} f_j(x) \leq 1, \qquad x \in E^n,$

(ii) $\sum_{j=1}^{q} f_j(x) = 1, \qquad x \in K,$

(iii) *each f_j vanishes outside some one of the sets U_α.*

Proof. By Lemma 1, for each point y in K there is a non-negative function $\tilde{f}_y$ in $C_0^\infty(E^n)$ with the properties that $\tilde{f}_y(y) > 0$, and $\tilde{f}_y(x) = 0$ if x lies outside some particular one of the sets U_α. Let $N_y = \{x \in E^n | \tilde{f}_y(x) > 0\}$. Then, since K is compact, a finite collection of the sets N_y cover K. If $\{\tilde{f}_1, \ldots, \tilde{f}_q\}$ is the corresponding collection of functions, we have $\sum_{j=1}^{q} \tilde{f}_j(x) > 0$ for each x in K. Let

$$N = \{x \in E^n | \sum_{j=1}^{q} \tilde{f}_j(x) > 0\}.$$

Then, using Lemma 1, let G be a function in $C_0^\infty(E^n)$ such that $0 \leq G(x) \leq 1$ for all x, $G(x) = 1$ for x in K, and $G(x) = 0$ for x not in N. If we let

$$f_j(x) = \{1 - G(x) + \sum_{j=1}^{q} \tilde{f}_j(x)\}^{-1} \tilde{f}_j(x),$$

it is evident that the set $\{f_1, \ldots, f_q\}$ of functions in $C_0^\infty(E^n)$ has the desired properties. Q.E.D.

The following lemma will be used in Section 7 below.

5 Lemma. *Let f be in $C_0^p(E^n)$. Then, as $t \to 1$, $f(t\,\cdot) \to f(\cdot)$ in the topology of $C^p(E^n)$.*

Proof. By the principle of uniform continuity, as t approaches 1, $g(tx) \to g(x)$ uniformly in x for each g in $C_0(E^n)$. Similarly, as $t \to 1$,

$t^{|J|}\partial^J g(tx) \to \partial^J g(x)$ uniformly in x for each g in $C_0^p(E^n)$, provided that $|J| \leq p$. Q.E.D.

If f is a Lebesgue-integrable function defined on a subset e of E^n, then

$$\int_e f(x)dx$$

will denote the integral of f over the set e. Let I be an open subset of E^n, τ a formal differential operator of order m, and f a function in $C^m(I)$. Then if g is in $C_0^m(I)$, and vanishes outside a small sphere contained in I, it follows immediately on integrating by parts with respect to all the variables $x_1, \ldots, x_n$ successively that

$$\int_I (\tau f)(x)\overline{g(x)}dx = \int_I f(x)\overline{(\tau^* g)(x)}dx. \tag{1}$$

$$\int_I (\tau f)(x)g(x)dx = \int_I f(x)(\tau^+ g)(x)dx. \tag{2}$$

It will now be shown that these two equations are still valid for an arbitrary function g in $C_0^m(I)$. Let K be a compact subset of I outside of which g vanishes. Let $K \subseteq S_1 \cup \ldots \cup S_p$ be a finite covering of K by closed spheres entirely contained in I, and let $S_i^0 \supseteq S_i$ be a family of open spheres contained entirely in I. Using Lemma 4, we can find a finite set $\{f_j\}$, $j = 1, \ldots, q$, of functions in $C_0^\infty(I)$, each of which vanishes outside some one of the spheres S_i^0, such that $\sum_{j=1}^q f_j(x) = 1$ for x in K. If we place $g_j = f_j g$, we have evidently $\sum_{j=1}^q g_j = g$. Since

$$\int_I (\tau f)(x)g_j(x)dx = \int_I f(x)(\tau^+ g_j)(x)dx, \qquad j = 1, \ldots, q,$$

by what has already been proved, it follows on adding all these equations that (2) is valid for each g in $C_0^m(I)$. The general validity of (1) follows in exactly the same way.

3. The Theory of Distributions

It will be most essential in subsequent sections to be able to apply partial differential operators to *complete* spaces of functions. Suppose, for instance, that we consider the operator defined for each function f in $C_0^\infty(E^2)$ by

$$(A_0 f)(x, y) = \frac{\partial^2 f}{\partial x \, \partial y}(x, y).$$

This is an operator densely defined in $L_2(E^2)$, but not a closed operator. If we let A be its closure, we find that $D(A)$ contains non-differentiable functions. Which non-differentiable functions? One might expect the answer to be "those (non-differentiable) functions f such that $\partial_x \partial_y f$ belongs to $L_2(E^2)$." In order for such an answer to make sense, it is desirable that we should be able to define $\partial_x \partial_y$ for every function, differentiable or not, and irrespective of whether $\partial_x \partial_y f$ belongs to $L_2(E^2)$ or not. Such a "derivative" can no longer be an element of any space of functions, but can only be a "function" in some generalized sense. Hence, we are led to the attempt to define some sort of "generalized function." A very complete and interesting development of such a theory of generalized functions was given by Laurent Schwartz; the generalized functions were called by him "distributions." It is the purpose of the present section to develop those parts of the theory of distributions which are needed in our subsequent development of the theory of partial differential operators.

1 Definition (i) Let I be an open set in E^n. Let $\{\varphi_n\}$ be a sequence of functions in $C_0^\infty(I)$ and let φ be in $C_0^\infty(I)$. Then, if there is a compact subset K of I such that all the function φ_n vanish outside K, and if, in addition, $\varphi_n \to \varphi$ in the topology of $C_0^\infty(I)$, we shall write

$$\varphi_n \rightrightarrows \varphi \quad \text{in} \quad I.$$

(ii) A linear functional F defined on $C_0^\infty(I)$ such that $F(\varphi_n) \to F(\varphi)$ whenever $\varphi_n \rightrightarrows \varphi$ in I is called a *distribution in* I.

(iii) The family of all distributions in I will be denoted by $D(I)$.

The following definition indicates the sense in which the space of distributions constitutes a space of generalized functions, by giving an exact sense in which functions may be regarded as particular distributions.

2 Definition. Let I be an open set in E^n. Let f be a function defined in I which is (Lebesgue) integrable over every compact subset of I. Then the distribution F defined by

$$F(\varphi) = \int_I \varphi(x) f(x) \, dx, \qquad \varphi \in C_0^\infty(I),$$

is called the ***distribution corresponding to f***.

It is clear that if F corresponds to the function f and G corresponds to the function g in the sense of the above definition, then $\alpha F+\beta G$ corresponds to $\alpha f+\beta g$. Thus the linear space of functions integrable on each compact subset of I may be considered to be imbedded as a subspace of the space of all distributions in I. The next lemma shows that this imbedding is essentially one-to-one.

8 LEMMA. *If, in the sense of the preceding definition, a distribution corresponds to two functions f and g, then $f(x) = g(x)$ for almost all x.*

PROOF. By considering the difference $f-g$, we may evidently suppose without loss of generality that $g = 0$. Thus, we must show that if

$$\int_I f(x)\varphi(x)dx = 0$$

for all φ in $C_0^\infty(I)$, then $f(x) = 0$ for almost all x in I. Now, if χ is the characteristic function of any Borel subset e of I whose closure $\bar{e}$ is compact and contained in I, it follows from Lemmas 2.2, and III.3.6, III.6.2 that there is a sequence $\{\varphi_n\}$ of elements in $C_0^\infty(I)$ with $\varphi_n \to \varphi$ almost everywhere, If, using Lemma 2.1, we let ψ be an element of $C_0^\infty(I)$ such that $\psi(x) = 1$ for x in $\bar{e}$, and replace φ_n by $\varphi_n\psi$, it follows that we may assume without loss of generality that all the functions φ_n vanish outside some compact subset K of I. If, using Lemma 2.1 again, we let ζ be a function in $C_0^\infty(E^*)$ such that $\zeta(t) = t$ for $0 \leq t \leq 1$, $\zeta(t) = 0$ for $|t| \geqq 2$, and replace $\varphi_n(\cdot)$ by $\zeta(\varphi_n(\cdot))$, it follows that without loss of generality we may assume also that $\{\varphi_n\}$ is a uniformly bounded sequence of functions. Then

$$\int_e f(x)dx = \int_I f(x)\chi(x)dx = \lim_{n\to\infty}\int_I f(x)\varphi_n(x)dx = 0$$

by the Lebesgue dominated convergence theorem for each Borel subset e of I whose closure $\bar{e}$ is contained in I. Thus, it follows readily (cf. III.2.15) that $f(x) = 0$ for almost all x in I. Q.E.D.

Lemma 3 enables us to make the following definition.

4 DEFINITION. A distribution F which corresponds to a function f in the sense of Definition 2 will be said to *be a function*. If f is continuous, differentiable, belongs to $L_p(I)$, $C^n(I)$, $C_0^\infty(I)$, etc., F will be

said to be continuous, differentiable, belong to $C^n(I)$, $C_0^\infty(I)$, etc. *In general, we will simply identify a distribution which is a function with the function to which it corresponds.*

In connection with Definition 4, it should be noted that two continuous functions defined in I which differ at most on a Lebesgue null set are in fact everywhere identical. Thus, by Lemma 3, a distribution F *corresponds to a unique continuous function if it corresponds to any continuous function at all.*

The following definition shows the way in which a distribution in I may be differentiated partially, and also the way in which it may be multiplied by an element of $C^\infty(I)$.

5 Definition. Let τ be a formal partial differential operator defined in an open subset I of E^n, and with coefficients in $C^\infty(I)$. Let F be a distribution in I. Then τF will denote the distribution defined by the equation

$$(\tau F)(\varphi) = F(\tau^+\varphi), \qquad \varphi \in C_0^\infty(I).$$

The evident fact that $\varphi_n \rightrightarrows \varphi$ implies $\tau^+\varphi_n \rightrightarrows \tau^+\varphi$ provides basic justification for this definition, by showing that τF, as defined by the above equation, satisfies condition (ii) of Definition (1). Additional justification for Definition 5 is contained in the following lemma.

6 Lemma. *Let I be an open subset of E^n.*

(i) *If the distribution F in I corresponds to the function f in $C^n(I)$ and if τ is a formal partial differential operator of order at most n defined in I, then τF corresponds to τf.*

(ii) $\tau(\alpha F+\beta G) = \alpha\tau F+\beta\tau G, \qquad F, G \in D(I).$

(iii) $(\alpha\tau_1+\beta\tau_2)F = \alpha(\tau_1 F)+\beta(\tau_2 F), \qquad F \in D(I).$

(iv) $(\tau_1\tau_2)F = \tau_1(\tau_2 F), \qquad F \in D(I).$

Proof. Part (i) follows from Definition 5 and the result on integration by parts proved in the final paragraph of Section 2.

Parts (ii), (iii) and (iv) follow from Definition 5 and the obvious identities

$$(\alpha\tau_1+\beta\tau_2)^+ = \alpha(\tau_1{}^+\varphi)+\beta(\tau_2{}^+\varphi),$$
$$(\tau_1\tau_2)^+\varphi = \tau_2{}^+(\tau_1{}^+\varphi),$$

which are valid for φ in $C_0^\infty(I)$. Q.E.D.

If the operator τ of Definition 5 is ∂^J, where J is any index for E^n, then Definition 5 assigns a meaning to $\partial^J F$ for each F in $D(I)$. If, on the other hand, the operator τ is of order zero, i.e., is the operation of multiplying by a function a in $C^\infty(I)$, then Definition 5 assigns a meaning to the product aF for each F in $D(I)$. A few examples may be illuminating. Suppose that F is the distribution on the real axis E^1 which corresponds to the non-differentiable function ζ (Heaviside function) defined by the equations

$$\begin{aligned} \zeta(s) &= 0, \qquad s \leqq 0, \\ \zeta(s) &= 1, \qquad s > 0. \end{aligned}$$

Then

$$F(\varphi) = \int_0^\infty \varphi(s)ds, \qquad \varphi \in C_0^\infty(E^1),$$

and

$$(\partial_s F)(\varphi) = -\int_0^\infty \varphi'(s)ds = \varphi(0) - \varphi(\infty) = \varphi(0),$$

so that

$$\partial_s F = \delta,$$

where δ is the Dirac distribution defined by the equation

$$\delta(\varphi) = \varphi(0).$$

The derivatives of this distribution are evidently defined by

$$\delta^{(n)}(\varphi) = \left(\left(\frac{d}{ds}\right)^n \delta\right)(\varphi) = (-1)^n \varphi^{(n)}(0).$$

We have

$$\begin{aligned} (s^m \delta^{(n)})(\varphi) &= (-1)^n (s^m \varphi)^{(n)}(0) = 0, && m > n, \\ &= \frac{n!}{(n-m)!}(-1)^n \varphi^{(n-m)}(0), && m \leq n. \end{aligned}$$

Thus

$$\begin{aligned} s^m \delta^{(n)} &= 0, && m > n, \\ &= \frac{n!}{(n-m)!}(-1)^m \delta^{(n-m)}, && m \leqq n, \end{aligned}$$

a set of identities first stated by Dirac.

It is also possible in an evident way to define the complex conjugate of a distribution.

7 Definition. Let I be an open set in E^n, and let F be a distribution in I. Then the distribution $\bar{F}$ in I defined by the equation

$$\bar{F}(\varphi) = F(\bar{\varphi}), \qquad \varphi \in C_0^\infty(I),$$

is called the *complex conjugate of F.*

8 Lemma. *Let I and F be as in the preceding definition, and let τ be a formal partial differential operator defined in I. Then*

(i) $\quad \bar{\bar{F}} = F,$

(ii) $\quad \overline{\alpha F} = \bar{\alpha}\bar{F},$

(iii) $\quad \overline{F_1+F_2} = \bar{F}_1+\bar{F}_2,$

(iv) $\quad \overline{\tau F} = \bar{\tau}\bar{F}.$

(v) *If F corresponds to a function f, then $\bar{F}$ corresponds to the complex conjugate of f.*

The proof of this lemma is left to the reader as an exercise.

Another operation on distributions which is of importance is that of *restriction.*

9 Definition. Let I be an open set in E^n, I_0 an open subset of I and F a distribution in I. Then the distribution $F|I_0$ in I_0 defined by the equation

$$(F|I_0)(\varphi) = F(\varphi), \qquad \varphi \in C_0^\infty(I_0),$$

is called the restriction of F to I_0.

10 Lemma. *Let I and I_0 be as in the preceding definition, and let F and G be distributions in I. Then*

(i) *the correspondence $F \to F|I_0$ is a linear mapping of $D(I)$ into $D(I_0)$;*

(ii) *if F corresponds to a function f, then $F|I_0$ corresponds to the function $f|I_0$;*

(iii) *if τ is a partial differential operator defined in I, and $\tau|I_0$ denotes its restriction to I_0, then*

$$(\tau|I_0)(F|I_0) = (\tau F|I_0);$$

(iv) $$\overline{(F|I_0)} = \bar{F}|I_0;$$

(v) *let I be an open set in E^n, and let $\{I_\alpha\}$ be a family of open*

subsets of I and let F be in $D(I)$. If F vanishes in each set I_α, it vanishes in $\bigcup_\alpha I_\alpha$.

Proof. The proofs of the first four parts of this lemma are left to the reader as an exercise.

To prove (v), we must show from our hypothesis that $F(\varphi) = 0$ if φ is in $C_0^\infty(\bigcup_\alpha I_\alpha)$. Let K be a compact subset of $\bigcup_\alpha I_\alpha$ outside of which φ vanishes. Using Lemma 2.4, let $\{\varphi_1, \ldots, \varphi_p\}$ be a finite set of functions in $C_0^\infty(E^n)$ such that $\varphi = \sum_{j=1}^p \varphi_j \varphi$, and such that each function φ_j vanishes outside some set I_α. Then

$$F(\varphi) = \sum_{j=1}^{p} F(\varphi_j \varphi) = 0. \qquad \text{Q.E.D.}$$

Lemma 10 enables us to make the following definition.

11 Definition. Let F be a distribution in the open subset I of E^n. Then the closed set C_F in I which is the complement in I of the largest open set in I in which F vanishes, i.e., which is the complement in I of the union of all the open subsets of I in which F vanishes, is called the *carrier* or *support* of F.

12 Lemma. *Let I be an open subset of E^n, F a distribution in I, and C_F its carrier. Let I_0 be an open subset of E^n whose closure does not intersect C_F. Then there exists a unique distribution $G \in D(I \cup I_0)$ such that $G|I = F$ and $C_G = C_F$.*

Proof. Let K be any compact subset of $I \cup I_0$. Then $K\bar{I}_0$ and KC_F are disjoint compact sets. Hence, by Lemma 2.1, there is a function ψ_K in $C_0^\infty(I \cup I_0)$ such that $\psi_K(x) = 1$ for x in a neighborhood of KC_F and $\psi_K(x) = 0$ for x in a neighborhood of $K\bar{I}_0$. Put $G(\varphi) = F(\psi_K \varphi)$ for every φ in $C_0^\infty(I \cup I_0)$ which vanishes outside of K. In order that this definition be legitimate, we must show that if K_0 is a second compact subset of $I \cup I_0$ outside of which φ vanishes, then $F(\psi_K \varphi - \psi_{K_0}\varphi) = 0$. But, since $\psi_K\varphi - \psi_{K_0}\varphi$ evidently vanishes outside of a compact subset of $I - C_F$ in this case, the assertion is obvious.

It follows immediately from Definition 1(i) that $G(\varphi_n) \to G(\varphi)$ if $\varphi_n \rightrightarrows \varphi$ in $C_0^\infty(I \cup I_0)$. Since any two functions in $C_0^\infty(I \cup I_0)$ both vanish outside some common compact subset of $I \cup I_0$, it is also clear that G is a linear functional. Thus G is in $D(I \cup I_0)$. If φ is in $C_0^\infty(I)$,

and φ vanishes outside K, then $\psi_K\varphi-\varphi$ vanishes outside a compact subset of $I-C_F$, so that $G(\varphi) = F(\psi_K\varphi) = F(\varphi)$. Thus $G|I = F$. If $KC_F = 0$ and the function φ in $C_0^\infty(I \cup I_0)$ vanishes outside K, then it is clear that $\varphi\psi_K$ vanishes outside a compact subset of $I - C_F$; thus $G(\varphi) = F(\psi_K\varphi) = 0$. This shows that $C_G \subseteq C_F$, and it is clear conversely that $C_F - C_G|I \subseteq C_G$. This completes the proof of the existence of G.

All that remains to prove is the uniqueness of G. Hence, let G_1 be a second element of $D(I \cup I_0)$ such that $C_{G_1} \subseteq C_F$, and $G_1|I = F$. Then it is evident from Lemma 10 and Definition 11 that $C_{(G_1-G)} \subseteq C_F$, while, since, by Lemma 10, $(G_1-G)|I = F-F = 0$, $G_1-G = 0$ in I. Hence, it follows from Lemma 10 that $G_1-G = 0$. Q.E.D.

Remark. If a distribution F is a restriction of a distribution G defined in an open set I, then G will be said to be an *extension of* F to I.

13 Lemma. *Let H be a linear combination of the distributions F and G defined in I, let τ be a formal partial differential operator defined in I, and let a be in $C^\infty(I)$. Then*

(i) $C_H \subseteq C_F \cup C_G$;

(ii) $C_{\tau F} \subseteq C_F$;

(iii) $C_{\overline{F}} = C_F$;

(iv) *if a vanishes outside the closed set K, then* $C_{aF} \subseteq KC_F$.

The proofs of these statements are left to the reader.

We now wish to take up the study of certain important subspaces of $D(I)$ which are at the same time spaces of functions. The following elementary lemma gives us a useful result in this direction.

14 Lemma. *Let I be an open subset of E^n, and F be in $D(I)$. Let $\infty \geqq p > 1$, and $1/p+1/p' = 1$. Then F is a function in $L_p(I)$ if and only if there is a finite constant K such that*

$$[*] \qquad |F(\varphi)| \leqq K|\varphi|_{p'}, \qquad \varphi \in C_0^\infty(I).$$

Proof. If F is in $L_p(I)$, the inequality $[*]$ is simply that of Hölder (cf. III.3.2). On the other hand, if the inequality $[*]$ is satisfied, then by the Hahn-Banach theorem (II.3.11), F may be extended to a continuous linear functional defined on all of $L_{p'}(I)$, and the present lemma follows immediately from Theorem IV.8.1. Q.E.D.

15 Definition. Let I be an open subset of E^n and let k be a non-negative integer. Then

(i) the set of all F in $D(I)$ such that $\partial^J F$ is in $L^2(I)$ for all $|J| \leqq k$ will be denoted by $H^{(k)}(I)$. For each pair F, G in $H^{(k)}(I)$ we write

$$(F, G)_{(k)} = \sum_{|J| \leqq k} \int_I (\partial^J F)(x)(\partial^J \bar{G})(x) dx$$

and

$$|F|_{(k)} = \{(F, F)_{(k)}\}^{1/2};$$

(ii) the symbol $H_0^{(k)}(I)$ will denote the closure in the norm of $H^{(k)}(I)$ of the subspace $C_0^\infty(I)$;

(iii) the symbol $A^{(k)}(I)$ will denote the set of all F in $D(I)$ such that $F|I_0$ is in $H^{(k)}(I_0)$ for each open subset I_0 of I whose closure is compact and contained in I.

16 Lemma. *Let I be an open set in E^n. Then the space $H^{(k)}(I)$ of the preceding definition is a complete Hilbert space, and the space $H_0^{(k)}(I)$ is a closed subspace of $H^{(k)}(I)$.*

Proof. Let $\{F_n\}$ be a Cauchy sequence in $H^{(k)}(I)$. Then, since $|F|_{(k)} \geqq |F|_2$ for each F in $H^{(k)}(I)$, it is clear that $\{F_n\}$ converges to some F in $L_2(I)$. Similarly, since $|F|_{(k)} \geqq |\partial^J F|_2$ for each F in $H^{(k)}(I)$ and each index J such that $|J| \leq k$, it is clear that if $|J| \leq k$, the sequence $\{\partial^J F_n\}$ converges to some F_J in $L_2(I)$. Let φ be in $C_0^\infty(I)$. Then

$$\int_I F_J(x)\varphi(x)dx = \lim_{n\to\infty} \partial^J F_n(\varphi) = \lim_{n\to\infty} (-1)^{|J|} F_n(\partial^J \varphi) = (-1)^{|J|} F(\partial^J \varphi),$$

proving that $\partial^J F = F_J$ is in $L_2(I)$ for $|J| \leq k$, so that F is in $H^{(k)}(I)$. Since $|\partial^J F_n - \partial^J F|_2 \to 0$ as $n \to \infty$ for each index J with $|J| \leq k$, it is clear that $|F_n - F|_{(k)} \to 0$ as $n \to \infty$. This proves that $H^{(k)}(I)$ is complete. That it is a Hilbert space follows from Definition 15 (i). That $H_0^{(k)}(I)$ is a closed subspace of $H^{(k)}(I)$ follows from Definition 15 (ii). Q.E.D.

17 Definition. Let I be an open subset of E^n and let k be a positive integer. Then

(i) the set of all F in $D(I)$ such that

$$|F|_{(-k)} = \sup_{\varphi \in C_0^\infty(I)} \frac{|F(\varphi)|}{|\varphi|_{(k)}} < \infty$$

will be denoted by $H^{(-k)}(I)$;

(ii) the set of all F in $D(I)$ whose restrictions $F|I_0$ are in $H^{(-k)}(I_0)$ for each open subset I_0 of I whose closure is compact and contained in I will be denoted by $A^{(-k)}(I)$.

18 Lemma. *Let I be an open subset of E^n. Then*

(i) $A^{(k)}(I) \supseteq A^{(k+1)}(I)$, $\quad +\infty > k > -\infty$;

(ii) $H^{(k)}(I) \supseteq H^{(k+1)}(I)$, $\quad +\infty > k > -\infty$;

(iii) $H_0^{(k)}(I) \supseteq H_0^{(k+1)}(I)$, $\quad \infty > k \geqq 0$.

Moreover, the identity mapping of $H^{(k+1)}(I)$ into $H^{(k)}(I)$ is norm-reducing and hence continuous.

Proof. Statement (i) follows from statement (ii) by Definitions 15 (iii) and 17 (ii). Statement (iii) follows from statement (ii) and the fact that $|F|_{(k+1)} \geqq |F|_{(k)}$ for all $k \geqq 0$ and F in $H^{(k+1)}(I)$, (cf. Definition 15 (i)). To prove (ii) and the final statement of the lemma, we first note that it is evident for $k \geqq 0$ from Definition 15 (i). If $k < 0$ and F is in $H^{(k+1)}(I)$, then

$$\frac{|F(\varphi)|}{|\varphi|_{(-k)}} \leqq \frac{|F(\varphi)|}{|\varphi|_{(-k-1)}} \leqq |F|_{(-k-1)}, \qquad \varphi \in C_0^\infty(I),$$

since $|\varphi|_{(-k-1)} \leqq |\varphi|_{(-k)}$ by Definition 15(i). Thus F is in $H^{(k)}(I)$. This also proves the final statement of the lemma for $k < 0$. Q.E.D.

19 Lemma. *Let I be an open set in E^n and let k be an integer, positive or negative. Then the space $H^{(k)}(I)$ is a complete B-space.*

Proof. For $k \geqq 0$, this is merely Lemma 16. For $k < 0$, $H^{(k)}(I)$ has been defined simply as the Banach space adjoint to $H^{(-k)}(I)$, so that the present lemma follows from Corollary II.3.2. Q.E.D.

20 Definition. Let I be an open set in E^n and let k be an integer, positive or negative. Let $\{I_m\}$, $m \geqq 1$, be a sequence of open subsets of I whose closures are compact and contained in I, such that $\bigcup_{m=1}^\infty I_m = I$. Then, for each F in $A^{(k)}(I)$, we place

$$\|F\|_{(k)} = \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{|(F|I_m)|_{(k)}}{1+|(F|I_m)|_{(k)}}.$$

21 LEMMA. *Let I and k be as in the preceding lemma. Then*

(i) $A^{(k)}(I)$, taken with the norm of Definition 20, is a complete F-space;

(ii) if $\{\hat{I}_m\}$, $m \geq 1$, is a second sequence of open subsets of I whose closures are compact and contained in I, and such that $\bigcup_{m=1}^{\infty} \hat{I}_m = I$, then the norm $\|F\|_{(k)}$ defined for $A^{(k)}(I)$ as in Definition 20 but in terms of the sequence $\{\hat{I}_m\}$ determines a topology for $A^{(k)}(I)$ equivalent to that defined in terms of the sequence $\{I_m\}$.

PROOF. Part (i) is an elementary consequence of Lemma 19, and the details of its proof will be left to the reader. Since both the topologies in question in (ii) are metric topologies, it follows that to prove (ii) it suffices to note the evident fact that $F_n \to 0$ in either of these two topologies if and only if $|(F_n|I_0)|_{(k)} \to 0$ for every open subset I_0 of I whose closure is compact and contained in I. Q.E.D.

Remark. In what follows, the phrase "the topology of $A^{(k)}(I)$" will mean the topology defined by the metric of Definition 20 and Lemma 21.

22 LEMMA. *Let I be an open set in E^n. Let τ be a formal partial differential operator of order k, with coefficients belonging to $C^\infty(I)$. Then*

(i) $F \to \tau F$ is a continuous linear mapping of $A^{(j)}(I)$ into $A^{(j-k)}(I)$, $-\infty < j < \infty$;

(ii) if I has a compact closure $\bar{I}$, and the coefficients of τ are infinitely often differentiable in some neighborhood of $\bar{I}$, then $F \to \tau F$ is a continuous linear mapping of $H^{(j)}(I)$ into $H^{(j-k)}(I)$, $-\infty < j < +\infty$;

(iii) if I has a compact closure $\bar{I}$, and the coefficients of τ are infinitely often differentiable in some neighborhood of $\bar{I}$, then $F \to \tau F$ is a continuous linear mapping of $H_0^{(j)}(I)$ into $H_0^{(j-k)}(I)$, $k < j < +\infty$.

PROOF. Suppose that (ii) is proved. Then, since τF is in $C_0^\infty(I)$ if F is in $C_0^\infty(I)$, (iii) follows from Definition 15 (ii). Since $\tau(F|I_0) = (\tau F)|I_0$ by Lemma 9 (iii), (i) also follows from Definition 20. Hence, we have only to prove (ii). To do this, first suppose that $j \geq k$.

Suppose that

$$\partial^{J_1}\tau = \sum_{|J|\le |J_1|+k} a_J(x; J_1)\partial^J,$$

and

$$A = \max_{|J_1|\le |J|-k}\ \max_{|J|\le |J_1|+k}\ \max_{x\in I} |a_J(x; J_1)|.$$

Then, from Definition 15 (i), we have

$$\begin{aligned}
|\tau F|_{(j-k)} &= \Big\{\sum_{|J_1|\le j-k} |\partial^{J_1}\tau F|^2\Big\}^{1/2}\\
&= \Big\{\sum_{|J_1|\le j-k} \Big|\sum_{|J|\le |J_1|+k} a_J(\cdot\,; J_1)\partial^J F\Big|^2\Big\}^{1/2}\\
&\le A|F|_{(j)}\Big\{\sum_{|J_1|\le j-k} 1\Big\}^{1/2}\\
&= An^{(j-k)/2}|F|_{(j)},
\end{aligned}$$

proving (ii) in this special case.

Next suppose that $j = -p$ is non-positive. Then by Definition 17,

$$\begin{aligned}
|\tau F|_{(j-k)} &= \sup_{\varphi\in C_0^\infty(I)} \frac{|(\tau F)(\varphi)|}{|\varphi|_{(p+k)}}\\
&= \sup_{\varphi\in C_0^\infty(I)} \frac{|F(\tau^+\varphi)|}{|\tau^+\varphi|_{(p)}}\,\frac{|\tau^+\varphi|_{(p)}}{|\varphi|_{(p+k)}}\\
&\le |F|_{(j)}\cdot \sup_{\varphi\in C_0^\infty(I)} \frac{|\tau^+\varphi|_{(p)}}{|\varphi|_{(p+k)}}.
\end{aligned}$$

It follows from the special case of (ii) already established that the final factor is finite, proving (ii) in the special case in which j is non-positive.

Finally, we must consider the case $k > j \ge 0$. We have

$$\tau F = \sum_{|J|\le k} a_J(\cdot)\partial^J F,$$

so that from what has already been proved and from Lemma 6 it follows that it is sufficient to show that $F \to \partial^J F$ is a continuous mapping of $H^{(j)}(I)$ into $H^{(j-k)}(I)$ for $|J| = m \le k$. Write $J = [l_1, \ldots, l_m]$ so that $\partial^J F = \partial_{l_m} \ldots \partial_{l_1} F$. Then, from the fact that (ii) has been shown for $j-k$ positive, it follows that the map

$F \to \partial_{i_j} \ldots \partial_{i_1} F$ is a continuous map of $H^{(j)}(I)$ into $L_2(I)$, and by using the fact that (ii) has been shown for j non-positive the correspondence $F \to \partial_{i_m} \ldots \partial_{i_1} F$ is seen to be a continuous map of $H^{(j)}(I)$ into $H^{(j-m)}(I)$. Since, by Lemma 18 (ii), the identity mapping of $H^{(j-m)}(I)$ into $H^{(j-k)}(I)$ is continuous, the present lemma is proved. Q.E.D.

The easy proof of the following complement to Lemma 22 is left to the reader.

23 Lemma. *Let I be an open set in E^n. Let I_0 be an open subset of I and let k be an integer. Then*

(i) *the correspondence $F \to \bar{F}$ is a continuous mapping of $A^{(k)}(I)$ into itself and of $H^{(k)}(I)$ into itself;*

(ii) *the correspondence $F \to F|I_0$ is a continuous mapping of $A^{(k)}(I)$ into $A^{(k)}(I_0)$ and of $H^{(k)}(I)$ into $H^{(k)}(I_0)$.*

24 Lemma. *Let I be an open subset of E^n. Let k be an integer and let F be a distribution in I.*

(i) *If each point p in I has a neighborhood U_p contained in I such that $F|U_p \in A^{(k)}(U_p)$, then $F \in A^{(k)}(I)$.*

(ii) *If $\bar{I}$ is compact and each point p in $\bar{I}$ has a neighborhood U_p such that $F|U_pI \in H^{(k)}(U_pI)$, then $F \in H^{(k)}(I)$.*

(iii) *If $\bar{I}$ is compact, $k \geqq 0$, and each point p in $\bar{I}$ has a neighborhood U_p such that $F|U_pI \in H_0^{(k)}(U_pI)$, then $F \in H_0^{(k)}(U_pI)$.*

Proof. Part (i) evidently follows from Part (ii) and Lemma 23. To prove (ii), we argue as follows. Let $F|U_pI = F_p$. Using Lemma 2.4, let $\{\varphi_m\}$, $m = 1, \ldots, M$, be a set of functions in $C^\infty(E^n)$ such that φ_m vanishes outside some neighborhood U_m, and such that $\sum_{m=1}^{M} \varphi_m(x) = 1$ identically for x in a neighborhood of $\bar{I}$. Then $F = \sum_{m=1}^{M} F_m \varphi_m$, and we have only to show that $F_m \varphi_m \in H^{(k)}(I)$ for each $m = 1, \ldots, M$. That is (cf. Lemma 13 (iv)) we may and shall assume without loss of generality that the closure of the carrier of F is contained in a single neighborhood U_p.

Now, suppose $k \geq 0$. Then, on the one hand, $\partial^J F$ is equal to a square-integrable function f_J in U_p. On the other hand, if we put $f_J(x) = 0$ for x in U_p, then $F = f_J$ in the complement of a closed subset of U_pI also. Hence, by Lemma 10, $F = f_J$ is in $L_2(I)$, proving that F is in $H^{(k)}(I)$.

Next let $k < 0$. Using Lemma 21, let ψ be a function in $C_0^\infty(U_p)$ which is identically one in a neighborhood of the closure of the carrier of F. Then

$$F(\varphi) = F(\psi\varphi) = (F|U_pI)(\psi\varphi), \qquad \varphi \in C_0^\infty(I),$$

so that by Definition 17,

$$\begin{aligned} |F|_{(k)} &= \sup_{\varphi \in C_0^\infty(I)} \frac{|(F|U_pI)(\psi\varphi)|}{|\psi\varphi|_{(-k)}} \\ &= \sup_{\varphi \in C_0^\infty(I)} \frac{|(F|U_pI)(\psi\varphi)|}{|\psi\varphi|_{(-k)}} \frac{|\psi\varphi|_{(-k)}}{|\varphi|_{(-k)}} \\ &\leqq |(F|U_pI)|_{(k)} \sup_{\varphi \in C_0^\infty(I)} \frac{|\psi\varphi|_{(-k)}}{|\varphi|_{(-k)}}. \end{aligned}$$

Since the final factor is finite by Lemma 22 (ii), statement (ii) is proved.

It follows in the same way that to prove (iii) it suffices to show that $F\varphi_m \in H_0^{(k)}(I)$. By hypothesis, there exists a sequence $\{\psi_l\}$ of elements of $C_0^\infty(E^n)$, all vanishing outside IU_m, such that

$$\lim_{l\to\infty} \sum_{|J|\leqq k} \int_{U_mI} |\partial^J F\varphi_m(x) - \partial^J \psi_l(x)|^2 dx = 0.$$

Since $\partial^J F\varphi_m(x) = 0$ for x in $I - U_mI$, it follows that

$$\lim_{l\to\infty} \sum_{|J|\leqq k} \int_{I} |\partial^J F\varphi_m(x) - \partial^J \psi_l(x)|^2 dx = 0.$$

Thus $F\varphi_m \in H_0^{(k)}(I)$. Q.E.D.

Similar use of Lemma 2.4 may be made in the proof of the following lemma. The details are left to the reader.

25 LEMMA. *Let F be a distribution in the open subset I of E^n, let $1 \leqq p < \infty$, and let k be a non-negative integer. Then*

(i) if each point q in I has a neighborhood U_p contained in I such that $F|U_q$ is a function, then F is a function;

(ii) if each point q in I has a neighborhood U_p contained in I such that $F|U_q$ is a function in $C^k(U_q)$, then F is in $C^k(I)$;

(iii) if $\bar{I}$ is compact, and each q in $\bar{I}$ has a neighborhood U_q such that $F|U_pI$ is in $L_p(U_qI)$, then F is in $L_p(I)$;

(iv) *if $\bar{I}$ is compact, and each q in $\bar{I}$ has a neighborhood U_q such that $F|U_qI$ is in $C^n(\overline{U_qI})$, then F is in $C^n(\bar{I})$.*

It is useful to define a topology in the space $D(I)$. This may be done by taking the weak topology of $D(I)$ regarded as a set of linear functionals on $C_0^\infty(I)$.

26 Definition. Let I be an open subset of E^n. Then *the topology in $D(I)$* will be the topology whose basic neighborhoods of F are the sets

$$N(\varphi_1, \ldots, \varphi_m, \varepsilon, F) = \{G \in D(I) \mid |F(\varphi_i) - G(\varphi_i)| < \varepsilon, \quad i = 1, \ldots, m\},$$

where ε is a positive number, m is a positive integer and $\varphi_1, \ldots, \varphi_m$ are elements of $C_0^\infty(I)$.

27 Lemma. *Let I be an open set in E^n, and I_0 an open subset of I. Then*

(i) *if τ is a formal partial differential operator with coefficients in $C^\infty(I)$, then $F \to \tau F$ is a continuous mapping of $D(I)$ into itself,*

(ii) *$F \to \bar{F}$ is a continuous mapping of $D(I)$ into itself,*

(iii) *$F \to F|I_0$ is a continuous mapping of $D(I)$ into $D(I_0)$.*

Proof. Let ζ_2 and ζ_3 denote the mappings in (ii) and (iii) respectively, and let $\Lambda: C_0^\infty(I_0) \to C_0^\infty(I)$ be defined by the equations

$$(\Lambda\varphi)(x) = 0, \quad x \in I - I_0; \qquad \Lambda\varphi(x) = \varphi(x), \quad x \in I_0.$$

Then (i), (ii), and (iii) are respectively consequences of the following three evident formulae.

$$\begin{aligned}
&\tau\{G \in D(I) \mid |F(\tau^+\varphi_i) - G(\tau^+\varphi_i)| < \varepsilon, && i = 1, \ldots, m\} \\
&\qquad \subseteq \{H \in D(I) \mid |(\tau F)(\varphi_i) - H(\varphi_i)| < \varepsilon, && i = 1, \ldots, m\}; \\
&\zeta_2\{G \in D(I) \mid |F(\bar{\varphi}_i) - G(\bar{\varphi}_i)| < \varepsilon, && i = 1, \ldots, m\} \\
&\qquad \subseteq \{H \in D(I) \mid |(\zeta_2 F)(\varphi_i) - H(\varphi_i)| < \varepsilon, && i = 1, \ldots, m\}; \\
&\zeta_3\{G \in D(I) \mid |F(\Lambda\varphi_i) - G(\Lambda\varphi_i)| < \varepsilon, && i = 1, \ldots, m\} \\
&\qquad \subseteq \{H \in D(I_0) \mid |(\zeta_3 F)(\varphi_i) - H(\varphi_i)| < \varepsilon, && i = 1, \ldots, m\}.
\end{aligned}$$
Q.E.D.

28 Lemma. *Let I be an open subset of E^n and let k be an integer.*

Let $\{a_m\}$ be a sequence of functions in $C^\infty(I)$ such that $a_m \to 0$ in the topology of $C^\infty(I)$. Then

(i) *for each F in $D(I)$, $a_m F \to 0$ in the topology of $D(I)$;*

(ii) *for each F in $A^k(I)$, $a_m F \to 0$ in the topology of $A^k(I)$;*

(iii) *if a_m and its partial derivatives of all orders approach zero uniformly on I as $m \to \infty$, then the sequence of mappings $F \to a_m F$ of $H^{(k)}(I)$ into $H^{(k)}(I)$ approaches zero in the uniform topology of operators.*

Proof. It is clear that $a_m \varphi \rightrightarrows 0$ as $m \to \infty$ for each φ in $C_0^\infty(I)$. Hence (i) follows immediately from Definition 26. Statement (ii) follows immediately from (iii) by virtue of Definition 20 and Lemma 10 (iii). To prove (iii), first suppose that $k \geq 0$. Then

$$|a_m F|_{(k)}^2 = \sum_{|J| \leq k} \int_I |\partial^J a_m F|^2 dx \leq A\{\max_{|J| \leq k} \sup_{x \in I} |\partial^J a_m(x)|\}|F|_{(k)}^2$$

where A is a constant depending only on k (by Lemma 6 (iv) which allows us to expand $\partial^J a_m F$ by Leibniz' rule). This proves (iii) for $k \geq 0$. If $k = -p$ where $p \geq 0$, we have

$$|a_m F|_{(k)} = \sup_{\varphi \in C_0^\infty(I)} \frac{|F(a_m \varphi)|}{|\varphi|_{(p)}} \leq |F|_{(k)} \sup_{\varphi \in C_0^\infty(I)} \frac{|a_m \varphi|_{(p)}}{|\varphi|_{(p)}},$$

so that the result (iii) for $k \leq 0$ evidently follows from the result (iii) for $p \geq 0$.

In addition to functions and distributions in the sense given in Definition 1 and in the various definitions and lemmas up to Lemma 27, we will need to consider classes of multiply periodic functions and the corresponding classes of distributions. Unless the contrary is definitely specified, we shall, in the next few definitions and lemmas, work within the cube

$$C = \{x \in E^n \,|\, |x_i| \leq \pi\}$$

in Euclidean n-space E^n. Opposite faces of this cube will be "identified"; that is, unless the contrary is explicitly stated, we shall suppose all functions of x to be multiply periodic, of period 2π, in the variable $x = [x_1, \ldots, x_n]$.

What this means is that we treat any two points x and $\tilde{x}$ on the boundary of C such that $|x_j - \tilde{x}_j| = 0$ or 2π for all j as identical,

and define a neighborhood of the point x (after identification with $\tilde{x}$ and possibly other points) as the intersection with C of an open set of E^n *containing x and all the points with which x is identified*. It is apparent that by this set of identifications C becomes topologically equivalent to the Cartesian product of n replicas of the unit circle in the plane. We shall, in what follows, always assume implicitly that those identifications have been carried out, and understand such topological phrases as "open set," "closed set," etc., in this slightly modified sense. Since we deal only with multiply periodic functions throughout, all our functions will be well-defined on the set C even after the indicated identifications are carried out. Conversely, this is the basic reason why we deal only with multiply periodic functions. Of course, if we deal only with subsets of the interior of the cube C in E^n, no identifications are made and the phrases "open set," "closed set," etc., have only their ordinary meanings. We shall be concerned with the spaces $F_\pi(C)$ of all functions defined in C which are multiply periodic of period 2π in the variable $x = [x_1, \ldots x_m]$ and with the spaces

$$C_\pi^\infty(C) = \{f \in C^\infty(C) | f \in F_\pi(C)\}$$

and

$$C_\pi^p(C) = \{f \in C^p(C) | f \in F_\pi(C)\}.$$

If I is an open subset of C, in the modified sense explained in the preceding paragraph, then $C_{\pi,0}^\infty(I)$ will denote the subspace of $C_\pi^\infty(C)$ consisting of all functions in $C_\pi^\infty(C)$ which vanish outside a compact subset of I. Let I be an open subset of C. We shall write $f_n \rightrightarrows f$ in I if f_n, f are in $C_{\pi,0}^\infty(I)$, if all the functions f_n vanish outside a fixed compact subset of I, and if $f_n \to f$ in the topology of $C^\infty(I)$ as $n \to \infty$. We may then make a definition corresponding to Definition 1 (ii) as follows.

29 Definition. Let I be an open subset of C. The space $D_\pi(I)$ consists of all linear functionals F on $C_{\pi,0}^\infty(I)$ such that $F(\varphi_n) \to F(\varphi)$ if $\varphi_n \rightrightarrows \varphi$ in I. The elements of $D_\pi(I)$ will be called *distributions multiply periodic in the set I*.

It should be especially noted that if I is an open subset of the interior of C, then $D_\pi(I)$ and $D(I)$ are evidently two notations for the same space.

The preceding development may now be paralleled in this new setting. Since this development differs from the previous development only in a few points (in many cases only by the substitution of $C^\infty_{\pi,0}(I)$ for $C^\infty_0(I)$), this development will only be indicated and the detailed adaption of the earlier development left to the reader.

30 Definition. Let I be an open subset of C. The element $F \in D_\pi(I)$ is said to *correspond to the function* f if f is (Lebesgue) integrable over every compact subset of I and if

$$F(\varphi) = \int_G f(x)\varphi(x)dx, \qquad \varphi \in C^\infty_{\pi,0}(I).$$

If F and f correspond in this way, they may be identified in contexts in which such identification cannot cause confusion.

The analogs of Lemma 3 and Definition 4 are now evident, and instead of stating them in the new context we shall simply refer to "Lemma 3 or Definition 4 as generalized to $D_\pi(I)$" wherever necessary.

31 Definition. Suppose that I is an open subset of C and that F is in $D_\pi(I)$.

(i) Let τ be a formal partial differential operator with coefficients in $C^\infty_\pi(C)$. Then τF will denote the element of $D_\pi(I)$ defined by the equation

$$(\tau F)(\varphi) = F(\tau^+\varphi), \qquad \varphi \in C^\infty_{\pi,0}(I).$$

(ii) The symbol $\bar{F}$ will denote the element of $D_\pi(I)$ defined by the equation

$$\bar{F}(\varphi) = F(\bar{\varphi}), \qquad \varphi \in C^\infty_{\pi,0}(I).$$

(iii) If I_0 is an open subset of I then the restriction $F|I_0$ will denote the element of $D_\pi(I)$ defined by the equation

$$(F|I_0)(\varphi) = F(\varphi), \qquad \varphi \in C^\infty_{\pi,0}(I_0).$$

The analogs of Lemmas 6 to 9 and Lemma 14 are evident, and instead of giving detailed statements which apply to our new context we shall simply refer to "Lemma 6," etc., as "generalized to $D_\pi(I)$" when necessary.

Lemma 10 applies similarly in an evident way to the new

situation, it being necessary to recall, however, that "open set" is to be understood in the sense explained in the paragraph preceding Definition 29. Keeping this same convention in mind, we may generalize Definition 11 as follows.

32 DEFINITION. Let I be an open subset of C, and let F be in $D_\pi(I)$. Then the closed set C_F in I, which is the complement of the largest open set in I in which F vanishes, i.e., which is the complement in I of the union of all the open subsets of I in which F vanishes, is called the *carrier* of F.

Lemma 12 now may be extended, with much the same proof, to the new setting. However, an evident consequence of Lemma 12, as generalized to $D_\pi(I)$, which will be of importance to us in subsequent discussions, is stated in the following lemma.

33 LEMMA. *Let I be an open subset of the interior of C, and let F be in $D(I)$. Suppose that F has a carrier C_F which is a compact subset of I. Then there is a unique distribution G in $D_\pi(C)$ such that $F = G|I$ and $C_G = C_F$.*

In later sections, the distribution G will be referred to as *the natural extension of F to C* or simply as *F, regarded as an element of $D_\pi(C)$.*

The following definitions generalize Definitions 15, 17, and 21 to the new setting in an evident way.

34 DEFINITION. Let I be an open subset of C. Let k be a non-negative integer.

(a) The set of all F in $D_\pi(I)$ for which $\partial^J F$ is in $L_2(I)$ for all $|J| \leq k$ will be denoted by $H_\pi^{(k)}(I)$. For each pair F, G in $H_\pi^{(k)}(I)$ we write

$$(F, G)_{(k)} = \sum_{|J| \leq k} \int_I (\partial^J F)(x)(\partial^J \bar{G})(x)dx$$

and

$$|F|_{(k)} = \{(F, F)_{(k)}\}^{1/2}.$$

(b) The symbol $H_{\pi,0}^{(k)}(I)$ will denote the closure in the norm of $H_\pi^{(k)}(I)$ of the subspace $C_{\pi,0}^\infty(I)$ of $H_\pi^{(k)}(I)$.

(c) The symbol $A_\pi^{(k)}(I)$ will denote the set of all F in $D_\pi(I)$ such

that $F|I_0$ is in $H_\pi^{(k)}(I_0)$ for each open subset I_0 of I whose closure is compact and contained in I.

35 Definition. Let I be an open subset of C and let k be a positive integer.

(i) The set of all F in $D_\pi(I)$ for which

$$|F|_{(-k)} = \sup_{\varphi \in C_{\pi,0}^{\infty}(I)} \frac{|F(\varphi)|}{|\varphi|_{(k)}} < \infty$$

will be denoted by $H_\pi^{(-k)}(I)$.

(ii) The set of all F in $D_\pi(I)$ for which $F|I_0$ is in $H_\pi^{(-k)}(I_0)$ for each open subset I_0 of I whose closure is compact and contained in I will be denoted by $A_\pi^{(-k)}(I)$.

36 Definition. Let I be an open subset of C. Let k be an integer, positive or negative. Let $\{I_m\}$, $m \geq 1$, be a sequence of open subsets of I, whose closures are compact and contained in I and such that $\bigcup_{m=1}^{\infty} = I$. Then, for each F in $A_\pi^{(k)}(I)$, we place

$$\|F\|_{(k)} = \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{|(F|I_m)|_{(k)}}{1+|(F|I_m)|_{(k)}}.$$

Once the definitions are made, it is an easy matter to adapt all of the lemmas from Lemma 13 to Lemma 28 to the new setting. The necessary details are left to the reader. Instead of giving detailed statements which apply to the new context we shall simply refer to "Lemma 13," etc., "as generalized to $D_\pi(I)$" whenever necessary. In the case of Lemma 27 the topology of $D_\pi(I)$ must be defined in a way generalizing that of Definition 26. This is done in the following definition.

37 Definition. Let I be an open subset of C. Then *the topology in* $D_\pi(I)$ will be the topology whose basic neighborhoods of F are the sets

$$N(\varphi_1, \ldots, \varphi_m, \varepsilon, F) = \{G \in D_\pi(I) \mid |F(\varphi_i) - G(\varphi_i)| < \varepsilon,\ i = 1, \ldots, m\},$$

where ε is a positive number, m is a positive integer, and $\varphi_1, \ldots, \varphi_m$ are elements of $C_{\pi,0}^{\infty}(I)$.

The very important possibility exists of expanding elements F

in $D_\pi(C)$ in Fourier series. The next definition and lemma show how this is to be done.

38 Definition. Let F be in $D_\pi(C)$ and let L be an index with $|L| = n$. Then the expression

$$F_L = F(e^{-iL\cdot x})$$

is called the Lth *Fourier coefficient of* F. The formal series

$$(2\pi)^{-n}\sum_L F_L e^{iL\cdot x}$$

is called the Fourier series of F.

39 Lemma. *The Fourier series of an element F in $D_\pi(C)$ converges unconditionally to F.*

Proof. It follows from the Definition 37 of the topology in $D_\pi(C)$ that it suffices to show that

$$(2\pi)^{-n}\sum_L F_L\int_C e^{iL\cdot x}\varphi(x)dx$$

converges unconditionally to $F(\varphi)$ for each φ in $C^\infty_\pi(C)$. For any set A of indices L, we have

$$(2\pi)^{-n}\sum_{L\in A} F_L\int_C e^{iL\cdot x}\varphi(x)dx = F\Big((2\pi)^{-n}\sum_{L\in A}\varphi_L e^{-iL\cdot x}\Big),$$

where

[†] $$\varphi_L = \int_C \varphi(x)e^{iL\cdot x}dx, \qquad \varphi\in C^\infty_\pi(C).$$

Thus, it is sufficient for us to show that for each φ in $C^\infty_\pi(C)$, the series

[*] $$(2\pi)^{-n}\sum_L \varphi_L e^{-iL\cdot x}$$

converges unconditionally in the topology of $C^\infty_\pi(C)$ to φ. By Plancherel's theorem (XI.3.9; see also XI.3.22 and the discussion following it), for every φ in $C^\infty_\pi(C)$ this series converges unconditionally to φ in the topology of $L_2(C)$. Hence, it suffices to show that the series [*] converges unconditionally in the topology of $C^\infty_\pi(C)$, i.e., converges absolutely no matter how often differentiated term by term. We shall actually show that

[**] $$|\varphi_L| = O\big((1+l_1^2+\ldots+l_n^2)^{-p}\big)$$

for every positive p no matter how large, from which the desired absolute convergence will be evident. To do this, we integrate equation [†] by parts $2p$ times with respect to each of the variables x_j for which l_j is not zero, which we assume without loss of generality are the variables $x_1, \ldots, x_m$. This gives

$$\varphi_L = (i)^{mp}(l_1 \ldots l_m)^{-2p}\int_C (\partial_1 \ldots \partial_m)^{2p}\varphi(x)e^{iL\cdot x}\,dx,$$

so that, since $(l_1 \ldots l_m)^{-2p} \leq n(l_1^2+\ldots+l_n^2)^{-p}$, if $l_{m+1}, \ldots, l_n = 0$, $m \neq 0$, we have

$$|\varphi_L| \leq n(l_1^2+\ldots+l_n^2)^{-p} \max_{\substack{x\in C\\ |J|\leq 2mp}} |\partial^J\varphi(x)|,$$

from which the estimate [**] evidently follows. Q.E.D.

40 Lemma. *Let F be in $D_\pi(C)$. Then*

(i) $F_L = F_{(-L)}$;

(ii) $(\partial^J F)_L = (iL)^J F_L$.

Proof. Both statements follow immediately from Definition 38. Q.E.D.

41 Lemma. *Let F be in $D_\pi(C)$ and let k be an integer.*

(i) *The distribution F is in $H_\pi^{(k)}(C)$ if and only if*

$$\|F\|_{(k)} = \{\sum_L (1+l_1^2+\ldots+l_n^2)^k|F_L|^2\}^{1/2} < \infty.$$

(ii) *The norm $\|F\|_{(k)}$ is a norm in $H_\pi^{(k)}(C)$ equivalent to the norm for this space specified in Definitions 34 and 35.*

(iii) *If F is in $H_\pi^{(k)}(C)$, its Fourier series converges unconditionally to F in the topology of $H_\pi^{(k)}(C)$.*

Proof. First suppose that $k = 0$. It follows from Lemma 39 that

$$F(\varphi) = (2\pi)^{-n}\sum_L \varphi_L F_L$$

where

$$\varphi_L = \int_C e^{iL\cdot x}\varphi(x)dx.$$

Thus, since

$$|\varphi|_2^2 = (2\pi)^{-n}\sum_L |\varphi_L|^2,$$

by Plancherel's theorem (XI.3.9) it follows from Lemma 14 that F

is in $L_2(C)$ provided that $\|F\|_0 < \infty$. In this case, we have also

$$|F|_2^2 = (2\pi)^n \sum_L |F_L|^2$$

by Plancherel's theorem.

Next suppose that $k \geqq 0$. Suppose that $\|F\|_{(k)} < \infty$. Then, by what has been proved and by Lemma 40 (ii), F is in $H_n^{(k)}(C)$ and $|F|_{(k)} \leqq A\|F\|_{(k)}$, where A is a finite constant depending only on k. Conversely, if $|F|_{(k)} < \infty$, then by Lemma 40 (ii) and Plancherel's theorem, we have

$$\sum_{|J| \leqq k} \sum_L |L^J|^2 |F_L|^2 = |F|_{(k)}^2.$$

Since there evidently exists a finite constant A depending only on k such that

$$(1+l_1^2+\ldots+l_n^2)^k \leqq A \sum_{|J| \leqq k} |L^J|^2,$$

it follows that $\|F\|_{(k)} \leqq A|F|_{(k)} < \infty$.

This proves (i) and (ii) for $k \geqq 0$. Next suppose that $k = -p$, where $p \geqq 0$. Let $|F|_{(k)} < \infty$ so that, by the result just proved and Definition 35, there is a finite constant $M \leqq A|F|_{(k)}$, where A depends only on k, such that

$$|F(\varphi)| \leqq M\|\varphi\|_p, \qquad \varphi \in C_n^\infty(C).$$

By Lemma 39, this may be written as

$$(1) \quad |\sum_L F_L \varphi_L| \leqq M\{\sum_L (1+l_1^2+\ldots+l_n^2)^p |\varphi_L|^2\}^{1/2}, \qquad \varphi \in C_n^\infty(C)$$

where we have set

$$\varphi_L = \int_C e^{iL\cdot x}\varphi(x)dx$$

as in [†] of the proof of Lemma 39. From (1), and from the Hahn-Banach theorem, II.3.11, it follows that the linear functional

$$(\varphi_L) \to \sum F_L \varphi_L$$

may be extended to a continuous linear functional of norm at most M defined on the Hilbert space $\mathfrak{H}$ of all multisequences (φ_L) such that

$$\|(\varphi_L)\| = \{\sum_L (1+l_1^2+\ldots+l_n^2)^p |\varphi_L|^2\}^{1/2} < \infty.$$

Since

$$\sum_L F_L \varphi_L = \sum_L [F_L(1+l_1^2+\ldots+l_n^2)^{-p}]\varphi_L(1+l_1^2+\ldots+l_n^2)^p$$

it follows from Theorem IV.8.1. that

$$\sum_L |F_L(1+l_1^2+\ldots+l_n^2)^{-p}|^2(1+l_1^2+\ldots+l_n^2)^p \leq M,$$

so that there is a constant A depending only on k such that $\|F\|_k < A|F|_k$ for each F in $H_n^{(k)}(C)$. Conversely, if F is in $D_n(C)$ and $\|F\|_k = M < \infty$, then it follows immediately from Schwarz' inequality that

$$|\sum F_L \varphi_L| \leqq \|F\|_{(k)}\{\sum_L (1+l_1^2+\ldots+l_n^2)^p|\varphi_L|^2\}^{1/2}.$$

Using what has been proved above, this may be written as

$$|F(\varphi)| \leq A\|F\|_{(k)}|\varphi|_{(k)}, \qquad \varphi \in C_n^\infty(C).$$

where A is a constant depending only on k. Thus by Definition 35, F is in $H_n^{(k)}(C)$ and

$$|F|_{(k)} \leqq A\|F\|_{(k)},$$

where A is a constant depending only on k. This proves (i) and (ii) in all cases.

It is now quite easy to prove (iii). By (i) and (ii), we must show that if $\{A_m\}$ is an arbitrary increasing sequence of finite subsets of the set $\{L \mid |L| = n\}$ such that

$$\bigcup_{m=1}^\infty A_m = \{L \mid |L| = n\},$$

and if

$$\sum_L (1+l_1^2+\ldots+l_n^2)^k|F_L|^2 < \infty,$$

then

$$\|F - \sum_{L\in A_m} F_L e^{iL\cdot x}\|_{(k)} = \{\sum_{L\notin A_m} |(1+l_1^2+\ldots+l_n^2)^k F_L|^2\}^{1/2} \to 0$$

as $m \to 0$. This statement follows immediately from the Lebesgue dominated convergence theorem. Q.E.D.

A theory of multiply periodic distributions may be developed

in an exactly similar way for functions defined in an arbitrary rectangular parallelepiped $\hat{C}$:

$$[\dagger] \qquad \hat{C} = \{x \in E^n \mid a_j \leqq x_j \leqq b_j, \qquad j = 1, \ldots, n\}.$$

One has only to replace the space $F_n(C)$ of functions multiply periodic of period 2π in the cube C by the space $F_n(\hat{C})$ of functions multiply periodic in the parallelepiped $\hat{C}$, and having period $b_1 - a_1$ in the variable $x_1, \ldots, b_n - a_n$ in the variable x_n. We leave it to the reader to carry out the details of this program. In subsequent sections, we shall simply refer wherever necessary to "Definition 1, Lemma 3," etc., "as generalized to $D_n(\hat{C})$, $H_n^{(k)}(\hat{C})$," etc. It is, however, worth stating the analog of certain parts of Definition 38 and Lemmas 39 and 41 somewhat more explicitly.

42 Definition. Let $\hat{C}$ be the rectangular parallelepiped [†]. Let $\xi_{\hat{C}}$ denote the vector $(2\pi)[(b_1 - a_1)^{-1}, \ldots, (b_n - a_n)^{-1}]$, and $v(\hat{C})$ denote the product $(b_1 - a_1)(b_2 - a_2) \ldots (b_n - a_n)$. Let L be an index such that $|L| = n$, and let $F \in D_n(\hat{C})$. Then the expression

$$F_L = F(e^{-iL \cdot \xi_{\hat{C}} x})$$

is called the Lth *Fourier coefficient of* F. The formal series

$$(v(\hat{C}))^{-1} \sum_L F_L e^{iL \cdot \xi_{\hat{C}} x}$$

is called *the Fourier series of* F.

43 Lemma. *Let the hypotheses of the preceding definition be satisfied.*

(i) *The Fourier series of an element* F *in* $D_n(\hat{C})$ *converges unconditionally to* F.

(ii) *If* F *is in* $H_n^{(k)}(\hat{C})$, *then the Fourier series of* F *converges unconditionally to* F *in the topology of* $H_n^{(k)}(\hat{C})$.

(iii) *Let* k *be an integer. The expression*

$$\|F\|_{(k)} = \{\sum_{|L|=n} (1 + l_1^2 + \ldots + l_n^2)^k |F_L|^2\}^{1/2}$$

is finite if and only if the element F *in* $D_n(\hat{C})$ *belongs to* $H_n^{(k)}(\hat{C})$, *and defines a norm in* $H_n^{(k)}(\hat{C})$ *equivalent to the norm for this space in (the evident generalizations of) Definitions 32 and 38.*

The next topic on which we wish to touch is that of the behavior of distributions under changes of variable.

44 DEFINITION. Let I_1 be a domain in E^{n_1}, and let I_2 be a domain in E^{n_2}. Let $M : I_1 \to I_2$ be a mapping of I_1 into I_2 such that

(a) $M^{-1}C$ is a compact subset of I_1 whenever C is a compact subset of I_2;

(b) $(M(\cdot))_j \in C^\infty(I_1)$, $\quad j = 1, \ldots, n_2$.

Then

(i) for each φ in $C^\infty(I_2)$, $\varphi \circ M$ will denote the function ψ in $C^\infty(I_1)$ defined, for x in I_1, by the equation $\psi(x) = \varphi(M(x))$;

(ii) for each F in $D(I_1)$ the symbol $F \circ M^{-1}$ will denote the distribution G in $D(I_2)$ defined, for φ in $C_0^\infty(I_2)$, by the equation $G(\varphi) = F(\varphi \circ M)$.

Remark. It should be observed that it follows from (b) of the above definition that $\varphi \circ M$ is in $C^\infty(I_1)$ if φ is in $C^\infty(I_2)$, and that the mapping $\varphi \to \varphi \circ M$ is a continuous mapping of $C^\infty(I_2)$ into $C^\infty(I_1)$. (See Section 2 for a definition of the topology in these spaces.) By (a), $\varphi \to \varphi \circ M$ maps $C_0^\infty(I_2)$ into $C_0^\infty(I_1)$. By (a) again, all the functions of the sequence $(\varphi_m \circ M)$ vanish outside a fixed compact subset of I_1 if all the functions of the sequence $\{\varphi_m\}$ vanish outside a fixed compact subset of I_2. It consequently follows from (a) and (b) that $\varphi_m \rightrightarrows \varphi$ implies that $\varphi_m \circ M \rightrightarrows \varphi \circ M$, so that part (ii) of the preceding definition does legitimately define an element G in $D(I_2)$.

45 LEMMA. *Let I_1 be a domain in E^{n_1} and let I_2 be a domain in E^{n_2}. Let $M : I_1 \to I_2$ be a mapping of I_1 into I_2 satisfying the hypotheses* (a) *and* (b) *of Definition 44. Then*

(a) *the mapping $F \to F \circ M^{-1}$ is a continuous linear mapping of $D(I_1)$ into $D(I_2)$;*

(b) $\overline{F \circ M^{-1}} = \bar{F} \circ M^{-1}$, $\quad F \in D(I_1)$;

(c) *if $\hat{I}_2$ is an open subset of I_2, and $\hat{I}_1 = M^{-1}\hat{I}_2$, then*

$$(F \circ M^{-1})|\hat{I}_2 = (F|\hat{I}_1) \circ M^{-1}.$$

The proof of this lemma follows readily from the definitions of all the various terms; the details of proof are left to the reader. The proof of the following lemma is also elementary and is left to the reader.

46 Lemma. *Let I_j be a domain in E^{n_j}, $j = 1, 2, 3$. Let $M_1 : I_1 \to I_2$ and $M_2 : I_2 \to I_3$ be mappings such that*

(a) *$M_1^{-1}C$ is a compact subset of I_1 whenever C is a compact subset of I_2; $M_2^{-1}C$ is a compact subset of I_2 whenever C is a compact subset of I_3;*

(b) $(M_1(\cdot))_j \in C^\infty(I_1)$, $\quad 1 \leq j < n_2$;
$(M_2(\cdot))_j \in C^\infty(I_2)$, $\quad 1 \leq j < n_3$.

Then for each F in $D(I_1)$,

$$F \circ (M_1 M_2)^{-1} = (F M_1^{-1}) \circ M_2^{-1}.$$

In case the mapping M of Lemma 45 is one-to-one and onto all of I_2, we may give additional information.

47 Lemma. *Let I_1 and I_2 be domains in E^n, and let $M : I_1$ be a one-to-one mapping of I_1 onto all of I_2. Suppose that*

(a) $(M(\cdot))_j \in C^\infty(I_1)$, $\quad j = 1, \ldots, n$;

(b) $(M^{-1}(\cdot))_j \in C^\infty(I_2)$, $\quad j = 1, \ldots, n$.

Then

(i) $(aF) \circ M^{-1} = a(M^{-1}(\cdot))(F \circ M^{-1}), \quad a \in C^\infty(I_1), \quad F \in D(I_1)$;

(ii) $$(\partial_j F) \circ M^{-1} = \sum_{k=1}^{n} \{\partial_k b_{kj}(F \circ M^{-1})\}, \quad j = 1, \ldots, n, \quad F \in D(I_1),$$

where

$$b_{kj}(x) = \big((\partial_j M)(M^{-1}(x))\big)_k, \qquad 1 \leq k, \; j \leq n, \qquad x \in I_2;$$

(iii) *if F corresponds to the function f, then $F \circ M$ corresponds to the function*

$$J(\cdot) f(M^{-1}(\cdot)),$$

where J is the absolute value of the Jacobian determinant of the mapping $x \to M^{-1}(x)$.

Proof. Let $\varphi \in C_0^\infty(I_2)$. Using Definition 44, we have

$$\begin{aligned}
\{(aF) \circ M^{-1}\}(\varphi) = (aF)(\varphi \circ M) &= F((\varphi \circ M)a) \\
&= F\big((\varphi \circ M)(a \circ M^{-1} \circ M)\big) \\
&= F\big(\{\varphi(a \circ M^{-1})\} \circ M\big) \\
&= (F \circ M^{-1})\big(\varphi(a \circ M^{-1})\big) \\
&= \{(a \circ M^{-1})(F \circ M)\}(\varphi),
\end{aligned}$$

which proves (i). Similarly, we have

$$\{(\partial_j F) \circ M^{-1}\}(\varphi) = (\partial_j F)(\varphi \circ M) = -F(\partial_j(\varphi \circ M))$$
$$= -F\Big(\sum_k (\varphi_k \circ M)\, a_{k_j}\Big)$$

where $a_{k_j}(x) = (\partial_j M(x))_k$; using (i), (ii) now follows readily.

If F corresponds to the function f, we have

$$(F \circ M^{-1})(\varphi) = F(\varphi \circ M) = \int f(x)\varphi(M(x))\,dx$$
$$= \int f(M^{-1}(x))\varphi(x) J(x)\,dx,$$

J denoting the absolute value of the Jacobian determinant of the mapping $x \to M^{-1}(x)$; this follows by the standard theorem on change of variables in a multiple integral. But then (iii) is evident. Q.E.D.

Lemma 47 allows us to describe the behavior of the spaces H^p, A^p, etc., under the changes of variable.

48 LEMMA. *Let I_1 and I_2 be domains in E^n, and let $M : I_1 \to I_2$ be a one-to-one mapping of I_1 onto all of I_2. Suppose that*

(a) $(M(\cdot))_j \in C^\infty(I_1)$, $\quad j = 1, \ldots, n$;

(b) $((M^{-1}(\cdot))_j \in C^\infty(I_2)$, $\quad j = 1, \ldots, n$.

Let k be an integer. Then

(i) *$F \to F \circ M^{-1}$ is a one-to-one continuous mapping of $D(I_1)$ onto all of $D(I_2)$ whose inverse is $F \to F \circ M$;*

(ii) *$F \to F \circ M^{-1}$ is a one-to-one continuous mapping of $A^{(k)}(I_1)$ onto all of $A^{(k)}(I_2)$;*

(iii) *if all the partial derivatives of $(M(\cdot))_j$ and $(M^{-1}(\cdot))_j$, $j = 1, \ldots, n$ are uniformly bounded in I_1 and I_2 respectively, then $F \to F \circ M^{-1}$ is a one-to-one continuous mapping of $H^{(k)}(I_1)$ onto all of $H^{(k)}(I_2)$; in this case, $F \to F \circ M^{-1}$ is also a one-to-one mapping of $H_0^{(k)}(I_1)$ onto all of $H_0^{(k)}(I_2)$.*

PROOF. In view of Lemma 46 and Lemma 45, statement (i) follows immediately. Our next step is to prove the first half of (iii). In view of (i), it is evidently sufficient to prove that $F \to F \circ M^{-1}$ is a continuous mapping of $H^{(k)}(I_1)$ into $H^{(k)}(I_2)$. In virtue of the ordinary formula for change of variables in a multiple integral and of (iii) of the preceding lemma,

$$\begin{aligned}|F\circ M^{-1}|^2_{(0)} &= \int_{I_2} |f(M^{-1}(x))|^2 J(x)^2 dx \\ &= \int_{I_1} |f(x)|^2 J(M(x)) dx \\ &\leqq A|F|^2_{(0)},\end{aligned}$$

where $A = \sup_{x\in I_1} J(M(x))$ is finite by the hypothesis of (iii). This proves the first part of (iii) for $k = 0$. Since by Definition 15

$$|F|^2_{(k)} = \sum_{|J|\leqq k} |\partial^J F|^2_{(0)},$$

the first part of (iii) follows for all $k \geqq 0$ from what has just been shown and from (ii) of the preceding lemma.

Using (i) of the present lemma, it follows that $F \to F\circ M^{-1}$ and $F \to F\circ M$ are continuous mappings of $H^{(k)}(I_2)$ into $H^{(k)}(I_1)$ and $H^{(k)}(I_1)$ into $H^{(k)}(I_2)$ respectively for $k \geqq 0$. Hence, there must exist finite positive constants A and B depending only on k such that

$$[*] \qquad B|F|_{(k)} \leqq |F\circ M^{-1}|_{(k)} \leqq A|F|_{(k)}.$$

Next let $k = -p$, where $p \geqq 0$. Then, if F is in $H^{(k)}(I_1)$, it follows from Definition 17 that

$$\begin{aligned}|F\circ M^{-1}|_{(k)} &= \sup_{\varphi\in C_0^\infty(I_2)} \frac{|F(\varphi\circ M)|}{|\varphi|_{(p)}} \\ &= \sup_{\varphi\in C_0^\infty(I_2)} \frac{F(\varphi\circ M)}{|\varphi\circ M|_{(p)}}\,\frac{|(\varphi\circ M)|_{(p)}}{|\varphi|_{(p)}} \\ &\leqq |F|_{(k)} \sup_{\varphi\in C_0^\infty(I_2)} \frac{|\varphi\circ M|_{(p)}}{|\varphi|_{(p)}} \\ &\leqq A|F|_{(k)},\end{aligned}$$

by formula [*]. Thus $F \to F\circ M^{-1}$ is a continuous mapping of $H^{(k)}(I_1)$ into $H^{(k)}(I_2)$ for k negative, which proves the first part of (iii) for all k.

Since $F \to F\circ M^{-1}$ maps the subspace $C_0^\infty(I_1)$ of $D(I_1)$ into the subspace $C_0^\infty(I_2)$ of $F(I_2)$ by part (iii) of the preceding lemma, the second part of (iii) follows immediately from the first part of (iii) and from Definition 15 (iii).

Let $\tilde{I}_1$ be an open subset of I_1 whose closure is a compact subset of I_1, and let $\tilde{I}_2 = M\tilde{I}_1$. Then $\tilde{I}_2$ is an open subset of I_2 whose closure

is a compact subset of I_2. Hence the restrictions $M|\hat{I}_1$ and $M^{-1}|\hat{I}_2 = (M|\hat{I}_1)^{-1}$ satisfy the hypotheses of (iii). Thus, (ii) follows from (iii) and from Definitions 15 (iii) and 20 of $A^k(I)$ and its topology. Q.E.D.

Definition 44 may readily be generalized from the spaces $D(I_1)$ and $D(I_2)$ to the corresponding spaces of multiply periodic distributions.

49 Definition. Let C_1 and C_2 be rectangular parallelepipeds of the form

$$C_1 = \{x \in E^{n_1} | a_j \leq x_j \leq b_j, \quad j = 1, \ldots, n_1\},$$
$$C_2 = \{x \in E^{n_2} | \hat{a}_j \leq x_j \leq \hat{b}_j, \quad j = 1, \ldots, n_2\}.$$

Let I_1 and I_2 be open subsets of C_1 and C_2 respectively, and let $M : I_1 \rightarrow I_2$ be a mapping of I_1 into I_2 such that $\varphi\big(M(\cdot)\big)$ is in $C^\infty_{\pi,0}(I_1)$ whenever φ is in $C^\infty_{\pi,0}(I_2)$. Then

(i) for each φ in $C^\infty_{\pi,0}(I_2)$ the symbol $\varphi \circ M$ will denote the function ψ in $C^\infty_{\pi,0}(I_1)$ defined by the equation $\psi(x) = \varphi\big(M(x)\big)$;

(ii) for each F in $D_\pi(I_1)$ the symbol $F \circ M^{-1}$ will denote the distribution G in $D_\pi(I_2)$ defined by the equation $G(\varphi) = F(\varphi \circ M)$ for φ in $C^\infty_{\pi,0}(I_2)$.

Lemmas 45 through 48 may readily be generalized from the spaces $D(I_1)$, $D(I_2)$, etc., to the spaces $D_\pi(I_1)$, $D_\pi(I_2)$, etc. We leave the details of this adaptation to the reader, and shall in what follows simply refer to Lemma 45, etc., as generalized to $D_\pi(I_1)$, etc. There is however one particularly important special mapping of $D_\pi(C)$ into itself, whose properties are discussed in detail in the following lemma.

50 Lemma. *Let C denote the parallelepiped*

$$C = \{x \in E^n | a_j \leq x_j \leq b_j, \quad 1 \leq j \leq n\}.$$

For each $\Delta > 0$ such that $\Delta < b_1 - a_1$, let M_Δ denote the mapping of C into itself defined by the equations

$$M_\Delta[x_1, \ldots, x_n] = [x_1+\Delta, x_2, \ldots, x_n], \qquad x_1+\Delta \leq b_1$$
$$M_\Delta[x_1, \ldots, x_n] = [x_1+\Delta-b_1+a_1, x_2, \ldots, x_n], \qquad x_1+\Delta > b_1.$$

Let k be an integer, and F be in $H^{(k)}_\pi(C)$. Then

(i) $\partial_1 F$ is in $H^{(k)}_\pi(C)$ if and only if $|\Delta^{-1}(F \circ M_\Delta^{-1} - F)|_{(k)}$ is uniformly bounded for $0 < \Delta < b_1 - a_1$;

(ii) *if* $\partial_1 F$ *is in* $H_\pi^{(k)}(C)$, *then*

$$\partial_1 F = \lim_{\Delta\to 0} \Delta^{-1}(F \circ M_\Delta^{-1} - F)$$

in the norm of $H_\pi^{(k)}(C)$.

Proof. It is clear that $\Delta^{-1}(\varphi \circ M_\Delta - \varphi)$ approaches $\partial_1\varphi$ uniformly for x in C as $\Delta \to 0$ for each φ in $C_\pi^\infty(C)$. If we apply this fact to each of the partial derivatives of φ, it follows that $\Delta^{-1}(\varphi\circ M_\Delta - \varphi)$ approaches $\partial_1\varphi$ as $\Delta \to 0$ in the topology of $C_\pi^\infty(C)$ for each $\varphi \in C_\pi^\infty(C)$. Thus, if G is in $D_\pi(C)$, it follows from Definitions 49 and 37 that

$$\lim_{\Delta\to 0} \Delta^{-1}(G\circ M_\Delta^{-1} - G)(\varphi) = \lim_{\Delta\to 0} G\big(\Delta^{-1}(\varphi\circ M_\Delta - \varphi)\big) = G(\partial_1\varphi),$$

so that $\Delta^{-1}(G\circ M_\Delta^{-1} - G)$ approaches $\partial_1 G$ in the topology of $D_\pi(C)$.

Now suppose that F is in $H_\pi^{(k)}(C)$ and that $|\Delta^{-1}(F\circ M_\Delta^{-1} - F)|_{(k)}$ is uniformly bounded for $0 < \Delta < b_1 - a_1$. First suppose that $k \geq 0$. Then, since the Hilbert space $H_\pi^{(k)}(C)$ is reflexive (cf. Lemma 16 and IV.4.6), it follows that each sequence (Δ_m) of positive real numbers approaching zero has a subsequence $(\hat\Delta_m)$ such that $\hat\Delta_m^{-1}(F\circ M_{\hat\Delta_m}^{-1} - F)$ converges weakly as $m \to \infty$ to an element $\hat F$ in $H_\pi^{(k)}(C)$. Now, it is clear that for each φ in $C_\pi^\infty(C)$, the mapping $G \to G(\varphi)$ is a continuous linear functional on $H_\pi^{(k)}(C)$. Thus, there is an element H_φ in $H_\pi^{(k)}(C)$ such that

$$G(\varphi) = (G, H_\varphi)_{(k)}, \qquad G \in H_\pi^{(k)}(C).$$

Thus,

$$\begin{aligned}\lim_{m\to\infty} \hat\Delta_m^{-1}(F\circ M_{\hat\Delta_m}^{-1} - F)(\varphi) &= \lim_{m\to\infty} (\hat\Delta_m^{-1}(F\circ M_{\hat\Delta_m}^{-1} - F), H_\varphi)_{(k)} \\ &= (\hat F, H_\varphi)_{(k)} = \hat F(\varphi), \qquad \varphi \in C_\pi^\infty(C)\end{aligned}$$

that is,

$$\lim_{m\to\infty} \hat\Delta_m^{-1}(F\circ M_{\hat\Delta_m}^{-1} - F) = \hat F$$

in the topology of distributions. It consequently follows, from what has been shown in the first paragraph of the present proof, that $\hat F = \partial_1 F$ so that $\partial_1 F$ is in $H_\pi^{(k)}(C)$.

Next let F be in $H_\pi^{(k)}(C)$, $k = -p$, where $p \geq 0$, and suppose that $|\Delta^{-1}(F\circ M_\Delta^{-1} - F)|_{(k)}$ is uniformly bounded by a constant A for $0 < \Delta < b_1 - a_1$. Then, by Definition 35 and by the Hahn-Banach theorem (II.3.11), $\Delta^{-1}(F\circ M_\Delta - F) = F_\Delta$ may be extended to a

continuous linear functional on the Hilbert space $H_\pi^{(p)}(C)$ of norm at most 1 for each $0 < \Delta < b_1 - a_1$. It follows from Theorem IV.4.6 and Corollary IV.4.7 that there is a sequence Δ_m of positive real numbers approaching zero and an $\hat{F}$ in $H_\pi^{(k)}(C)$ such that

$$\lim_{m\to\infty} (\Delta_m^{-1}(F \circ M_{\Delta_m}^{-1} - F))(\varphi) = \hat{F}(\varphi), \qquad \varphi \in C_\pi^\infty(C).$$

By what has been shown in the first paragraph of the present proof, $\hat{F} = \partial_1 F$, so that $\partial_1 F$ is in $H_\pi^{(k)}(C)$. This completes the proof of the direct part of (i) of the present lemma.

To prove the converse, let F be in $H_\pi^{(k)}(C)$ and let $\partial_1 F$ be in $H_\pi^{(k)}(C)$. Let us agree to consider that each φ in $C_\pi^\infty(C)$ is extended by periodicity to a multiply periodic function defined on all of E^n, of period $b_1 - a_1$ in the variable x_1, $b_2 - a_2$ in the variable x_2, etc.; which extended function we continue to denote by the symbol φ. Then, for each $\Delta > 0$, the mapping $\int_\Delta$ defined by the equation

$$\left(\int_\Delta \varphi\right)(x_1, \ldots, x_n) = \Delta^{-1} \int_{x_1}^{x_1+\Delta} \varphi(y, x_2, \ldots, x_n)\,dy \tag{1}$$

is a continuous mapping of $C_\pi^\infty(C)$ into itself. It is readily evident that $\partial^J \int_\Delta = \int_\Delta \partial^J$ for each index J and each $\Delta > 0$. From this it follows immediately that for each $\Delta > 0$, $\int_\Delta$ is a continuous mapping of $C_\pi^\infty(C_1)$ into itself. Moreover, it is clear that $\partial_1 \int_\Delta \varphi = \Delta^{-1}(\varphi \circ M_\Delta - \varphi)$ for $0 < \Delta < b_1 - a_1$ and φ in $C_\pi^\infty(C)$. Thus, if we define $\int_\Delta F$ for F in $D_\pi(C)$ by the formula

$$\left(\int_\Delta F\right)(\varphi) = F\left(\int_\Delta \varphi\right), \qquad \varphi \in C_\pi^\infty(C),$$

it follows that

$$\int_\Delta \partial_1 F - \partial_1 \int_\Delta F = \Delta^{-1}(F \circ M_\Delta^{-1} - F).$$

Hence if we show that $\int_\Delta$ is a mapping of $H_\pi^{(k)}(C)$ into itself of norm at most 1 for each $\Delta > 0$, this will establish the converse part of (i) of the present lemma.

First suppose that $k = -p$, where $p \geq 0$. Then, by Definition 85, to show that $|\int_\Delta F|_{(k)} \leq |F|_{(k)}$ for F in $H_0^{(k)}(C)$, it is sufficient to show that $|\int_\Delta \varphi|_{(p)} \leq |\varphi|_{(p)}$ for φ in $C_\pi^\infty(C)$. It is clear from (1), Schwarz's inequality, and Fubini's theorem that

$$\int_C \left|\left(\int_\Delta \varphi\right)(x_1, \ldots, x_n)\right|^2 dx \leq \Delta^{-1} \int_C \int_{x_1}^{x_1+\Delta} |\varphi(s, x_2, \ldots, x_n)|^2 \, ds\, dx$$
$$\leq \Delta^{-1} \int_{s-\Delta}^{s} dx_1 \left\{\int_C |\varphi(s, x_2, \ldots, x_n)|^2 \, ds\, dx_2 \ldots dx_n\right\}$$
$$= \int_C |\varphi(x)|^2 dx.$$

Thus $|\int_\Delta \varphi|_{(0)} \leq |\varphi|_{(0)}$ for each φ in $C_\pi^\infty(C)$. Since $\partial^J \int_\Delta = \int_\Delta \partial^J$ for each index J, we have also $|\int_\Delta \partial^J \varphi|_{(0)} \leq |\partial^J \varphi|_{(0)}$ for φ in $C_\pi^\infty(C)$. Thus it follows from Definition 34 that $|\int_\Delta \varphi|_{(p)} \leq |\varphi|_{(p)}$ and the converse part of (i) is proved for $k \leq 0$.

Next we turn to the proof of (ii). Let G be in $C_\pi(C) \subset D_\pi(C)$, and suppose that we agree to consider that each F in $C_\pi(C)$ is extended by periodicity to a multiply periodic function defined on all E^n, of period $b_1 - a_1$ in the variable x_1, $b_2 - a_2$ in the variable x_2, etc.; which extended function we continue to denote by the symbol F. Then by Fubini's theorem,

$$\left(\int_\Delta F\right)(\varphi) = -\Delta^{-1} \int_C F(x) \left\{\int_{x_1}^{x_1+\Delta} \varphi(s, x_2, \ldots, x_n)\, ds\right\} dx$$
$$= -\Delta^{-1} \int_C \left\{\int_{s-\Delta}^{s} F(x_1, \ldots, x_n) \varphi(s, x_2, \ldots, x_n)\, dx_1\right\} ds\, dx_2 \ldots dx_n$$
$$= \int_C \left\{-\Delta^{-1} \int_{x_1-\Delta}^{x_1} F(t, x_2, \ldots, x_n)\right\} \varphi(x) dx.$$

Thus, it is clear that for G in $C_\pi(C) \subseteq D_\pi(C)$, $(\int_\Delta G)(x) \to G(x)$ uniformly for x in C as $\Delta \to 0$. Hence $|\int_\Delta G - G| \to 0$ as $\Delta \to 0$ for each G in $C_\pi(C) \subseteq D_\pi(C)$. Since $C_\pi(C)$ is dense in $L_2(C) \subseteq D_\pi(C)$ by Lemma 2.2, it follows from Theorem II.3.6 that we have $|\int_\Delta G - G| \to 0$ as $\Delta \to 0$ for each G in $H_\pi^{(0)}(C) = L_2(C)$. Since $\int_\Delta \partial^J = \partial^J \int_\Delta$, it follows immediately from Definition 34 that if $k \geq 0$ $|\int_\Delta G - G|_{(k)} \to 0$ as $\Delta \to 0$ for each G in $H_\pi^{(k)}(C)$. In particular, $|\int_\Delta \partial_1 F - \partial_1 F|_{(k)}$ approaches zero as $\Delta \to 0$ if F is in $H_\pi^{(k)}(C)$ and $\partial_1 F$ is in $H_\pi^{(k)}(C)$. Since we have seen that $\int_\Delta \partial_1 F = \Delta^{-1}(F \circ M_\Delta^{-1} - F)$, this proves (ii) in the special case $k \geq 0$.

Putting $k = 0$, it follows that $|\int_\Delta F| \leq |F|$ for each F in $L_2(I) \subseteq D_\pi(I)$. Since $\partial^J \int_\Delta = \int_\Delta \partial^J$ for each index J, we have also $|\int_\Delta \partial^J F| \leq |\partial^J F|$ for $l \geq 0$, each F in $H_\pi^{(l)}(I)$, and each index J such that $|J| \leq l$. Thus it follows from Definition 34 that $|\int_\Delta F|_{(l)} \leq |F|_{(l)}$ for each $l \geq 0$, which proves the converse part of (i) for $k \geq 0$.

To complete the proof of the present lemma, we have only to prove (ii) for $k \leq 0$. This may be done as follows. Let $k = -p$ where $p \geq 0$. Let F be in $H_\pi^{(k)}(C)$, and $\partial_1 F$ be in $H_\pi^{(k)}(C)$. Let G denote an arbitrary element of $H_\pi^{(k)}(C)$. Then, by Definition 35, and by the Hahn-Banach theorem (II.3.11), the mapping $\varphi \to G(\varphi)$ can be extended to a continuous linear functional (which we shall continue to denote by the symbol G) on the Hilbert space $H_\pi^{(p)}(C)$. Thus, (cf. IV.4.5) there is an element $\hat{G}$ in $H_\pi^{(p)}(C)$ such that

$$G(\varphi) = (\varphi, \hat{G})_{(p)}, \qquad \varphi \in C_\pi^\infty(C). \tag{2}$$

Using Definitions 34 and 31 (i), and placing

$$\zeta = \sum_{|J| \leq p} (-1)^{|J|} \partial^J \partial^J,$$

we have $(\varphi, \hat{G})_{(p)} = (\varphi, \zeta\hat{G})$. Thus, it follows from (ii) that $G = \zeta\bar{G}$, where $\bar{G}$ is the complex conjugate of $\hat{G}$, and so that $\bar{G}$ is in $H_\pi^{(p)}(C)$. We have $\int_\Delta G = \zeta \int_\Delta \bar{G}$. It follows from what was proved in the previous paragraph that $\int_\Delta \bar{G} \to \bar{G}$ in the norm of $H_\pi^{(p)}(C)$ as $\Delta \to 0$. Thus, by Lemma 22 (as generalized to $H_\pi^{(p)}(C)$), $\int_\Delta G \to G$ in the norm of $H_\pi^{(-p)}(C) = H_\pi^{(k)}(C)$ as $\Delta \to 0$. Applying this to $\partial_1 F$ and noting that $\int_\Delta \partial_1 F = \Delta^{-1}(F \circ M_\Delta^{-1} - F)$, it follows that

$$\lim_{\Delta \to 0} |\Delta^{-1}(F \circ M_\Delta^{-1} - F)|_{(k)} = 0,$$

proving (ii) in case $k \leq 0$. Q.E.D.

There is an important sense in which certain distributions may be taken to apply to particular infinitely often differentiable functions which do not belong to C_0^∞. This sense is defined and studied in the following definition and lemma.

51 Definition. Let I be an open subset of E^n, and let F be in $D(I)$. Let φ be in $C^\infty(I)$, and let $\varphi(x)$ vanish for all x outside a certain closed subset K of I. Suppose that $K_1 = C(F) \cap K$ is a compact subset of I. Then we place

$$F(\varphi) = F(\psi\varphi),$$

where ψ denotes any function in $C_0^\infty(I)$ which is identically equal to one in a neighborhood of K_1.

52 Lemma. *Let I, F, and φ be as in the preceding definition. Then $F(\varphi)$ is independent of the particular function ψ used to define it. If*

$\hat{\varphi}$ is a second function in $C^\infty(I)$ vanishing outside a closed subset $\hat{K}$ of I such that $\hat{K}_1 = C(F) \cap K$ is compact, and α and $\hat{\alpha}$ are complex numbers, then

$$F(\alpha\varphi+\hat{\alpha}\hat{\varphi}) = \alpha F(\varphi)+\hat{\alpha}F(\hat{\varphi}).$$

PROOF. Let $\hat{\psi}$ be a second function in $C_0^\infty(I)$ such that $\hat{\psi}(x) = 1$ for x in a neighborhood of K_1. Then $\psi\varphi - \hat{\psi}\varphi$ vanishes in a neighborhood of $K \cap C(F)$, and vanishes in a neighborhood of $C(F)-K$ since φ vanishes in the complement of K. Hence $\psi\varphi-\hat{\psi}\varphi$ vanishes in a neighborhood of $C(F)$, so that $F(\psi\varphi) = F(\hat{\psi}\varphi)$ by Definition 11.

By Lemma 2.1, there is a function $\tilde{\psi}$ in $C_0^\infty(I)$ with $\tilde{\psi}(x) = 1$ for x in a neighborhood of $K_1 \cup \hat{K}_1$. Then

$$F(\alpha\varphi+\hat{\alpha}\hat{\varphi}) = F(\tilde{\psi}(\alpha\varphi+\hat{\alpha}\hat{\varphi})) = \alpha F(\tilde{\psi}\varphi)+\hat{\alpha}F(\tilde{\psi}\hat{\varphi}) = \alpha F(\varphi)+\hat{\alpha}F(\hat{\varphi})$$

by Definition 51 and the first paragraph of the present proof. Q.E.D.

The following lemma will be useful in what follows.

53 LEMMA. *Let I be an open subset of E^n, and let F be in $D(I)$. Let φ be in $C_0^\infty(I)$, and let f be in $L_1(I)$. Let K be a closed subset of I such that for x not in K, $\varphi(x-y)f(y) = 0$ for all y in I, and suppose that $K_1 = C(F) \cap K$ is compact. Then*

$$(*) \qquad F\left(\int_I \varphi(\cdot-y)f(y)dy\right) = \int_I F(\varphi(\cdot-y))f(y)dy.$$

Remark. If $f(y) \neq 0$, then $\varphi(x-y) = 0$ for $x \notin K$ by hypothesis. Hence, if $f(y) \neq 0$, $F(\varphi(\cdot+y))f(y)$ is well defined by the preceding definition and lemma. If $f(y) = 0$, we agree to set $F(\varphi(\cdot+y))f(y) = 0$, thereby giving a meaning to the integrand on the right of formula $(*)$ in all cases. In the course of the following proof, it will be shown that the integral on the right of $(*)$ exists.

PROOF. Using Lemma 2.1, let ψ be a function in $C_0^\infty(I)$ such that $\psi(x) = 1$ for x in a neighborhood of K_1. From Definition 51, we have

$$(1) \qquad \begin{aligned} F\left(\int_I \varphi(\cdot-y)f(y)dy\right) &= F\left(\psi^2(\cdot)\int_I \varphi(\cdot-y)f(y)dy\right) \\ &= F\left(\psi(\cdot)\int_I \psi(\cdot)\varphi(\cdot-y)f(y)dy\right) \end{aligned}$$

and

$$(2) \qquad F(\varphi(\cdot-y))f(y) = F(\psi^2(\cdot)\varphi(\cdot-y))f(y).$$

As $y_n \to y$, y_n and y remaining interior to I, it is clear that $\varphi(\cdot-y_n) \to$

$\varphi(\cdot - y)$ in the topology of $C^m(I)$ for each m. Thus, $\psi(\cdot)\varphi(\cdot - y_n) \rightrightarrows \psi(\cdot)\varphi(\cdot - y)$. This shows that $F(\psi(\cdot)\varphi(\cdot - y))$ is continuous in y for y in I (cf. Definition 1). Hence the integrand on the right of formula $(*)$ is integrable, and

$$\text{(3)} \qquad \int_I F(\varphi(\cdot - y))f(y)dy = \int_I F(\psi(\cdot)\psi(\cdot)\varphi(\cdot - y))f(y)dy.$$

Using (1) and (3), we see that to establish the present lemma it suffices to show that

$$\text{(4)} \qquad G\left(\int_I \psi(\cdot)\varphi(\cdot - y)f(y)dy\right) = \int_I G(\psi(\cdot)\varphi(\cdot - y))f(y)dy,$$

where $G = \psi F$. Let K_2 be a compact subset of I containing in its interior a second compact set outside of which the function ψ vanishes; we may evidently and shall henceforth suppose that K_2 is the closure of its interior $\mathring{K}_2$. Then, for sufficiently large m, G may be extended from $C_0^\infty(\mathring{K}_2)$ to a continuous linear functional on $C^m(K_2)$. Indeed, if this is not the case, then by Theorem II.3.11, there is, for each $m \geqq 1$, a function f_m in $C_0^\infty(\mathring{K}_2)$ whose norm in the space $C^m(K_2)$ is at most $1/m$, and such that $G(f_m) = 1$. But then it is clear that $\psi f_m \rightrightarrows 0$ so that $F(\psi f_m) = G(f_m) = 1$, which contradicts Definition 1.

It is clear that $\psi(\cdot)f(\cdot - y)$ varies continuously in the topology of $C^m(\mathring{K}_2)$ with y as y varies over I. Thus (4) finally follows from Theorem III.2.18. Q.E.D.

In subsequent sections of the present chapter, whenever τ denotes a formal partial differential operator defined in a domain I, $T_0(\tau)$ and $T_1(\tau)$ will denote the operators in $L_2(I)$ defined by the equations

$$\begin{aligned} &\mathfrak{D}(T_0(\tau)) = C_0^\infty(I); \qquad T_0(\tau)f = \tau f, \qquad f \in \mathfrak{D}(T_0(\tau)). \\ &\mathfrak{D}(T_1(\tau)) = \{f \in D(I) | f \in L_2(I), \qquad \tau f \in L_2(I)\}; \\ &\qquad\qquad T_1(\tau)f = \tau f, \qquad f \in \mathfrak{D}(T_1(\tau)). \end{aligned}$$

It should be noted that by Definition 5, $T_1(\tau) = (T_0(\tau^*))^*$. So that (cf. XII.1.6) $T_1(\tau)$ is always closed. By Lemma 6(i), $T_0(\tau) \subseteq T_1(\tau)$. By Lemma 6(iv), $T_1(\tau_1)T_1(\tau_2) \subseteq T_1(\tau_1\tau_2)$. These elementary facts will be used implicitly in what follows.

A final notational convention must be mentioned before the present section is concluded. If F is in $D(I)$, and F corresponds to a function f which is integrable over every compact subset of I, then, as noted above, we may write $F(\varphi)$ as

$$(*)\qquad F(\varphi) = \int_I f(x)\varphi(x)dx, \qquad \varphi \in C_0^\infty(I).$$

It is often suggestive to write $F(\varphi)$ in this form even in the general case; that is, to introduce for each F in $D(I)$ an ideal "function" f which is defined by formula $(*)$. Of course, this amounts simply to a notational convention, and nothing more; in the general case, the "function" f has no defined values, and indeed no existence beyond that which it has by virtue of appearing formally on the right of equation $(*)$. It is in this sense that Dirac wrote

$$\varphi(0) = \int_{-\infty}^{+\infty} \delta(x)f(x)dx$$

in terms of his δ-"function," and similar equations for $\varphi^{(k)}(0)$ in terms of the "functions" obtained by differentiating the δ-"function."

4. The Theorem of Sobolev

In the present section, we shall prove a very useful theorem of Sobolev relating the analytic properties of the derivatives of various orders of a given function f to the corresponding properties of the function f itself.

1 Lemma. *Let $n \geq 1$, and let E_+^n denote the half-space of Euclidean n-space E^n defined by the equation*

$$E_+^n = \{x \in E^n | x_1 > 0\}.$$

Let $\infty \geqq p' \geqq p \geqq 1$, and

$$\frac{1}{p'} > \frac{1}{p} - \frac{1}{n}.$$

Let F be a distribution defined in E_+^n, having bounded support, and with $(\partial/\partial x_i)F$ in $L_p(E_+^n)$, $i = 1, \ldots, n$. Then F is in $L_{p'}(E_+^n)$.

Proof. Let us agree to write y for a variable point in E^n, and to write $r = r(y) = |y|$ and $\omega = \omega(y) = y/|y|$ for the corresponding "radial" and "angular" variables. (Cf. the seventh paragraph of Section XI.7 for a discussion of spherical polar coordinates in Euclidean n-space; in the course of the present proof we shall adopt the notational conventions of that paragraph respecting hypersurface

area of Borel subsets of the unit sphere, transformation of integrals in E^n to spherical polar coordinates, etc.)

Let K be so large that the support of f is contained in the hemisphere $\{x \in E^n_+ \mid |x| \leq K/2\}$. Let a be a non-negative infinitely differentiable function defined in E^n, vanishing outside the set $\{x \in E^n \mid |x - [1, 0, \ldots, 0]| < 1/10\}$, and satisfying the equation

$$\int_S a(\omega)\mu(d\omega) = 1.$$

Let b be an infinitely often differentiable non-negative function of the positive real variable r such that $b(r) = 1$ for $r \leq 2K$ and $b(r) = 0$ for $r \geq 3K$. Then, if g is an infinitely differentiable function with support in the set $B = \{x \in E^n_+ \mid |x| \leq 3K/4\}$, we have clearly

$$\begin{aligned}
\text{(i)}\quad g(x) &= \int_S \left\{\int_\infty^0 \frac{\partial}{\partial r} g(x - r\omega)b(r)dr\right\} a(\omega)\mu(d\omega) \\
&= \sum_{j=1}^n \int_S \int_0^\infty \frac{\partial g}{\partial x_j}(x - r\omega)\omega_j b(r)a(\omega)\mu(d\omega)dr \\
&= \sum_{j=1}^n \int_{E^n_+} \frac{\partial g}{\partial x_j}(x - y)h_j(y)dy, \qquad x \in E^n_+
\end{aligned}$$

where $h_j(y)$ is the function defined by the equation

$$h_j(y) = \frac{a(\omega)\omega_j b(r)}{r^{n-1}}.$$

Let Ω_n denote the hypersurface area of the unit sphere in n-space. Then the functions h_j, $j = 1, \ldots, n$ satisfy the inequality

$$\int_{E^n} |h_j(y)|^s dy \leq \Omega_n \sup_{\omega \in S} |a(\omega)|^s \int_0^{3K} r^{(n-1)(1-s)} dr < \infty,$$

provided that $(n-1)(1-s) > -1$; i.e., h_j is in $L_s(E^n)$ if $1 \leq s < n/(n-1)$. It follows from (i) and Lemmas 3.52 and 3.53 that if F_j is the function in $L_p(E^n_+)$ corresponding to the distribution $\partial_j F$, we have

$$\text{(ii)}\quad F(g) = -\sum_{j=1}^n \int_{E^n_+} \int_{E^n_+} F_j(x) g(x - y) h_j(y) dy dx, \qquad g \in C_0^\infty(E^n_+).$$

Thus, using Fubini's theorem and writing $x - y = u$, we have

$$F(g) = -\sum_{j=1}^{n} \int_{E_+^n} \int_{E_+^n} F_j(x) h_j(x-u) g(u)\, dx\, du, \qquad g \in C_0^\infty(E_+^n),$$

so that the distribution F corresponds to the function

$$\text{(iii)} \qquad \sum_{j=1}^{n} \int_{E_+^n} F_j(x) h_j(x-a)\, dx.$$

We have only to show that this function belongs to $L_{p'}(E_+^n)$.

Now consider the convolution

$$\text{(iv)} \qquad c(u) = \int_{E^n} a(x) b(x-u)\, du.$$

If b is in $L_1(E^n)$ then Lemma XI.3.1 shows that equation (iv) defines a continuous map $a \to c$ from $L_1(E^n)$ into $L_1(E^n)$ as well as one from $L_\infty(E^n)$ into $L_\infty(E^n)$. It follows from the Riesz convexity theorem (VI.10.11) that this map also takes $L_p(E^n)$ into $L_p(E^n)$ in a continuous manner. Moreover, if a is in $L_p(E^n)$ and b is in $L_q(E^n)$ where $p^{-1}+q^{-1} = 1$ then it follows from Hölder's inequality (III.3.2) that c is in $L_\infty(E^n)$. Now fix a in $L_p(E^n)$, let $c = Tb$ be the linear map defined by equation (iv), and let $|T|_{p,q}$ be its norm as a mapping from $L_p(E^n)$ into $L_q(E^n)$. We have seen that the norms $|T|_{1,p}$ and $|T|_{q,\infty}$ where $p^{-1}+q^{-1} = 1$, are both finite. Consider the points $u = (q^{-1}, 0)$, $v = (1, p^{-1})$ in the unit square $0 \leq u, v \leq 1$. It follows from the Riesz convexity theorem that $|T|_{s,p'}$ is finite provided that

$$\left(\frac{1}{s}, \frac{1}{p'}\right) = \alpha u + (1-\alpha) v,$$

for some α in the interval $0 \leq \alpha \leq 1$. Now this equation defines s and α in terms of p and p' and the solutions are

$$\alpha = 1 - \frac{p}{p'}, \qquad s = 1 + \frac{1}{p'} - \frac{1}{p}.$$

Since $p' \geq p \geq 1$ the number α is in the interval $0 \leq \alpha \leq 1$ and since $(p')^{-1} > p^{-1} - n^{-1}$ and $p' \geq p$ we have $1 \leq s < n/(n-1)$. Thus, as shown above, h_j is in $L_s(E^n)$ and, since $|T|_{s,p'}$ is finite, Th_j is in $L_{p'}(E^n)$. If we let $a = F_j$ on E_+^n and $a = 0$ elsewhere on E^n it follows that the function given in (iii) is in $L_{p'}(E^n)$. Q.E.D.

2 Corollary. *Let $n \geq 1$, and let E_+^n denote the half-space of Euclidean n-space E_+^n defined by the equation*

$$E_+^n = \{x \in E^n | x_1 > 0\}.$$

Let $\infty \geqq p' \geqq p \geqq 1$, let $k \geqq 1$ be an integer, and let

$$\frac{1}{p'} > \frac{1}{p} - \frac{k}{n}.$$

Let F be a distribution defined in E_+^n and having bounded support. Then, if every derivative of order k of F belongs to $L_p(E_+^n)$, it follows that F is in $L_{p'}(E_+^n)$.

Proof. Let $\varepsilon = (p')^{-1} - p^{-1} + kn^{-1}$, and let the numbers $p = p_0, p_1, p_2, \ldots, p_k = p'$ be defined by the equations

$$\frac{1}{p_{j+1}} = \frac{1}{p_j} - \frac{1}{n} + \frac{\varepsilon}{k}, \qquad j = 0, \ldots, k-1.$$

Then, by Lemma 1, every derivative of order $k-1$ of F belongs to $L_{p_1}(E_+^n)$, every derivative of order $k-2$ of F belongs to $L_{p_2}(E_+^n), \ldots,$ and finally F belongs to $L_{p_k}(E_+^n) = L_{p'}(E_+^n)$. Q.E.D.

Remark. It is readily seen by much the same argument that Lemma 1 and Corollary 2 remain valid if the domain E_+^n is replaced by any unbounded domain E_0 having the following property:

There exists an open set U on the unit sphere in E^n such that any line l parallel to a line through the origin and through a point of U intersects the boundary of E_0 in at most one point.

Moreover, if we make use of the inequalities of Thorin described in Section XI.11, it may be shown that if $\infty > p' \geqq p > 1$ in the hypotheses of Lemma 1 and Corollary 2, then the conclusion of these two results may be strengthened to assert that F is in $L_{p'}(E_+^n)$ if

$$\frac{1}{p'} = \frac{1}{p} - \frac{1}{n}$$

in the case of Lemma 1 and if

$$\frac{1}{p'} = \frac{1}{p} - \frac{k}{n}$$

in the case of Corollary 2.

Next we give the stronger result of Sobolev which applies if, in Corollary 2, $p^{-1} - n^{-1}$ is negative.

3 LEMMA. *Let $n \geqq 1$, and let E^n_+ denote the half-space of Euclidean n-space E^n defined by the equation*

$$E^n_+ = \{x \in E^n | x_1 > 0\}.$$

Let $\infty \geqq p \geqq 1$, let $k \geqq 1$ and $m \geqq 0$ be integers, and let

$$m < k - \frac{n}{p}.$$

Let F be a distribution in E^n_+ having bounded support. Then, if every partial derivative of F of order k belongs to $L_p(E^n_+)$, it follows that every partial derivative of F of order not more than m is continuous in the closure of E^n_+.

PROOF. By Corollary 2 and Hölder's inequality, each $(k-m)$th derivative of any lth derivative of F belongs to $L_p(E^n_+)$ (and has compact carrier) for $l \leqq m$. Hence, it is quite sufficient to prove the present lemma for the special case $m = 0$. By Corollary 2 again, each derivative g of order 1 of F belongs to $L_{p'}(E^n_+)$ (and has compact carrier), for every p' satisfying the inequality

$$\frac{1}{p'} > \frac{1}{p} - \frac{(k-1)}{n}. \tag{i}$$

If p' is chosen so that $1 < p' < \infty$, and so that (i) and the equation

$$0 < 1 - \frac{n}{p'}$$

are satisfied, it follows immediately that it is sufficient to prove the present lemma for the special case $k = 1$, $m = 0$. We shall consequently assume for the remainder of the present proof that $k = 1$, $m = 0$.

Our hypothesis then is the fact that every derivative of order 1 of F belongs to $L_p(E^n_+)$, where

$$0 < 1 - \frac{n}{p};$$

and what is to be shown is that F is continuous in the closure of E^n_+. By Lemma 1, F is a function in $L_\infty(E^n_+)$.

Let the functions h_j and F_j be defined as they were in the proof

of Lemma 1 and recall that h_j is in $L_s(E^n)$ provided that $1 \leq s < n/(n-1)$. Since $p > n$ the number q defined by the equation $p^{-1}+q^{-1} = 1$ is in the interval $1 \leq q < n/(n-1)$ and so h_j is in $L_q(E^n)$. Since F_j is in $L_p(E^n_+)$ it follows from Hölder's inequality that $F_j(\cdot)h_j(\cdot)$ is in $L_1(E^n_+)$. We have, by an elementary change of variable,

$$\sum_{j=1}^{n} \int_{E^n_+} F_j(x-y)h_j(y)dy = \sum_{j=1}^{n} \int_{E^n_+} F_j(y)h_j(x-y)dy.$$

Thus, for a function g in $C_0^\infty(E^n_+)$,

$$\begin{aligned}
\int_{E^n_+} \left\{ \sum_{j=1}^{n} \int_{E^n_+} F_j(x-y)h_j(y)dy \right\} g(x)dx &= \sum_{j=1}^{n} \int_{E^n_+} \left\{ \int_{E^n_+} F_j(y)h_j(x-y)g(x)dx \right\} \\
&= \sum_{j=1}^{n} F\left(\frac{\partial}{\partial y_j} \int_{E^n_+} h_j(x-y)g(x)dx \right) \\
&= -\sum_{j=1}^{n} F\left(\frac{\partial}{\partial y_j} \int_{E^n_+} h_j(x)g(y+x)dx \right) \\
& F\left(-\sum_{j=1}^{n} \int_{E^n_+} \frac{\partial}{\partial y_j} h_j(x)g(y+x)dx \right) \\
&= -F\left(\sum_{j=1}^{n} \int_{E^n_+} h_j(x) \frac{\partial g}{\partial x_j}(y+x)dx \right) \\
&= -F(g).
\end{aligned}$$

In deriving this series of equations we have used the fact that h_j is in $L_q(E^n)$, Hölder's inequality, Lemma X.3.1(a), and Fubini's theorem to justify the change of order of integration which is the first equation; Definitions 3.4 and 3.5(a) to obtain the second equation; an elementary change of variable to obtain the third equation; the facts that h_j is in $L_1(E^n)$ and that g is in $C_0^\infty(E^n)$ to obtain the fourth equation; and equation (i) to obtain the final equation. Thus, according to Definition 3.4, the distribution F corresponds to the function

$$\text{(ii)} \qquad f(x) = \sum_{j=1}^{n} \int_{E^n_+} F_j(x+y)h_j(y)dy.$$

The n-dimensional analogue of Lemma XI.3.1(f) and Hölder's inequality show that this function is everywhere defined on E^n_+, is continuous in x, and has a unique continuous extension to the closure $\bar{E}^n_+$. Q.E.D.

4 Definition. Let p be a point of the subset A of Euclidean n-space E^n. Then A is said to be *smooth in the vicinity of* p if there exists a neighborhood U of p, and a mapping φ of U on a spherical neighborhood V of the origin, such that

(i) φ is one-to-one, φ is infinitely often differentiable, and φ^{-1} is infinitely often differentiable.

(ii) $$\varphi(AU) = V \cap \{x \in E^n \mid x_1 = 0\}.$$

If the set A is smooth in the vicinity of each of its points, it is said to be *smooth*, or to be a *smooth surface*.

In terms of this definition we are able to state a general form of Sobolev's theorem.

→ 5 Theorem. *Let $n \geqq 1$, and let D be a bounded open set in Euclidean space E^n. Suppose that the boundary of D is a smooth surface and that no point in the boundary of D is interior to the closure of D. Let $k \leqq 1$ and $m > 0$ be integers. Let $\infty > p \geqq 1$, $\infty \geq p' \geqq 1$, and let $1 > \varepsilon > 0$. Let F be a distribution in D, and let every derivative of order not more than k of F belong to $L_p(D)$. Then, if*

(i) $$\frac{1}{p'} > \frac{1}{p} - \frac{k}{n},$$

F is in $L_{p'}(D)$; while if

(ii) $$m < k - \frac{n}{p},$$

then every derivative of order not more than m of F is continuous in the closure of D.

Proof. Cover the closure of D with a finite collection of bounded open sets U each of which is either disjoint from the boundary of D or is differentiably equivalent to a spherical neighborhood V of the origin of E^n as in Definition 4. Let $\{h_j\}$, $j = 1, \ldots, N$ be a family of non-negative functions in $C_0^\infty(E^n)$ such that $\sum_{j=1}^N h_j(x) = 1$ for x in a neighborhood of the closure of D and such that each function h_j vanishes outside a compact subset of some set of this covering (cf. Lemma 2.3). Let h_j be a function in this partition of unity with support in U. We shall show that if (i) holds, h_jF is in $L_p(D)$, while if (ii) holds, every derivative of order not more than m of h_jF is con-

tinuous in the closure of D. This will evidently imply the truth of the present theorem.

First suppose that U is disjoint from the boundary of D. Then $h_j F$ is a distribution whose carrier is a compact set contained in U (cf. Lemma 3.13 (iv)). It is clear from Lemma 3.6 that every derivative of $h_j F$ of order not more than k belongs to $L_p(E^n)$. Translating E^n sufficiently far to the right along the x_1-axis (cf. Lemmas 3.47 and 3.48), we may assume without loss of generality that U is contained in the set E^n_+ of Corollary 2 and Lemma 3. Thus, by Lemma 3.12 $h_j F$ may be regarded as a distribution in E^n_+, and our assertion follows from Corollary 2 and Lemma 3.

Next consider the case in which U intersects the boundary of D, so that there exists a one-to-one mapping φ of U onto a spherical neighborhood V of the origin of E^n, φ being assumed to have the following properties:

(i) φ and φ^{-1} are infinitely often differentiable transformations.

(ii) $\varphi(AU) = V \cap \{x \in E^n | x_1 = 0\}$, A denoting the boundary of D.

Since, by hypothesis, no point in V but the points in $\varphi(AU)$ belong to the boundary of $\varphi(UD)$, and no point in the boundary of D is interior to the closure of D, it follows that $\varphi(UD)$ must consist of one or another of the hemispheres $V_+ = \{x \in V | x_1 > 0\}$ or $V_- = \{x \in V | x_1 < 0\}$. For the sake of definiteness, we will consider the case in which $\varphi(UD) = \{x \in V | x_1 < 0\}$; the other case is equivalent to this by a change of variables. By Lemmas 3.13 (iv) and 3.45(c) and Definition 3.11, the distribution $(h_j F)\circ\varphi^{-1}$ is a distribution in V_+ the closure in E^n of whose support is disjoint from the curved boundary of V_+, and all of whose derivatives of order at most k belong to $L_p(V_+)$ by Lemmas 3.47 and 3.48. Hence, by Lemma 3.12, $(h_j F)\circ\varphi^{-1}$ may be regarded as a distribution in E^n_+ all of whose derivatives of order not more than k belong to $L_p(E^n_+)$. By Lemma 3, $(h_j F)\circ\varphi^{-1}$ and all its derivatives of order not more than m are continuous in the closure of V_+. From this and Lemma 3.47 it is evident that $h_j F = (h_j F)\circ\varphi^{-1}\circ\varphi$ and all its derivatives of order at most m are continuous in the closure of D.

Since $f = \sum_{j=1}^{N} h_j f$, the theorem now follows immediately from the evident fact that both $L_{p'}(D)$, and the set of all functions all of whose derivatives of order at most m are continuous in the closure of

D, are linear spaces. Q.E.D.

Remark It is easy to show, making use of the first part of the remark following Corollary 2 and of the fact that a similar observation may be made in connection with Lemma 3, that Theorem 5 is valid even if the boundary of D contains "corners," "edges," etc. These configurations may be defined, as in Definition 4, as configurations differentiably equivalent to the local configuration at the intersection of a finite number of hyperplanes in E^n.

6 Corollary. *Let I be a bounded open subset of E^n, whose boundary is a smooth surface Σ. Suppose that no point of Σ belongs to the interior of $\bar{I}$. Let $1 \leq p, q < \infty$ and let k be a positive integer with*

$$\frac{1}{q} > \frac{1}{p} - \frac{k}{n}.$$

Let f, f_m, $m = 1, 2, \ldots$, be functions defined in I, such that

$$\lim_{m\to\infty} \int_I |\partial^J f_m(x) - \partial^J f(x)|^p dx = 0, \qquad |J| \leq k.$$

Then

$$\lim_{m\to\infty} \int_I |f_m(x) - f(x)|^q dx = 0.$$

Moreover, there is a constant K depending on I, q, p, k, n, but not on f, such that

$$\left\{\int_I |f(x)|^q dx\right\}^{1/q} \leq K \sum_{|J| \leq k} \left\{\int_I |\partial^J f(x)|^p dx\right\}^{1/p}.$$

If

$$\frac{1}{p} < \frac{1}{n},$$

then we have

$$\lim_{m\to\infty} \operatorname*{ess\,sup}_{x\in I} |f_m(x) - f(x)| = 0.$$

Moreover, there exists a constant K depending on I, q, p, k, n, but not on f, such that

$$\operatorname*{ess\,sup}_{x\in I} |f(x)| \leq K \sum_{|J| \leq k} \left\{\int_I |\partial^J f(x)|^p dx\right\}^{1/p}.$$

Proof. Let $L_p^k(I)$ denote the subspace of $L_p(I)$ consisting of all

functions f such that $\partial^J f$ is in $L_p(I)$ if $|J| \leq k$, the derivatives being taken in the sense of the theory of distributions. Put

$$\text{(i)} \qquad |f|_{(p,k)} = \sum_{|J| \leq k} \left\{ \int_I |\partial^J f(x)|^p dx \right\}^{1/p}$$

Then $L_p^k(I)$ is a B-space. Indeed, if $\{f_m\}$ is a Cauchy sequence in $L_p^k(I)$, it is clear from (i) that $\{\partial^J f_m\}$ is a Cauchy sequence in $L_p(I)$ for $|J| \leq k$, so that there exist functions g, g^J in $L_p(I)$ such that $\lim_{m\to\infty}|f_m - g|_p = 0$ and $\lim_{m\to\infty}|\partial^J f_m - g^J|_p = 0$. It is then clear from Definition 3.26 that $\lim_{m\to\infty} f_m = g$ and $\lim_{m\to\infty} \partial^J f_m = g^J$ for $|J| < k$ in the sense of the topology of distributions, so that $g^J = \partial^J g$ by Lemma 3.27. Thus g is in $L_p^k(I)$ and $\lim_{m\to\infty}|f_m - g|_{(p,k)} = 0$ by (i).

According to Theorem 5, we have $L_p^k(I) \subseteq L_q(I)$ if $q^{-1} > p^{-1} - kn^{-1}$, and $L_p^k(I) \subseteq L_\infty(I)$ if $p^{-1} > kn^{-1}$. The identity mapping $f \to f$ is easily seen to be a closed mapping of $L_p^k(I)$ into $L_q(I)$ in the first case and of $L_p^k(I)$ into $L_\infty(I)$ in the second. Thus, the present lemma follows immediately from the closed graph theorem (II.2.4). Q.E.D.

7 Lemma. *Let $p \geq 1$. Let F be in $L_p(E^n)$ and vanish outside a compact set K. Suppose that k is an integer greater than zero, and that $\partial^J F$ is in $L_p(E^n)$ for $|J| < k$, the partial derivatives being taken in the sense of the theory of distributions, and suppose that*

$$\sum_{|J| \leq k} \int_{E^n} |\partial^J F(x)|^p dx \leq 1.$$

Let $1 \leq q \leq \infty$, and $q^{-1} > p^{-1} - k/n$. Then for each $\varepsilon > 0$ there exists a $\delta > 0$ depending only on K, ε, q, and p, but not on F, such that if y is in E^n and $|y| < \delta$, then

$$|F(\cdot) - F(\cdot + y)|_q < \varepsilon.$$

Proof. Making a shift of coordinates, it is clear that we may assume without loss of generality that K is a subset of the set $E_\delta^n = \{x \in E^n | x_1 > \delta\}$. This being the case, it follows as shown in the first three paragraphs of the proof of Lemma 1 that there exist functions h_j defined in E^n and vanishing outside $E_+^n = \{x \in E^n | x_1 > 0\}$ such that

$$\text{(i)} \qquad \partial^J F(x) = \sum_{j=1}^{n} \int_{E^n} \partial_j \partial^J F(u) h_j(u - x) du,$$

$$\text{(ii)} \qquad \partial^J F(x+y) = \sum_{j=1}^{n} \int_{E^n} \partial_j \partial^J F(u+y) h_j(u-x)\,du.$$

for $|J| < k-1$, $x \in E^n_+$, and $|y| < \delta$. The functions h_j themselves belong, as shown in the first three paragraphs of the proof of Lemma 1, to the space $L_s(E^n)$ for each s in the interval $1 \leqq s < n/(n-1)$.

From (i) and (ii) it follows immediately that

$$\text{(iii)} \qquad \begin{aligned} &\partial^J(F(x) - F(x+y)) \\ &\qquad = \sum_{j=1}^{n} \int_{E^n} \partial_j \partial^J F(u)\{h_j(u-x) - h_j(u-x-y)\}\,du \end{aligned}$$

for $|J| \leq k-1$, $x \in E^n_+$, and $|y| < \delta$.

Now, if a is in $L_p(E^n)$, and b is in $L_1(E^n)$, then for $p = \infty$ or $p = 1$, the function

$$\text{(iv)} \qquad c(u) = \int_{E^n} a(x) b(x-u)\,dx$$

belongs to $L_p(E^n)$, and $|c|_p \leqq |a|_p |b|_1$ by the n-dimensional version of Lemma XI.3.1. Hence, by the Riesz convexity theorem (VI.10.11), this statement holds for all p such that $1 \leq p \leq \infty$. If a is in $L_p(E^n)$ and b is in $L_{\bar{p}}(E^n)$, where $p^{-1}+\bar{p}^{-1} = 1$, then the function c in (iv) belongs to $L_\infty(E^n)$, and $|c|_\infty \leqq |a|_p |b|_{\bar{p}}$ by Hölder's inequality (III.3.2). Thus, by the Riesz convexity theorem (VI.10.11), if a is in $L_p(E^n)$ and b is in $L_s(E^n)$ for each s such that $1 < s < n(n-1)^{-1}$ we have c in $L_{p'}(E^n)$ and $|c|_{p'} < |a|_p |b|_s$, where $p' = p'(s)$ is defined by the equation

$$\frac{1}{p'} = \frac{1}{p} + \frac{1}{s} - 1.$$

Thus it follows from (iii) and from our hypothesis that

$$\text{(v)} \qquad |\partial^J\{F(\cdot) - F(\cdot+y)\}|_p \leqq \sum_{j=1}^{n} \left\{ \int_{E^n} |h_j(u) - h_j(u-y)|^s du \right\}^{1/s},$$

$|J| < k-1$, for $|y| < \delta$ and $1 < s < n/(n-1)$. By Corollary 6, it follows that if $q_0^{-1} > (p')^{-1} - (k-1)n^{-1}$, there exists a constant $K(q_0, p')$ depending only on q_0 and p' such that

$$\text{(vi)} \qquad |F(\cdot) - F(\cdot+y)|_{q_0} \leqq K(q_0, p') \sum_{j=1}^{n} \left\{ \int_{E^n} |h_j(u) - h_j(u-y)|^s du \right\}^{1}$$

here $1 \leqq q_0 \leqq \infty$. Now let q be as in the hypothesis of the present lemma. Then it is clear that we may choose some definite $s = s_0$ such that $q^{-1} > \{p'(s_0)\}^{-1} - (k-1)n^{-1}$, and then the conclusion of the present lemma follows immediately from formula (vi) and from Lemma IV.8.21. Q.E.D.

8 LEMMA. *Let I be a bounded open subset of E^n, whose boundary is a smooth surface Σ. Suppose that no point of Σ belongs to the interior of $\bar{I}$. Let $1 < p < \infty$, and let k be a positive integer. Let $1 \leqq q < \infty$, and let*

$$\frac{1}{q} > \frac{1}{p} - \frac{k}{n}.$$

Then if $\{f_m\}$ is a sequence of functions defined in I such that

$$\int_I |\partial^J f_m(x)|^p dx$$

is bounded in m for $|J| \leqq k$, it follows that $\{f_m\}$ has a subsequence convergent in the norm of $L_q(I)$.

PROOF. Let $\{K_m\}$ be an increasing sequence of compact subsets of I whose union is I. Using Lemma 2.1, let $\{\varphi_m\}$ be a sequence of functions in $C_0^\infty(I)$ such that $\varphi_m(x) = 1$ for x in K_m and such that $0 \leqq \varphi_m(x) \leqq 1$ for all x. Then it is clear from Lemma 3.22 that

$$\int_I |\partial^J \{\varphi_j(x) f_m(x)\}|^p dx$$

is bounded in m for each fixed j. By Lemmas 3.12 and 3.13, it follows that if we put $\varphi_j(x) f_m(x) = 0$ for x not in I, we may regard $\varphi_j f_m$ as a distribution defined in all of E^n, to which the preceding lemma will apply. Making use of the preceding lemma, and of Theorem IV.8.21, we find that for each $j < \infty$, $\{f_m|K_j\}$ has a subsequence converging in the norm of $L_p(K_j)$. Making use of Cantor's diagonal construction, we may find a subsequence of $\{f_m\}$ such that $\{f_m|K_j\}$ converges in the norm of $L_p(K_j)$ for each $j \geqq 0$. For simplicity in notation, we shall henceforth assume that the sequence $\{f_m\}$ itself has this property.

Since $q^{-1} > p^{-1} - kn^{-1}$, we can find a $q_1 > q$ such that $q_1^{-1} > p^{-1} - kn^{-1}$. It then follows from Corollary 6 that $|f_m|_{q_1}$ is bounded by a finite constant A. Hence, by Hölder's inequality (III.3.2), we have

$$\text{(i)} \qquad \int_{I-K_j} |f_m(x)|^q dx < A^q\{\mu(I-K_j)\}^{1-q/q_1}$$

where $\mu(e)$ denotes the Lebesgue measure of the Borel set e. It follows from (i) that

$$\lim_{j\to\infty} \int_{I-K_j} |f_m(x)|^q dx = 0$$

uniformly in m. Since we have already seen that

$$\lim_{m, m_1\to\infty} \int_{K_j} |f_m(x) - f_{m_1}(x)|^q dx = 0$$

for each j, it follows that

$$\lim_{m, m_1\to\infty} \int_{I} |f_m(x) - f_{m_1}(x)|^q dx = 0,$$

proving the present lemma. Q.E.D.

9 Corollary. *The conclusions of Corollary 6 and Lemma 8 remain valid even if the open set I of these results is replaced by the cube*

$$C = \{x \in E^n \mid |x_j| < \pi, \quad j = 1, \ldots, n\}.$$

Proof. It was observed, in the remark following Theorem 5, that Theorem 5 was valid even if the boundary of the domain D of that theorem contained corners, edges, etc. Using this remark, the proofs of Corollary 6 and Lemma 8 may immediately be extended to the present case. Details are left to the reader. Q.E.D.

10 Corollary. *Let C denote either a bounded domain in E^n whose boundary is a smooth surface Σ no point of which is interior to the closure of C, or else denote the cube*

$$C = \{x \in E^n \mid |x_i| < \pi, \quad i = 1, \ldots, n\}$$

in E^n. Let p be a positive integer. Then the natural identity mapping of $H^{(p)}(C)$ into $H^{(p-1)}(C)$ is a compact linear mapping.

Proof. Let $\{f_m\}$ be a bounded sequence in $H^p(C)$. We must show that there exists a subsequence $\{f_{m_j}\}$ of $\{f_m\}$ such that $\{\partial^J f_{m_j}\}$ converges in $L_2(I)$ for each J such that $|J| \leq p-1$. Since $\{\partial_j \partial^J f_m\}$ is bounded in $L_2(C)$ for each J with $|J| \leq p-1$ and for each j such that $1 \leq j \leq n$, it is clear that the present lemma will follow immediately from its own special case, $p = 1$. However, this case $p = 1$ is the

special case $k = 1$, $p = 2$ either of Lemma 8 or of the preceding corollary. Q.E.D.

11 Corollary. *Let I be a bounded domain in E^n. Let p be a positive integer. Then the natural identity mapping of the space $H_0^{(p)}(I)$ into $H_0^{(p-1)}(I)$ is a compact linear mapping.*

Proof. Let $\{f_n\}$ be a bounded sequence in $H_0^{(p)}(I)$. We must show that there exists a subsequence $\{f_n\}$ which converges in the topology of $H_0^{(p-1)}(I)$. Since $C_0^\infty(I)$ is dense in $H_0^{(p)}(I)$ by the definition of $H_0^{(p)}(I)$, we can find a sequence $\{g_n\}$ of elements of $C_0^\infty(I)$ such that $|g_n - f_n|_{(p)} < 1/n$, and this implies that $|g_n - f_n|_{(p-1)} < 1/n$ (cf. Definition 3.15). Thus, it is sufficient to show that $\{g_n\}$ has a subsequence which converges in the topology of $H_0^{(p-1)}(I)$. That is, we may (and shall) suppose without loss of generality that f_n is in $C_0^\infty(I)$ for all n. This being the case, we may put $f_n(x) = 0$ for $x \notin I$, so that f_n is in $C_0^\infty(E^n)$. We wish to show that for each J such that $|J| \leq p-1$, $\partial^J f_n$ has a subsequence converging in $L_2(I)$. Since $\{\partial_j \partial^J f_n\}$ is bounded in $L_2(I)$ for each J with $|J| \leq p-1$ and $1 \leq j \leq n$ by hypothesis, it is clear that the present lemma will follow immediately if we can establish the following statement

Let $\{h_m\}$ be a sequence of functions in $C_0^\infty(E^n)$, all vanishing outside I. Suppose that $|h_m|_{(1)} \leq 1$, $m \geq 1$. Then $\{h_n\}$ contains a subsequence converging in $L_2(I)$.

To prove this statement, let C be a cube of the form

$$C = \{x \in E^n \mid |x_i| < a, \qquad i = 1, \ldots, n\},$$

a being chosen so large that $C \supseteq I$. Making a dilation of coordinates (cf. Lemma 3.48), we may and shall suppose without loss of generality that $a = \pi$. The present corollary then follows immediately from the preceding corollary. Q.E.D.

12 Lemma. *Let I be a bounded domain in E^n. Let p be a positive integer and let $\varepsilon > 0$. Then there exists a finite positive constant $K(\varepsilon)$ such that*

$$|f|_{(p)}|f|_{(p-1)} \leq \varepsilon |f|_{(p)}^2 + K(\varepsilon)\|f\|^2, \qquad f \in C_0^\infty(I)$$

Proof. Suppose that the statement is false. Then it is clear that there exists a sequence $\{f_m\}$ in $C_0^\infty(I)$ such that

$$\text{(i)} \qquad |f_m|_{(p)}|f_m|_{(p-1)} > \varepsilon |f_m|^2_{(p)} + m|f_m|^2.$$

Multiplying $\{f_m\}$ by a suitable sequence of constants, we may evidently suppose without loss of generality that $|f_m|_{(p)} = 1$, $m > 1$. Then it is clear that $|f_m| \to 0$ as $m \to \infty$. By the preceding corollary, we may suppose without loss of generality that $\{f_m\}$ converges in the topology of $H_0^{(p-1)}(I)$ to an element g. Then we clearly have $|f_m - g|_{(0)} \to 0$ as $m \to \infty$, and since $|f_m|_{(0)} \to 0$ as $m \to \infty$, it follows that $g = 0$. Thus, $|f_m|_{(p-1)} \to 0$ as $m \to \infty$. Since by (i) we have $|f_m|_{(p-1)} \geq \varepsilon > 0$ for all m, this is a contradiction. Q.E.D.

Sobolev's theorems enable us to complete our discussion of the theory of distributions in several important respects. The next two lemmas give us useful information on the structure of the set $D(I)$ of distributions.

13 LEMMA. *Let I be an open subset of E^n, and F a distribution in I whose carrier is a compact subset of I. Then F is in $H^{(k)}(I)$ for some sufficiently large negative k.*

PROOF. Let C denote the compact carrier of F, and, using Lemma 2.2, let φ denote a function in $C_0^\infty(I)$ which is identically equal to one in a neighborhood of C. If our assertion is false, it follows from Definition 3.17 that for each $n > 0$ there is a function ψ_n in $C_0^\infty(I)$ with $|\psi_n|_{(n)} \leq n^{-1}$, such that $|F(\psi_n)| \geq 1$. Now, we have $\psi_n = \varphi\psi_n$ in a neighborhood of the carrier of F, so that (cf. Definition 3.11) $F(\psi_n) = F(\psi_n\varphi)$. On the other hand, it is clear from Lemma 3.22 that $|\psi_n\varphi|_{(k)} \to 0$ as $n \to \infty$ for each k. We may consequently apply the second part of Corollary 6 to show that $\partial^J\psi_n\varphi \to 0$ uniformly in I for all J. Thus, since all the functions $\psi_n\varphi$ evidently vanish outside a fixed compact subset of I, $\psi_n\varphi \to 0$ as $n \to \infty$ in the sense of Definition 3.1. Thus $F(\psi_n\varphi) \to 0$ by Definition 3.1, which contradiction proves the present lemma. Q.E.D.

We may prove the following closely related lemma in precisely the same way.

14 LEMMA. *Let I be a rectangle in E^n, and F a distribution in $D_\pi(I)$. Then F is in $H_\pi^{(k)}(I)$ for some sufficiently large negative k.*

The detailed modification of the proof of Lemma 13 to give a proof of this lemma is left to the reader as an exercise.

Lemma 13 and the following lemma taken together give considerable insight into the nature of distributions in general.

15 LEMMA. *Let F be a distribution in the open subset I of E^n. Let $\{I_n\}$ be a sequence of open subsets of I whose union is I, such that $\bar{I}_m$ is compact and contained in I, and such that $\bar{I}_m \cap \bar{I}_p = \phi$ unless $|m-p| = 1$. Then F may be written as the sum $F = \sum_{m=1}^{\infty} F_m$ of a convergent infinite series of distributions for which the carrier of F_m is a subset of I_m.*

PROOF. It is easily seen that we may find a sequence $\{C_j\}$ of compact sets with $C_j \subseteq I_j$, $j = 1, \ldots, n$, and such that $\bigcup_{j=1}^{n} C_j = I$. Using Lemma 2.1, let the function ψ_j in $C_0^{\infty}(I_j)$ be such that $0 \leq \psi_j(x) \leq 1$ for all x, and such that $\psi_j(x) = 1$ for x in C_j. Then it is evident that the series

$$\psi(x) = \sum_{j=1}^{\infty} \psi_j(x)$$

converges to a function ψ in $C^{\infty}(I)$ which is everywhere positive. Thus, if we put $\eta_j(x) = \psi(x)^{-1}\psi_j(x)$, we have η_j in $C_0^{\infty}(I_j)$, and

$$\sum_{j=1}^{\infty} \eta_j(x) = 1.$$

Since only a finite number of the terms of this series fail to vanish in any compact subset of I, it is evident that

$$\varphi = \lim_{m \to \infty} \sum_{j=1}^{m} \eta_j \varphi, \qquad \varphi \in C_0^{\infty}(I).$$

Thus, by Definitions 3.5 and 3.26,

$$F = \sum_{j=1}^{\infty} \eta_j F, \qquad F \in D(I).$$

Since the carrier of $\eta_j F$ is a subset of I_j by Lemma 3.13, the proof of the present lemma is complete. Q.E.D.

16 COROLLARY. *Let I be a bounded open subset of E^n. Then the subspace $C_0^{\infty}(I)$ of $D(I)$ is dense in $D(I)$.*

PROOF. Let F be in $D(I)$. We wish to construct a sequence $\{\varphi_m\}$ of elements of the subspace $C_0^{\infty}(I)$ of $D(I)$ such that $\varphi_m \to F$

as $m \to \infty$. By Lemma 14 there is a sequence $\{F_m\}$ of elements of $D(I)$, each of which has a carrier which is a compact subset C_m of I, and such that $F_m \to F$ as $m \to \infty$. Hence, we can evidently suppose without loss of generality that the carrier C of F is a compact subset of I. If I is included in a cube D, it follows from Lemmas 13, 3.43 and 3.12 that there is a unique extension G of F to a distribution in $D_n(D)$ such that the carrier of G is C, and a sequence of elements $\hat{\varphi}_m$ in $C_0^\infty(\bar{D})$ such that $\hat{\varphi}_m \to G$ as $m \to \infty$. Let ψ be in $C_0^\infty(I)$ and have $\psi(x) = 1$ for all x in a neighborhood of C. Then $\tilde{\varphi}_m = \psi\hat{\varphi}_m$ is in $C_0^\infty(I)$. It is clear (cf. 3.22) that $\tilde{\varphi}_m \to \psi G = G$ as $m \to \infty$. If $\varphi_m = \tilde{\varphi}_m|I$, then by Lemma 3.23, $\varphi_m \to F = G|I$ as $m \to \infty$. Q.E.D.

17 Corollary. *Let I be a bounded open subset of E^n. Let F be in $H^{(k)}(I)$, where k is a non-negative integer, and let the carrier C of F be a compact subset of I. Then F is in $H_0^{(k)}(I)$.*

Proof. Let D be a sphere including I, and, using Lemmas 3.12 and 3.24, let G be an extension of F to a distribution in $H^{(k)}(D)$, G having the same carrier C. Using the preceding lemma, let φ_m be a sequence in $C^\infty(\bar{D})$ with $\varphi_m \to G$ in the topology of $H^{(k)}(D)$ as $m \to \infty$. Using Lemma 2.1, let the function ψ in $C_0^\infty(I)$ be such that $\psi(x) = 1$ for all x in a neighborhood of C. Then, by Lemma 3.10 and Definition 3.11, $\psi G = G$. It follows from Lemma 3.22 that $\varphi_m\psi \to G$ in the topology of $H^{(k)}(D)$ as $m \to \infty$. Hence, by Lemma 3.23, $\varphi_m\psi|I \to F = G|I$ as $m \to \infty$, so that F is in $H_0^{(k)}(I)$ by Definition 3.15. Q.E.D.

We conclude the present section with the following lemma, which gives a "density" result that will be useful in what follows.

18 Lemma. *Let I denote a bounded open set in E^n whose boundary is a smooth surface Σ. Suppose that no point of Σ belongs to the interior of $\bar{I}$. Let p be a non-negative integer. Then the subspace $C^\infty(\bar{I})$ of $H^{(p)}(I)$ is dense in $H^{(p)}(I)$.*

Proof. Let f be in $H^{(p)}(I)$. We wish to show that f can be approximated arbitrarily well in the norm of $H^{(p)}(I)$ by an element of $C^\infty(\bar{I})$. Let $\{U_j\}$, $j = 1, \ldots, m$, be an arbitrary covering of $\bar{I}$ by a finite collection of neighborhoods of points in $\bar{I}$, and, using Lemma 2.4, let $\{f_j\}$, $j = 1, \ldots, k$, be a collection of functions in $C^\infty(E^n)$ such that each function f_j vanishes outside a compact subset of some one of the neighborhoods $\{U_i\}$ and such that $\sum_{j=1}^k f_j(x) = 1$ for x in a

neighborhood of $\bar{I}$. Suppose that the collection of neighborhoods is chosen so that for each j there is a mapping φ_j of U_j into the unit spherical neighborhood V of the origin of E^n such that

(i) φ_j is one-to-one, is infinitely often differentiable, and φ^{-1} is infinitely often differentiable;

(ii) if $U_j \cap \Sigma \neq \phi$, then $\varphi_j(U_j \cap \Sigma) = \{x \in V | x_1 = 0\}$.

By Lemma 3.6 we have $f = \sum_{j=1}^{k} f_j f$, each separate term belonging to $H^p(I)$ by Lemma 3.22. Hence it is sufficient for us to show that each element $f_j f$ may be approximated arbitrarily closely in the norm of $H^{(p)}(I)$ by an element of $C^\infty(\bar{I})$. For the sake of definiteness, we shall take $j = 1$. Passing without loss of generality from consideration of f to consideration of $f_1 f$, it follows that we may (and shall) suppose that f vanishes outside a compact subset of U_1. We shall show that $f|U_1 I$ is the limit in the norm of $H^{(p)}(U_1 I)$ of a sequence $\{g_j\}$ of functions in $C_0^\infty(U_1)$, from which the present lemma evidently follows immediately on extending g_j to a function in $C_0^\infty(E^n)$ by putting $g_j(x) = 0$ for $x \notin U_1$. Write $f|U_1 I = \hat{f}$.

We have then two cases to consider.

(a) $U_1 \cap \Sigma = \phi$.

In this case it is clear that $U_1 \subseteq I$, so that $\varphi_1(U_1 I) = V$.

(b) $\varphi_1(U_1 \cap \Sigma) = \{x \in V | x_1 = 0\}$.

In this case, since it is clear that no point in $\varphi_1(U_1 \cap I)$ but the points $\varphi_1(U_1 \cap \Sigma)$ belong to the boundary of $\varphi_1(U_1 \cap I)$, and since from the hypothesis it follows that no point in $\varphi_1(U_1 \cap \Sigma)$ is interior to the closure of $\varphi_1(U_1 \cap I)$, we can conclude that $\varphi_1(I \cap U_1)$ must consist of one or another of the hemispheres $V_+ = \{x \in V | x_1 > 0)$ or $V_- = \{x \in V | x_1 < 0\}$. For the sake of definiteness, we shall suppose that $\varphi_1(IU_1) = V_+$; the other case is equivalent to this by a change of variables.

First consider case (a). Using Lemma 3.33, extend $\hat{f} \circ \varphi_1^{-1}$ to an element F of $H_\pi^{(p)}(C)$, whose carrier K is the same as the carrier of $\hat{f} \circ \varphi_1^{-1}$; here as below, C denotes the cube

$$C = \{x \in E^n | \, |x_j| \leq \pi, \qquad j = 1, \ldots, n\}.$$

By Lemma 3.41, F is the limit in the norm of $H_\pi^{(p)}(C)$ of a sequence of elements of $C_\pi^\infty(C)$. Thus, by Lemma 3.28, $\hat{f} \circ \varphi_1^{-1}$ is the limit in the

norm of $H^{(p)}(V)$ of a sequence $\{h_m\}$ of elements of $C^{\infty}(\bar{V})$. Using Lemma 2.1, let ψ be a function in $C_0^{\infty}(V)$ with $\psi(x) = 1$ for x in K. Then, by Lemmas 3.22 and 3.10, $\hat{f}\circ\varphi_1^{-1} = \psi(\hat{f}\circ\varphi_1^{-1}) = \lim_{m\to\infty}\psi h_m$ in the norm of $H^{(p)}(V)$, so that in case (a) we have shown that $\hat{f}\circ\varphi_1^{-1}$ is the limit in the norm of $H^{(p)}(V)$ of a sequence of functions in $C_0^{\infty}(V)$. Thus, in case (a), the present lemma follows immediately from Lemma 3.48.

Next consider case (b). Using Lemma 3.12, extend $\hat{f}\circ\varphi_1^{-1}$ to an element F of $H^{(p)}(C_+)$ whose carrier K is the same as the carrier of $\hat{f}\circ\varphi_1^{-1}$; here as below, C_+ denotes the cylinder

$$C_+ = \{x\in E^n \mid 0 < x_1 < \infty,\ |x_j| < \pi, \quad j = 2, \ldots, n\}.$$

For each ε such that $0 < \varepsilon < \pi$, let τ_ε denote the mapping of E^n into itself defined by the formula

$$\tau_\varepsilon[x_1, \ldots, x_n] = [x_1 - \varepsilon, x_2, x_3, \ldots, x_n].$$

By Lemmas 3.47 and 3.9, and by Theorem IV.8.20,

$$\lim_{\varepsilon\to 0} |\partial^J\{(F\circ\tau_\varepsilon^{-1})|C_+\} - \partial^J F| = 0, \qquad |J| \leq p.$$

Thus, by Definition 3.15,

$$\text{(i)} \qquad \lim_{\varepsilon\to 0} |\{(F\circ\tau_\varepsilon^{-1})|C_+\} - F|_{(p)} = 0.$$

By Lemma 3.48, $F\circ\tau_\varepsilon^{-1}$ is an element of $H^{(p)}(C_\varepsilon)$, where

$$C_\varepsilon = \{x\in E^n \mid \quad -\varepsilon < x_1 < \infty, \quad |x_j| < \pi, \quad j = 2, \ldots, n\}.$$

By Lemma 3.46, the carrier K_ε of $F\circ\tau_\varepsilon^{-1}$ is $K_\varepsilon = \{x - [\varepsilon, 0, \ldots, 0] | x\in K\}$. Using Lemma 2.1, let φ_ε be a function in $C_0^{\infty}(C_\varepsilon)$ with $\varphi_\varepsilon(x) = 1$ for x in a neighborhood of K, and let $F_\varepsilon = \varphi_\varepsilon(F\circ\tau_\varepsilon^{-1})$. Then by Lemma 3.13 F_ε has a compact carrier which is a subset of K_ε. Let L be a set of the form

$$L = \{x\in E^n \mid \quad a < x_1 < b, \quad |x_j| < \pi, \quad j = 2, \ldots, n\}$$

including the sets K_ε for all ε satisfying $0 < \varepsilon < 1$, a condition on ε which we shall continue to impose throughout the following. Using Lemma 3.33, extend F_ε to an element $\tilde{F}_\varepsilon$ of $H_\pi^{(p)}(L)$ whose carrier is the same as the carrier of F_ε. By Lemma 3.43, $\tilde{F}_\varepsilon$ is the limit in norm of $H_\pi^{(p)}(L)$ of a sequence of functions in $C_\pi^{\infty}(L)$. Consequently,

by Lemma 3.22, $\varphi_\varepsilon \hat{F}_\varepsilon$ is the limit in the norm of $H^{(p)}(L)$ of a sequence $\{\hat{g}_j\}$ of functions in $C_0^\infty(L)$. Putting $\hat{g}_j(x) \quad 0$ for x in $C_\varepsilon \quad L$, it follows from Definition 3.15 that $\varphi_\varepsilon F_\varepsilon$ is the limit in the norm of $H^{(p)}(C_\varepsilon)$ of the sequence $\{\hat{g}_j\}$ of elements of $C_0^\infty(C_\varepsilon)$. Hence, by Lemma 3.28, $(\varphi_\varepsilon F_\varepsilon)|C_+ \quad \varphi_\varepsilon^2(F\circ\tau_\varepsilon^{-1})|C_+$ is the limit in the norm of $H^{(p)}(C_+)$ of the sequence $\{g_j|C_+\}$ of functions. It then follows from (i) that F is also the limit in the norm of $C^\infty(C_+)$ of a sequence $\{h_j\}$ of functions in $C_0^\infty(E^n)$. Let ψ be a function in $C_0^\infty(E^n)$ which vanishes outside V, and which is identically equal to one in a neighborhood of the carrier K of F. Then, by Lemma 3.22, $\tilde{f}\circ\varphi_1^{-1}$ is the limit in the norm of $H^{(p)}(V_+)$ of the sequence of functions $\tilde{g}_j = \psi h_j|V_+$. Applying Lemma 3.48 we have a complete proof of case (b) of the present lemma. Q.E.D.

5. Some Geometric Considerations

Let I be a domain in E^n whose boundary B contains a part Σ which is a smooth surface. Suppose that no point of Σ is interior to the closure of I. For a function f in $C(\bar{I})$, the condition that f vanish on Σ is typical of the ordinary sort of boundary conditions imposed on functions in the theory of boundary-value problems. We wish in the present short section to indicate the way in which corresponding notions involving derivatives of higher order may be introduced, and to discuss a few elementary properties of these notions.

1 Definition. Let I be a domain in E^n whose boundary B contains a part Σ which is a smooth surface. Suppose that no point in Σ is interior to the closure of I. Let k be a positive integer. Then, if f is in $C^{k-1}(\bar{I})$ and $\partial^J f(x)$ vanishes for all x in Σ and all J with $|J| \leq k-1$, we will say that *f satisfies a Dirichlet condition of order k on Σ*, or that *f and its first $k \quad 1$ normal derivatives vanish on Σ*, and write

$$(\partial_\nu(\Sigma))^j f(x) \quad 0, \qquad x\in\Sigma, \qquad 0 \leq j < k \quad 1.$$

Remark. The subscript ν in the preceding formula indicates "normal." The sense in which the condition of Definition 1 may be regarded as a condition on directional derivatives of f taken in directions normal to Σ will be indicated below.

2 Lemma. *Let I be a domain in E^n whose boundary B contains a*

part Σ which is a smooth surface. Suppose that no point in Σ is interior to the closure of I. Let k be a positive integer, and let f be in $C^{k-1}(\bar{I})$. Then

(i) *if I_0 is a subdomain of I whose boundary contains Σ, then the conditions*

$$(\partial_\nu(\Sigma))^j f(x) = 0, \qquad x \in \Sigma, \qquad 0 \leq j \leq k-1,$$

and

$$(\partial_\nu(\Sigma))^j (f|I_0)(x) = 0, \qquad x \in \Sigma, \qquad 0 \leq j \leq k-1,$$

are equivalent;

(ii) *if I_0 is a subdomain of I whose boundary contains a smooth surface Σ_0 which forms part of Σ, then*

$$(\partial_\nu(\Sigma))^j f(x) = 0, \qquad x \in \Sigma, \qquad 0 \leq j \leq k-1,$$

implies

$$(\partial_\nu(\Sigma_0))^j (f|I_0)(x) = 0, \qquad x \in \Sigma_0, \qquad 0 \leq j \leq k-1;$$

(iii) *if for each α in a family A of indices α, I_α is a subdomain of I, if the boundary of I_α contains a smooth surface Σ_α, and if $\bigcup_\alpha \Sigma_\alpha = \Sigma$, then the condition*

$$(\partial_\nu(\Sigma))^j f(x) = 0, \qquad x \in \Sigma, \qquad 0 \leq j \leq k-1,$$

is equivalent to the family of conditions

$$(\partial_\nu(\Sigma_\alpha))^j (f|I_\alpha)(x) = 0, \qquad x \in \Sigma_\alpha, \qquad 0 \leq j \leq k-1; \qquad \alpha \in A.$$

This lemma follows immediately from Definition 1; the details of its proof are left to the reader to elaborate.

3 Lemma. *Let I be a domain in E^n whose boundary B contains a part Σ which is a smooth surface. Suppose that no point in Σ is interior to the closure of I. Let k and k_1 be positive integers, and let f be in $C^{k+k_1-1}(\bar{I})$ and f_1 be in $C^{k+k_1-1}(\bar{I})$. Then, if*

$$(\partial_\nu(\Sigma))^j f(x) = 0, \qquad x \in \Sigma, \qquad 0 \leq j \leq k-1$$

and

$$(\partial_\nu(\Sigma))^j f_1(x) = 0, \qquad x \in \Sigma, \qquad 0 \leq j \leq k_1-1,$$

it follows that

$$(\partial_\nu(\Sigma))^j (ff_1)(x) = 0, \qquad x \in \Sigma, \qquad 0 \leq j \leq k+k_1-1.$$

Proof. It is clear from Leibniz' rule that we may write

$$\text{(i)} \qquad \partial^J(ff_1)(x) = \sum_{|J_0|+|J_1|\le |J|} c_{J_0,J_1,J}\,\partial^{J_0}f(x)\partial^{J_1}f_1(x)$$

for each J such that $|J| \le k_1+k_2-1$ and for each x in I, in terms of certain constant coefficients $c_{J_0,J_1,J}$. Since both sides of this equation are continuous in $\bar{I}$, the equation (i) must hold for all x in $\bar{I}$. Since by hypothesis

$$\partial^{J_0}f(x)\partial^{J_1}f_1(x) = 0, \qquad x \in \Sigma, \qquad |J_0|+|J_1| \le k+k_1-1,$$

the conclusion follows. Q.E.D.

4 Lemma. *Let I, I_1 be two domains in E^n whose boundaries B, B_1 contain parts Σ, Σ_1 which are smooth surfaces. Suppose that no point in Σ (in Σ_1) is interior to the closure of I (of I_1). Let k be a positive integer, and let f be in $C^{k-1}(\bar{I})$. Let $M : I \to I_1$ be a one-to-one mapping such that*

$$\text{(i)} \quad (M(\cdot))_j \in C^{k-1}(\bar{I}), \qquad j = 1, \ldots, n,$$

$$\text{(ii)} \quad (M^{-1}(\cdot))_j \in C^{k-1}(\bar{I}_1), \qquad j = 1, \ldots, n.$$

Then the conditions

$$(1) \qquad (\partial_\nu(\Sigma))^j f(x) = 0, \qquad x \in \Sigma, \qquad 0 \le j \le k-1,$$

and

$$(2) \qquad (\partial_\nu(\Sigma_1))^j f(M^{-1}(x)) = 0, \qquad x \in \Sigma, \qquad 0 \le j \le k-1,$$

are equivalent.

Proof. Using the chain rule for partial differentiation and Leibniz' rule, it follows that we may write

$$\partial^J f(M^{-1}(x)) = \sum_{|J_1|\le|J|} a_{J_1,J}(x)(\partial^{J_1}f)(M^{-1}(x))$$

for each J such that $|J| \le k-1$ and each x in I, in terms of certain coefficients $a_{J_1,J}(x)$ which may be expressed as polynomials in the coordinates of the vector $M^{-1}(x)$ and the first $k-1$ derivatives of these coordinates. Since both sides of this equation are continuous in $\bar{I}$, (1) must hold for all x in $\bar{I}$. Since by hypothesis $\partial^{J_1}f(x) = 0$ for x in Σ and $|J_1| \le k-1$, it follows that $\partial^J f(x) = 0$ for all x in Σ and $|J| \le k-1$. Thus (1) implies (2). By symmetry, (2) implies (1). Q.E.D.

Lemma 4 establishes the invariance under suitable differentiable

mappings of a Dirichlet condition of order k. The following lemma exhibits the form that such a Dirichlet condition takes on in the neighborhood of a plane boundary of a domain I, and shows why the symbolic notation of "normal derivatives" introduced in Definition 1 is appropriate.

5 LEMMA. *Let I be a domain in E^n whose boundary B contains an open part Σ of the hyperplane $E^{n-1} = \{x \mid x \in E^n, x_1 = 0\}$. Suppose that no point in Σ is interior to the closure of I. Let k be a positive integer, and let f be in $C^{k-1}(\bar{I})$. Then the conditions*

$$\text{(i)} \qquad (\partial_\nu(\Sigma))^j f(x) = 0, \qquad x \in \Sigma, \qquad 0 \leqq j < k-1,$$

and

$$\text{(ii)} \qquad \partial_1^j f(x) = 0, \qquad x \in \Sigma, \qquad 0 < j \leqq k-1,$$

are equivalent.

PROOF. That (i) implies (ii) follows immediately from Definition 1. Conversely, let (ii) be satisfied. Let $|J| < k-1$, and write $J = J_1 \cup J_2$, where $\max J_1 = \min J_1 = 1$ and where $2 \leqq \min J_2 \leqq \max J_2 \leqq n$. Then if $|J_1| = j_1$, $|J_2| = j_2$, $\partial^J f(x) = \partial^{J_2} \partial_1^{j_1} f(x)$ for all x in I. Since both sides of this equation are continuous in $\bar{I}$, it must also hold for x in $\bar{I}$. The set Σ forms an open subset of the plane E^{n-1}. By hypothesis, $g(x) = \partial_1^{j_1} f(x)$ vanishes for x in Σ. Thus, what we have to show is that $\partial^{J_2} g(x)$, which is defined for x in Σ (cf. the sixth paragraph of Section 2) as the extension by continuity of $\partial^{J_2} g(x)$ from I to $\bar{I}$, is identical for each x in Σ with the derivative $\partial^{J_2}(g|\Sigma)(x)$ of the restriction of g to Σ, the derivative being taken in the open set Σ of E^{n-1}. Since $j_1 + j_2 \leqq k-1$, we have $g \in C^{k-1-j_1}(\bar{I}) \subseteq C^{j_2-1}(\bar{I})$. Let x_0 be in Σ, and let U be a neighborhood in E^{n-1} of x_0 so small that $\bar{U}$ is compact and included in Σ. Since Σ is not a subset of the closure of I, all the points of I in the immediate neighborhood of x must lie in one of the half-spaces

$$E_+ = \{x \mid x \in E^n, x_1 > 0\}$$

or $E_- = \{x \mid x \in E^n, x_1 < 0\}$. For definiteness, suppose the former to be the case. Then, if ε is sufficiently small, the set

$$U_\varepsilon = \{x \mid x \in E^n, 0 < x_1 < \varepsilon, [x_2, \ldots, x_n] \in U\}$$

is contained in I. Since g is in $C^{j_2-1}(\bar{I})$, then

$$\lim_{\delta\to 0} \partial^{J_3} g(\delta, x_2, \ldots, x_n)$$

exists uniformly for $[x_2, \ldots, x_n]$ in U_1, and for all J_3 such that $2 < \min J_3 < \max J_3 \leq n$, $|J_3| < j_2-1$. Consequently we have

$$\begin{aligned}\partial^{J_3} g(0, x_2, \ldots, x_n) &= \partial^{J_3} \lim_{\delta\to 0} g(\delta, x_2, \ldots, x_n) \\ &= \lim_{\delta\to 0} \partial^{J_3} g(\delta, x_2, \ldots, x_n)\end{aligned}$$

by a well-known elementary theorem on the interchange of limits and derivatives, proving the present lemma. Q.E.D.

6. The Elliptic Boundary Value Problem

Can the boundary value theory and the spectral theory of Chapter XIII be generalized to partial differential operators? In the present section it will be seen that it can, at least for the class of elliptic partial differential operators to be defined below. A crucial theorem in the development of the theory of Chapter XIII was Theorem XIII.2.10, which was based on Lemma XIII.2.9; what is most essential is to find a generalization of this theorem and lemma to partial differential operators. If Theorem XIII.2.10 and Lemma XIII.2.9 are viewed in the new distribution-theoretic terminology, it becomes apparent that the problem is to find a class of formal partial differential operators τ for which the distribution-theoretic equation $\tau f = g$ implies that f is smoother than g in a suitable sense. We have seen in the introduction to the present chapter that no such principle can possibly be valid for a formal partial differential operator for which the Cauchy problem can be solved, e.g., for a hyperbolic partial differential operator such as

$$\left(\frac{\partial}{\partial x_1}\right)^2 - \left(\frac{\partial}{\partial x_2}\right)^2.$$

(It will be shown in Section 7 that the Cauchy problem for this particular operator can be solved.) However, for elliptic operators, as defined in Definition 1, to follow, a generalization of Theorem XIII.2.10 will be given. This will enable us to develop generalized forms of some of the more elementary parts of the theory presented

in Chapter XIII. Thus, the descendants in the present chapter of the closely related Theorems XIII.1.8 and XIII.2.10 are far apart.

After these few remarks in the way of a preliminary orientation, let us proceed to the fundamental definition of this chapter.

1 Definition. Let

$$\tau = \sum_{|J| \leq p} a_J(x)\partial^J$$

be a formal partial differential operator of order p defined in a domain I of Euclidean n-space E^n. Then, if for each non-zero vector ξ in E^n, we have

$$\sum_{|J|=p} a_J(x)\xi^J \neq 0, \qquad x \in I,$$

the operator τ is said to be *elliptic*.

Thus, the requirement of ellipticity for a partial differential operator is the analogue of the condition that the leading coefficient should be non-vanishing, which was imposed on formal ordinary differential operators in Chapter XIII. The proof of Theorem 2 below will make the utility of the requirement of ellipticity evident.

→ 2 Theorem. *Let τ be an elliptic formal partial differential operator of order p in a domain I of n-space. Let f and g be distributions in I, and suppose that g is in $A^{(m)}(I)$ and that $\tau f = g$. Then f is in $A^{(m+p)}(I)$.*

Theorem 2 will be deduced from the following lemma.

3 Lemma. *Suppose that the hypotheses of the preceding theorem are satisfied, that $j \leq m+p-1$, and that f is in $A^{(j)}(I)$. Then f is in $A^{(j+1)}(I)$.*

Proof (of Lemma 3). Our idea is to regard τ as the sum of two operators, the operator $\tau_0 = \sum_{|J|=p} a_J(0)\partial^J$ and a residual operator which is relatively small, at least if considered in a small neighborhood of the point 0. We examine a small neighborhood of 0 by making a drastic "shrinkage" of coordinates. The technical development of these two simple ideas is as follows.

By Lemma 3.24 it suffices to show that each point q in I has a neighborhood U such that the restriction $f|U$ of f to U belongs to $A^{(j+1)}(U)$. Shifting coordinates in E^n, (cf. Lemmas 3.47 and 3.48), we may clearly assume without loss of generality that $q = 0$. For

each $\varepsilon > 0$, let S_ε be the map of E^n into itself defined by the equation $S_\varepsilon x = \varepsilon x$. It follows from Lemma 3.47 that $f \circ S_\varepsilon^{-1}$ is a solution of the partial differential equation

$$\tau_\varepsilon(f \circ S_\varepsilon^{-1}) = \sum_{|J| \leq p} a_J(\varepsilon x)\varepsilon^{p-|J|}\partial^J(f \circ S_\varepsilon^{-1}) = \varepsilon^p(g \circ S_\varepsilon^{-1}), \tag{1}$$

in the domain $\varepsilon^{-1}I$. Let ε be so small that the domain $\varepsilon^{-1}I$ contains the interior of the unit sphere Σ_1 in E^n, where we write $\Sigma_a = \{x \in E^n \mid |x| < a\}$.

Let φ be a function in $C^\infty(E^n)$ which vanishes outside $\Sigma_{1/2}$ and which is identically equal to one in $\Sigma_{1/4}$. It follows from Leibniz' rule that $\tau_\varepsilon\varphi = \varphi\tau_\varepsilon + \tau(\varepsilon)\varphi$, where $\tau(\varepsilon)$ is a partial differential operator of order at most $p-1$.

It follows from (1), from Lemmas 3.22, 3.18 and 3.6(iv), and on placing $(f \circ S_\varepsilon^{-1}) = f_\varepsilon$, that the distribution $f_\varepsilon\varphi$ satisfies the partial differential equation

$$\sum_{|J| = p} a_J(\varepsilon x)\partial^J(f_\varepsilon\varphi) = \tilde{g}_\varepsilon, \tag{2}$$

where $\tilde{g}_\varepsilon$ is in $A^{(m)}(I)$; here Lemma 3.6(iv) permits the use of the Leibniz formula to show that $\partial^J(f_\varepsilon\varphi) = \varphi\partial^J f_\varepsilon + g(J, \varepsilon)$ for $|J| \leq p$, where, by Lemmas 3.22 and 3.18, $g(J, \varepsilon) \in A^{(m)}(I)$. Suppose that we let τ_0 be the partial differential operator

$$\tau_0 = \sum_{|J| = p} a_J(0)\partial^J. \tag{3}$$

(The operator τ_0 may be regarded as the "principal part of τ at $x = 0$".) If we set

$$\tilde{a}_J(x) = a_J(x) - a_J(0),$$

then, as ε approaches zero, the functions $a_J(\varepsilon x)$ converge to zero in the topology of $C^\infty(\Sigma_1)$. By (2), $f_\varepsilon\varphi$ satisfies a partial differential equation of the form

$$(\tau_0 + \sum_{|J| \leq p} \tilde{a}_J(\varepsilon x)\partial^J)(f_\varepsilon\varphi) = \tilde{g}_\varepsilon, \tag{4}$$

where all the coefficients $\tilde{a}_J(\varepsilon x)$ converge to zero in the topology of $C^\infty(\Sigma_1)$ as ε approaches 0. Since f is in $A^{(m+p-1)}(I)$ by assumption, it follows from (4) that $f_\varepsilon\varphi$ satisfies the partial differential equation

$$\left((\tau_0 + \lambda) + \sum_{|J| \leq p} \tilde{a}_J(\varepsilon x)\partial^J\right)(f_\varepsilon\varphi) = \hat{g}_\varepsilon, \tag{5}$$

where λ is a complex constant entirely at our disposal which will be chosen below and where $\hat{g}_\varepsilon$ is a distribution in $A^{(m)}(\Sigma_1)$. By Lemma 3.13(ii), the carrier of $\hat{g}_\varepsilon$ is contained entirely in $\Sigma_{1/2}$. Let C denote the cube

$$C = \{x \in E^n \,|\, |x_i| < \pi,\ i = 1, \ldots, n\}.$$

Then, by Lemmas 3.12, 3.33, and 3.24, as generalized to $D_\pi(C)$, $f_\varepsilon \varphi \in H_\pi^{(m+p-1)}(C)$, $\hat{g}_\varepsilon \in H_\pi^{(m)}(C)$. Let $\psi \in C^\infty(E^n)$ be a function such that $\psi(x) = 1$ if $x \in \Sigma_{1/2}$, and $\psi(x) = 0$ if $x \in \Sigma_{3/4}$. Then, placing $\hat{a}_J(\varepsilon x) = \tilde{a}_J(\varepsilon x)\psi(x)$, the functions $\hat{a}_J(\varepsilon x)$ are defined in C, belong to $C_\pi^\infty(C)$, and converge to zero in the topology of $C_\pi^\infty(C)$ as ε approaches 0.

Put

$$\sigma_\varepsilon = \sum_{|J|=p} \hat{a}_J(\varepsilon x)\partial^J,$$

and

$$\tau_1 = \tau_0 + \lambda.$$

Then, since $\hat{a}_J(\varepsilon x) = \tilde{a}_J(\varepsilon x)$ for $x \in \Sigma_{1/2}$, and since by Lemma 3.13 the carrier of $f_\varepsilon \varphi$ is contained in $\Sigma_{1/2}$, it follows from (5) and from Lemmas 3.10 and 3.9 that

$$(\tau_1 + \sigma_\varepsilon)(f_\varepsilon \varphi) = \hat{g}_\varepsilon. \tag{6}$$

Let

$$H \sim \sum_{|L|=n} H_L e^{iL \cdot x} \tag{7}$$

be the Fourier series expansion of a distribution $H \in D_\pi(C)$ (cf. Lemma 3.39). By Lemma 3.40 and (3),

$$(\tau_0 + \lambda)H \sim \sum_{|L|=n} H_L \Big(\sum_{|J|=p} a_J(0) L^J + \lambda \Big) e^{iL \cdot x}. \tag{8}$$

The complex constant λ was left undetermined above. We now choose λ so that $\{\sum_{|J|=p} a_J(0) L^J\} + \lambda \neq 0$ for each index L with $|L| = n$; since λ can be chosen arbitrarily from an *uncountable* set, this offers no difficulty. The functions

$$f_1(\xi) = |\xi|^p, \qquad f_2(\xi) = \sum_{|J|=p} a_J(0)\xi^J$$

are both homogeneous of order p, and are non-vanishing on the surface of the unit sphere Σ_1 in E^n (here the ellipticity of τ is used for the first and last time, but in a critical way). Consequently, there exist two positive constants K_1 and K_2 such that

$$K_1 \leqq \frac{f_1(\xi)}{|f_2(\xi)|} \leqq K_2, \qquad \xi \in E^n.$$

Thus

$$K_1 \leqq \liminf_{|\xi|\to\infty} \frac{f_1(\xi)+1}{|f_2(\xi)+\lambda|}$$

$$\leqq \limsup_{|\xi|\to\infty} \frac{f_1(\xi)+1}{|f_2(\xi)+\lambda|} \leqq K_2.$$

Since λ has been chosen so that $f_2(L)+\lambda \neq 0$ for each index L with $|L| = n$, it follows immediately that the ratio of $|\xi|^p+1$ to $f_2(\xi)+\lambda$ is bounded between two positive constants as ξ ranges over the set of all vectors in E^n with *integral* coordinates. Hence, by Lemma 3.41, $\tau_0+\lambda$ is a continuous mapping with a continuous inverse of $H_\pi^{(k+p)}(C)$ onto $H_\pi^{(k)}(C)$, for all k between $-\infty$ and $+\infty$. Let ν_k and $\hat{\nu}_k$ be the norms of the map

$$\tau_0+\lambda : H_\pi^{(k+p)}(C) \to H_\pi^{(k)}(C)$$

and of its inverse, respectively.

Let $\tau_1 = \tau_0+\lambda$. Using the fact, established above, that all the functions $\hat{a}_J(\varepsilon x)$ converge to zero in the topology of $C_\pi^\infty(C)$ as ε approaches 0, and also using Lemma 3.28(iii), as generalized to $D_\pi(C)$, let ε be so small that if σ_ε denotes the formal partial differential operator

$$\sigma_\varepsilon = \sum_{|J|=p} \hat{a}_J(\varepsilon x)\partial^J,$$

then the norm of σ_ε, both as a mapping of $H_\pi^{(m+p)}(C)$ into $H_\pi^{(m)}(C)$, and of $H_\pi^{(m+p-1)}(C)$ into $H_\pi^{(m-1)}(C)$, is less than min $(\hat{\nu}_m, \hat{\nu}_{m-1})$. Then, by Lemma VII.3.4, the mapping

$$(I+\sigma_\varepsilon\tau_1^{-1}),$$

regarded either as a mapping of $H_\pi^{(m)}(C)$ or of $H_\pi^{(m-1)}(C)$ into itself, has a bounded everywhere-defined inverse. We then have

$$(\tau_1+\sigma_\varepsilon)\tau_1^{-1}(I+\sigma_\varepsilon\tau_1^{-1})^{-1} = (I+\sigma_\varepsilon\tau_1^{-1})(I+\sigma_\varepsilon\tau_1^{-1})^{-1} = I$$

and

$$\tau_1^{-1}(I+\sigma_\varepsilon\tau_1^{-1})^{-1}(\tau_1+\sigma_\varepsilon) = \tau_1^{-1}(I+\sigma_\varepsilon\tau_1^{-1})^{-1}(I+\sigma_\varepsilon\tau_1^{-1})\tau_1 = \tau_1^{-1}\tau_1 = I,$$

whether τ_1 and σ_ε are regarded as mappings of $H_\pi^{(m+p)}$ into $H_\pi^{(m)}$ or as

mappings of $H_n^{(m+p-1)}$ into $H_n^{(m-1)}(C)$. Thus $\tau_p+\sigma_\varepsilon$ is a one-to-one mapping both of $H_n^{(m+p)}$ into $H_n^{(m)}$ and of $H_n^{(m+p-1)}$ into $H_n^{(m-1)}$. Since $\tilde{g}_\varepsilon$ is in $H_n^{(m)}$, it follows that there exists some F in $H_n^{(m+p)}$ such that $(\tau_1+\sigma_\varepsilon)F=\tilde{g}_\varepsilon$. However, since $f_\varepsilon\varphi$ is in $H_n^{(m+p-1)}$, and since by (5), $(\tau_1+\sigma_\varepsilon)f_\varepsilon\varphi=\tilde{g}_\varepsilon$, it follows that $f_\varepsilon\varphi = F$ is in $H_n^{(m+p)}(C)$ so that *a fortiori*, $f_\varepsilon\varphi$ is in $A_n^{(m+p)}(C)$. Hence, since $\varphi(x)=1$ for x in $\Sigma_{1/4}$, it follows from Lemmas 3.9 and 3.23 that the restriction $f_\varepsilon|\Sigma_{1/4}$ belongs to $A^{(m+p)}(\Sigma_{1/4})$. Thus (cf. 3.48) $f|\Sigma_{\varepsilon/4}$ belongs to $A^{(m+p)}(\Sigma_{\varepsilon/4})$, and the proof of Lemma 3 is complete. Q.E.D.

PROOF (OF THEOREM 2). Let J be a domain whose closure is contained in I. Then, by Lemmas 4.13 and 3.13 and by Definition 3.15, $f|J$ is in $A^{(n)}(J)$ for some n. Since by Lemma 3.18 and Lemma 3.23 $g|J$ is in $A^{(s)}(J)$ for each $s\leq m$, and since by Lemma 3.10, $\tau(f|J)=(g|J)$ by Lemma 3.9, it follows from Lemma 3 that if $n=s+p-1$ and $s\leq m$, then $f|J$ is in $A^{(n+1)}(J)$. But, as long as $n<m+p$, we may always find an integer s such that $s\leq m$ and $n<s+p-1$. Thus, if $f|J$ is in $A^{(n)}(J)$ and $n<m+p$, $f|J$ is in $A^{(n+1)}(J)$. It follows by induction that $f|J$ is in $A^{(m+p)}(J)$. Since J is an arbitrary open set whose closure is contained in I, we have f in $A^{(m+p)}(I)$ by Definitions 3.15 and 3.17. Q.E.D.

4 COROLLARY. *Let the hypotheses of Theorem 2 be satisfied, and let $m+p-[n/2]-1\geq 0$. Then f is a function belonging to $C^{m+p-[n/2]-1}(I)$. The mapping $(f, T_1(\tau)f)\to f(x_0)$ is a continuous linear functional defined on the graph of $T_1(\tau)$ for each x_0 in I. In particular, if $g=0$, so that f is a solution of $\tau f=0$, then f is infinitely often differentiable, i.e., f is in $C^\infty(I)$.*

PROOF. This follows immediately from Theorem 2, Sobolev's theorem (4.5), and the closed graph theorem (II.2.4) Q.E.D.

5 COROLLARY. *Let the hypotheses of Theorem 2 be satisfied, and suppose that $p\geq[n/2]+1$. Then every function f in $\mathfrak{D}(T_1(\tau))$ is in $C^{p-[n/2]-1}(I)$.*

PROOF. This follows immediately from the preceding corollary and the remark immediately following the proof of Lemma 3.53. Q.E.D.

As a first application of the fundamental Theorem 2 and Corollary 5, we prove the following theorem of Mautner, Gårding,

and Browder, which generalizes Theorem XIII.5.1 to the case of a partial differential operator.

6 THEOREM. *Let τ be an elliptic formally self adjoint formal partial differential operator in a domain I of Euclidean n space, and let T be a self adjoint extension of $T_0(\tau)$. Let U be an ordered representation of $L_2(I)$ relative to T, with measure μ, multiplicity sets e_i, and multiplicity $m \leq \infty$. Then there exist kernels $W_i(t, \lambda)$, $1 \leq i < m$, measurable with respect to the product of Lebesgue measure and μ, which vanish on the complement of e_i, belong to $C^\infty(I)$, and satisfy the differential equation $(\tau - \lambda)W_i(\cdot, \lambda) = 0$ for each fixed λ. Moreover, the kernels W_i have the property that*

$$\nu\text{-ess}\sup_{t \in J} \int_e |W_i(t, \lambda)|^2 \mu(d\lambda) < \infty$$

for each compact subset J of the interior of I and each bounded Borel set e, and are such that

$$\text{(i)} \quad (Uf)_i(\lambda) = \int_I f(t)\overline{W_i(t, \lambda)}dt, \qquad f \in L_2(I), \qquad 1 \leq i \leq m,$$

the integral existing in the mean square sense in $L_2(\mu, e_i)$.

(ii) *For each Borel function F,*

$$U\mathfrak{D}(F(T)) = \left\{[f_i] \,\middle|\, \sum_{i=1}^{m} \int_{-\infty}^{+\infty} |F(\lambda)|^2 |f_i(\lambda)|^2 \mu(d\lambda) < \infty \right\}$$

and

$$[UF(T)g]_i(\lambda) = F(\lambda)(Ug)_i(\lambda), \qquad g \in \mathfrak{D}(F(T)), \qquad -\infty < \lambda < +\infty.$$

PROOF. Since $T = T^* \supseteq T_0(\tau)$, we have $T = T^* \subseteq T_0(\tau^*)^* = T_0(\tau^*)^*$ by Lemma XII.4.1(a), $\tau = \tau^*$ being formally self adjoint by hypothesis. Since, as observed in the paragraph immediately following the proof of Lemma 3.53, $T_0(\tau^*)^* = T_1(\tau)$, it follows that $T \subseteq T_1(\tau)$. If

$$\text{(1)} \qquad f \in \bigcap_{m=1}^{\infty} \mathfrak{D}(T^m) \subseteq \bigcap_{m=1}^{\infty} \mathfrak{D}((T_1(\tau))^m) \subseteq \bigcap_{n=1}^{\infty} \mathfrak{D}(T_1(\tau^n)),$$

(the last inclusion following by the remark in the paragraph of Section 3 cited above) then, from Corollary 5, f is in $C^\infty(I)$. Hence, by Corollaries XII.3.13 and XII.3.14, we find that there exists a μ-null set N and

kernels W_i, $i = 1, \ldots, m \leq \infty$, satisfying formulae (i) and (ii), such that for λ in $e_j - N$, $(T_1(\tau) - \lambda)W_j(\cdot, \lambda) = 0$. It follows from Corollary 4 that if we put $W_j(\cdot, \lambda) = 0$ for $\lambda \in N$ and modify $W_j(t, \lambda)$ on a suitable Lebesgue null set for each λ in N, we obtain a function $\tilde{W}_i$ such that $\tilde{W}_j(\cdot, \lambda)$ is in $C^\infty(I)$ for all λ, and such that

$$\tau\tilde{W}_j(\cdot, \lambda) = \lambda\tilde{W}_j(\cdot, \lambda), \qquad t \in I, \qquad \text{for all } \lambda.$$

If $\tilde{W}_i$ is measurable with respect to the product of μ and Lebesgue measure, then, since the functions W of Corollary XII.3.13 and XII.3.14 are only defined up to arbitrary changes on a null set, it will follow from Fubini's theorem that we may take $W_j = \tilde{W}_j$. To see that $\tilde{W}_j$ is measurable in this sense, note the evident fact that if C_m denotes the cube

$$\left\{x \in E^n \,\middle|\, |x_i| \leq \frac{1}{m}, \qquad i = 1, \ldots, n\right\}$$

in Euclidean n-space, then

$$\begin{aligned}\tilde{W}_i(t, \lambda) &= \lim_{m\to\infty} (2m)^{-n} \int_{t+C_m} \tilde{W}_i(s, \lambda)ds \\ &= \lim_{m\to\infty} (2m)^{-n} \int_{t+C_m} W_i(s, \lambda)ds\end{aligned}$$

for every t interior to I and $\lambda \notin N$. Q.E.D.

7 COROLLARY. (Inversion Formula) *Let I, W_i, etc., be as in the preceding theorem. Then, for each f in $L_2(I)$, we have*

$$f(t) = \sum_{i=1}^{m} \lim_{A\to\infty} \int_{-A}^{+A} (Uf)_i(\lambda)W_i(t, \lambda)\mu(d\lambda),$$

the limits existing in the mean square sense in $L_2(I)$, and the series converging in the norm of $L_2(I)$.

PROOF. The corollary follows from the preceding theorem and from Corollary XII.3.12. Q.E.D.

8 COROLLARY. *Let the operator T, the kernels W_i, etc., be as in Theorem 6, and let F be a bounded Borel measurable function vanishing outside a bounded Borel set e of the real axis. Then the bounded operator $F(T)$ has the representation*

$$(F(T)f)(t) = \int_I f(s)K(F; t, s)ds, \qquad f \in L_2(I),$$

where

$$K(F; t, s) = \sum_{i=1}^{m} \int_e F(\lambda)W_i(t, \lambda)\overline{W_i(s, \lambda)}\mu(d\lambda),$$

the series converging in $L_2(I)$ for almost all fixed t in I. Moreover,

$$\operatorname*{ess\,sup}_{t \in J} \int_I |K(F; t, s)|^2 ds < \infty,$$

where J is any compact subset of I.

Proof. It is clear from Theorem XII.2.6 and from formula (1) of the proof of Theorem 6 that if f is in $L_2(I)$, then $F(T)f$ is in $\bigcap_{n=1}^{\infty} \mathfrak{D}(T^n) \subseteq \bigcap_{n=1}^{\infty} \mathfrak{D}(T_1(\tau^n))$. Thus it follows from Corollary 5 that $F(T)f$ is in $C^\infty(I)$. The map $f \to F(T)f$ of $L_2(I)$ into the F-space $C^\infty(I)$ is evidently closed. Hence, by the closed graph theorem (II.2.4) it is continuous. Thus, there is a constant $M(J)$ such that

$$\text{[†]} \qquad \sup_{t \in J} |(F(T)f)(t)| \leq M(J)|f|, \qquad f \in L_2(I).$$

It follows from Theorem 6(ii) and from the preceding corollary that

$$\text{[*]} \qquad (F(T)f)(t) = \sum_{i=1}^{m} \int_e F(\lambda)W_i(t, \lambda) \int_I W_i(s, \lambda)f(s)ds\,\mu_i(d\lambda),$$

where the integral $\int_e f(s)W_i(s, \lambda)ds$ exists in the mean square sense in $L_2(\mu)$ and the series converges in the norm of $L_2(I)$. Let H_0 denote the dense set of all those f in $L_2(I)$ each of which vanishes outside a compact subset of I. First suppose that $m < \infty$. Then the proof may be concluded as follows. If f is in H_0 we may use Fubini's theorem, valid in view of the properties of the kernels W_j stated in Theorem 6, to interchange the order of integrations in [*], obtaining the equation

$$\text{[**]} \qquad (F(T)f)(t) = \int_I f(s)K(F; t, s)ds, \qquad f \in H_0,$$

and hence, using [†], the inequality

$$\sup_{t \in J} \left| \int_I f(s)K(F; t, s)ds \right| \leq M(J)|f|, \qquad f \in H_0.$$

It then follows from Theorem IV.8.1 that

$$\left[\int_I |K(F; t, s)|^2 ds\right]^{1/2} \leqq M(J),$$

and that equation [**] holds for all f in $L_2(I)$, so that the proof is complete if $m < \infty$.

If $m = \infty$, we can complete the proof in exactly the same way once we succeed in showing that the series defining $K(F; t, s)$ converges in the norm of $L_2(I)$ for almost all fixed t in I, and that equation [**] holds. This can be established by the following auxiliary argument. Let $\mathfrak{H} = \sum_{n=1}^{\infty} L_2(\mu, e_i)$ be the direct sum space onto which $L_2(I)$ is mapped by the ordered representation U of Theorem XII.3.11. For each $k < \infty$, let the orthogonal projection P_k in $\mathfrak{H}$ be defined by

$$P_k[f_1, \ldots, f_k, f_{k+1} \ldots] = [f_1, \ldots, f_k, 0, 0, \ldots].$$

Then clearly $P_k u \to u$ as $k \to \infty$ for each u in $\mathfrak{H}$. Since the map $f \to F(T)f$ is a continuous mapping of $L_2(I)$ into $C^{\infty}(I)$, it follows that for each t in I, the formula

$$\varphi_t(f) = (F(T)f)(t)$$

defines a continuous linear functional in $L_2(I)$. Hence, there exists a vector g_t in $L_2(I)$, such that

$$[\dagger] \qquad (F(T)f)(t) = (f, g_t), \qquad f \in L_2(I).$$

We have

$$\begin{aligned}(F(T)U^{-1}P_k Uf)(t) &= (U^{-1}P_k UF(T)f)(t) \\ &= \sum_{j=1}^{k} \int_e F(\lambda) W_i(t, \lambda) \int_I \overline{W_i(s, \lambda)} f(s) ds \mu(d\lambda)\end{aligned}$$

for each f in $L_2(I)$ by Theorem 6(ii) and the preceding corollary. It now follows from the argument used in the previous paragraph for $m < \infty$ that if

$$K_k(F; t, s) = \sum_{i=1}^{k} \int_e F(\lambda) W_i(t, \lambda) \overline{W_i(s, \lambda)} \mu(d\lambda),$$

then

$$\int_I |K_k(F; t, s)|^2 ds < \infty, \qquad t \in I,$$

and

$$[\dagger\dagger] \quad \big(F(T)U^{-1}P_kUf\big)(t) = \int_I K_k(F; t, s)f(s)ds, \qquad f \in L_2(I).$$

Now, we have

$$\begin{aligned}\big(F(T)U^{-1}P_kUf\big)(t) &= (U^{-1}P_kUf, g_t)\\ &= (f, U^{-1}P_kUg_t)\end{aligned}$$

by the definition [†] of g_t and the fact that the mapping U preserves inner products. Thus it follows from [††] that

$$\overline{K_k(F; t, \cdot)} = U^{-1}P_kUg_t.$$

It follows from the definition of the sequence P_k of projections, then, that $K_k(F; t, \cdot)$ converges in the norm of $L_2(I)$, as $k \to \infty$, and that its limit is $\bar{g}_t$. This shows that the series defining the kernel $K(f; t, s)$ converges in the norm of $L_2(I)$ for each fixed t, and that

$$\int_I K(F; t, s)f(s)ds = (f, g_t) = \big(F(T)f\big)(t),$$

which completes the proof. Q.E.D.

We now wish to develop certain portions of the theory of boundary value problems for elliptic partial differential operators. We shall deduce a number of results which have been obtained in this direction. It is only fair to remark, however, that the available theory of boundary values and boundary conditions for elliptic partial differential operators is somewhat fragmentary, and not entirely comparable to the very perfect theory available for ordinary differential operators, which was given in Section XIII.2. We shall in particular give a detailed treatment of the theory of one special but important set of boundary conditions, the so-called "Dirichlet" boundary conditions. For this purpose the fundamental Theorem 2, which pertains only to the interior of a domain I, is ultimately insufficient, and must be supplemented by an investigation of the corresponding differentiability properties in the neighborhood of the boundary of I of distributions satisfying an elliptic partial differential equation

$$\tau f = g,$$

and a corresponding set of boundary conditions. We begin, however, by treating those aspects of the Dirichlet boundary problem which

can be studied without detailed information on the question of differentiability at the boundary.

9 LEMMA. *Let τ be an elliptic formal partial differential operator of even order $2p$ defined in a domain I_0 in E_n, and suppose that*

$$\tau = \sum_{|J| \leq 2p} a_J(x)\partial^J.$$

Let I_0 be the union of two open sets I_1 and I_2, and suppose that there exist constants K_1, k_1; $k_1 > 0$, $K_1 < \infty$, such that

$$\mathscr{R}(\tau f, f) + K_1(f, f) \geqq k_1|f|^2_{(p)}, \qquad f \in C_0^\infty(I_1) \cup C_0^\infty(I_2).$$

Then, if I is a bounded subset of I_0 such that $\bar{I} \subseteq I_0$, there exists a pair K, k of constants with $k > 0$, $K < \infty$ and such that

$$\mathscr{R}(\tau f, f) + K(f, f) \geqq k|f|^2_{(p)}, \qquad f \in C_0^\infty(I).$$

Remark. The norm $|f|_{(p)}$ is defined in Definition 3.15.

PROOF. The compact sets $\bar{I} - I_1 = C_1$ and $\bar{I} - I_2 = C_2$ are disjoint. Let $\tilde{C}_1$ and $\tilde{C}_2$ be disjoint open sets containing C_1 and C_2 respectively. Then $D_1 = \bar{I} - \tilde{C}_1$ and $D_2 = \bar{I} - \tilde{C}_2$ are a pair of compact subsets whose union is $\bar{I}$; moreover, $D_1 \subseteq I_1$ and $D_2 \subseteq I_2$. By Lemma 2.1, there exists a pair of non-negative functions ψ_1 in $C_0^\infty(I_1)$ and ψ_2 in $C_0^\infty(I_2)$, such that $\psi_1(x) = 1$ for x in D_1 and $\psi_2(x) = 1$ for x in D_2. Using Lemma 2.1 again, let η be a non-negative function in $C_0^\infty(E^n)$ which is equal to one for x satisfying the inequality $\psi_1(x) + \psi_2(x) \geqq \frac{3}{4}$, and equal to zero if $\psi_1(x) + \psi_2(x) \leqq \frac{1}{4}$, and put $\psi_3(x) = 1 - \eta(x)$. Put $\psi(x) = \sum_{i=1}^3 (\psi_i(x))^2$, and $\varphi_i(x) = (\psi(x))^{-1}\psi_i(x)$, $i = 1, 2, 3$. Then φ_1 is in $C_0^\infty(I_1)$, φ_2 is in $C_0^\infty(I_2)$, and $(\varphi_1(x))^2 + (\varphi_2(x))^2 = 1$ for all x in $D_1 \cup D_2 \supseteq I$. Let f be in $C_0^\infty(I)$. Then

$$\begin{aligned} \mathscr{R}(\tau f, f) + K_1(f, f) &= \mathscr{R}((\varphi_1^2 + \varphi_2^2)\tau f, f) + K_1((\varphi_1^2 + \varphi_2^2)f, f) \\ &= \mathscr{R}(\varphi_1 \tau f, \varphi_1 f) + K_1(\varphi_1 f, \varphi_1 f) \\ &\quad + \mathscr{R}(\varphi_2 \tau f, \varphi_2 f) + K_1(\varphi_2 f, \varphi_2 f). \end{aligned} \tag{1}$$

By Leibniz' rule we have

$$\varphi_1 \tau f = \tau \varphi_1 f + \tau_1 f, \qquad \varphi_2 \tau f = \tau \varphi_2 f + \tau_2 f,$$

where τ_1 and τ_2 are formal differential operators of order at most

$2p-1$. Thus, putting $f_1 = f\varphi_1$, $f_2 = f\varphi_2$, and $\tau_3 = \tau_1+\tau_2$ it follows from (1) and from our hypothesis that

(2) $$\mathscr{R}(\tau f, f)+K_1(f, f) \geqq k_1|f_1|^2_{(p)} + k_1|f_2|^2_{(p)} + \mathscr{R}(\tau_3 f, f).$$

In the same way, it follows from (1) and Schwarz' inequality that there exists a finite constant $A(\tau)$, depending only on τ, such that

(3) $$\mathscr{R}(\tau f, f) \leqq A(\tau)\{|f_1|^2_{(p)}+|f_2|^2_{(p)}\}+\mathscr{R}(\tau_3 f, f).$$

Define the formal partial differential operator μ of order $2p$ by the equation

(4) $$\mu = \sum_{|J|\leqq p} (-1)^{|J|}\partial^J\partial^J.$$

Then it is clear on integrating by parts (cf. the final paragraph of Section 2) that

$$(\mu f, f) = |f|^2_{(p)}, \qquad f \in C_0^\infty(I).$$

Hence, applying (3) to the operator μ, we find that

(5) $$|f|^2_{(p)} \leqq A(\mu)\{|f_1|^2_{(p)}+|f_2|^2_{(p)}\}+\mathscr{R}(\mu_3 f, f),$$

where μ_3 is a certain partial differential operator of order at most $2p-1$. It follows from (2) and (5) that there exists a finite positive constant k_2 such that

(6) $$k_2|f|^2_p+(\tau_4 f, f) \leqq \mathscr{R}(\tau f, f)+K_1(f, f), \qquad f \in C_0^\infty(I),$$

where τ_4 is a certain partial differential operator defined in I_0 of order at most $2p-1$.

Using induction on $|J_1|$ we can readily show that an identity of the form

$$\partial^{J_1}G(x)\partial^{J_2} = G(x)\partial^{J_1}\partial^{J_2}+\sum_{|J|<|J_1|+|J_2|} C_J(x)\partial^J$$

holds (with suitable coefficients C_J) for every function G in $C^\infty(I_0)$. This makes it evident that the formal partial differential operator τ_4 may be written in the form

$$\tau_4 = \sum_{|J_1|<p,\,|J_2|\leqq p} \partial^{J_1}C_{J_1,J_2}(x)\partial^{J_2}$$

in terms of suitable coefficients C_{J_1,J_2} in $C^\infty(I_0)$. It follows upon integration by parts (cf. the final paragraphs of Section 2) that

$$(\tau_4 f, f) = \sum_{|J_1|<p,\, |J_2|\le p} (-1)^{|J_1|} \int_{I_0} C_{J_1, J_2}(x)\partial^{J_2} f(x)\overline{\partial^{J_1} f(x)}\,dx,$$

for each f in $C_0^\infty(I)$. Hence, it follows immediately from Hölder's inequality that there exists a finite constant M such that

$$|(\tau_4 f, f)| \le M|f|_{p-1}|f|_p, \qquad f \in C_0^\infty(I).$$

From Corollary 4.12 it then follows that for each $\varepsilon > 0$ there exists a finite constant $K(\varepsilon)$ such that

$$|(\tau_4 f, f)| \le \varepsilon|f|_p^2 + K(\varepsilon)|f|^2, \qquad f \in C_0^\infty(I). \tag{7}$$

Choosing $\varepsilon < k_2$, we see that our lemma is now a consequence of (6) and (7). Q.E.D.

10 LEMMA. (Gårding's Inequality) *Let τ be an elliptic operator of even order $2p$ defined in a domain I_0 in E^n. Let I be a bounded open set whose closure is contained in I_0. Let*

$$\tau = \sum_{|J|\le p} a_J(x)\partial^J,$$

and suppose that

$$(-1)^p \mathscr{R} \sum_{|J|\le 2p} a_J(x)\xi^J > 0, \qquad x \in I_0$$

for every $\xi \ne 0$ in E^n. Then there exist constants $K < \infty$ and $k > 0$, such that

$$\mathscr{R}(\tau f, f) + K(f, f) \ge k|f|_p^2, \qquad f \in C_0^\infty(I).$$

PROOF. We shall reduce our assertion to simpler and simpler subcases of itself, finally proving the simplest. For this purpose, we employ a sequence of elementary reductions, as follows. Let $\hat\tau = (\tau + \tau^*)/2$. Then from the formulae of Section 2 immediately following the definitions of τ^*, τ^+, and $\bar\tau$, and by applying Leibniz' rule to the formula defining τ^* we have $\hat\tau = \hat\tau^*$ and

$$\hat\tau = \tfrac{1}{2}(\tau + \tau^*) = \sum_{|J|=2p} \mathscr{R} a_J(x)\partial^J + \sum_{|J|<2p} b_J(x)\partial^J,$$

in terms of certain coefficients b_J. Moreover, we have

$$\mathscr{R}(\tau f, f) = \tfrac{1}{2}\{(\tau f, f) + (f, \tau f)\} = \tfrac{1}{2}((\tau + \tau^*)f, f) = (\hat\tau f, f)$$

for f in $C_0^\infty(I)$ by Green's formula (1) of the last paragraph of Section 2. Thus, what we must show is that

$$(\hat{\tau}f, f)+K(f, f) \geq k|f|^2_{(p)}, \qquad f \in C_0^\infty(I).$$

We may, therefore, assume without loss of generality that $\tau = \hat{\tau}$; i.e., that $\tau = \tau^*$ is formally self adjoint, and that $\sum_{|J|=2p} a_J(x)\xi^J > 0$ for each x in I_0 and $\xi \neq 0$ in E^n.

By induction on $|J_1|$, we can readily show that a formal identity

$$(1) \qquad \partial^{J_1}C(x)\partial^{J_2} = C(x)\partial^{J_1}\partial^{J_2} + \sum_{|J|<|J_1|+|J_2|} C_{J,J_1}a_J(x)\partial^J,$$

with suitable coefficients C_{J,J_1}, holds for every function C in $C_0^\infty(I_0)$. Making use of identities of the type (1), we may evidently proceed to prove by induction on the order of τ that τ may be written in the form

$$(2) \quad \tau = \sum_{|J_1|=p,\, |J_2|=p} \partial^{J_1}d_{J_1,J_2}(x)\partial^{J_2} + \sum_{|J|<p,\, |J_2|\leq p} \partial^{J_1}d_{J_1,J_2}(x)\partial^{J_2},$$

where the coefficients d_{J_1,J_2} belong to $C^\infty(I_0)$. It follows immediately from the formal identity (1) that

$$\sum_{|J_1|=p,\, |J_2|=p} d_{J_1,J_2}(x)\partial^{J_1}\partial^{J_2} = \sum_{|J|=2p} a_J(x)\partial^J.$$

From this identity between formal differential operators there follows the identity

$$\sum_{|J_1|=p,\, |J_2|=p} d_{J_1,J_2}(x)\xi^{J_1}\xi^{J_2} = \sum_{|J|=2p} a_J(x)\xi^J, \qquad \xi \in E^n,$$

so that

$$(-1)^p \sum_{|J_1|=|J_2|=p} d_{J_1,J_2}(x)\xi^{J_1}\xi^{J_2} > 0, \qquad x \in I_0, \xi \in E^n.$$

Since

$$\begin{aligned}\tau = \tau^* = \tfrac{1}{2}(\tau+\tau^*) &= \sum_{|J_1|=|J_2|=p} \partial^{J_1}\overline{d_{J,J_1}(x)}\partial^{J_2} + \tau_1 \\ &= \sum_{|J_1|=|J_2|=p} \partial^{J_1}\left(d_{J_1,J_2}(x)+\overline{d_{J_2,J_1}(x)}\right)\partial^{J_2} + \tau_2,\end{aligned}$$

where τ_1 and τ_2 are formal operators of order $2p-1$ at most, it is clear that we may assume without loss of generality that $d_{J_1,J_2}(x) = \overline{d_{J_2,J_1}(x)}$ for x in I_0, $|J_1| = |J_2| = p$. Thus, if we put

$$\tau_0 = \sum_{|J_1|=|J_2|=p} \partial^{J_1}d_{J_1,J_2}(x)\partial^{J_2},$$

τ_0 is formally symmetric; $\tau_0 = \tau_0^*$. We shall show below that there exist constants $K_1 < \infty$ and $k_1 > 0$ such that

$$(3) \qquad (\tau_0 f, f) + K_1(f, f) \geq k_1(f, f)_{(p)}, \qquad f \in C_0^\infty(I).$$

It follows from (2) and from Green's formula that

$$(\tau f, f) = (\tau_0 f, f) + \sum_{|J_1| < p,\, |J_2| \leq p} \int_I (-1)^{J_1} d_{J_1, J_2}(x) \partial^{J_2} f(x) \overline{\partial^{J_1} f(x)} dx$$

for each f in $C_0^\infty(I)$. Thus, since the coefficients d_{J_1, J_2} are uniformly bounded in I, from Schwarz' inequality (cf. III.3.2), there exists a constant $K_2 < \infty$ such that

$$|(\tau f, f) - (\tau_0 f, f)| \leq K_2 |f|_{(p)} |f|_{(p-1)}, \qquad f \in C_0^\infty(I).$$

From Lemma 4.12 it follows that for each $\varepsilon > 0$ there exists a finite constant $K(\varepsilon)$ such that

$$|f|_p |f|_{p-1} \leq \varepsilon |f|_{(p)}^2 + K(\varepsilon) |f|_{(0)}^2, \qquad f \in C_0^\infty(I).$$

Hence

$$(4) \qquad |(\tau f, f) - (\tau_0 f, f)| \leq \varepsilon K_2 |f|_{(p)}^2 + K_2 K(\varepsilon) |f|_{(0)}^2.$$

Suppose that (3) is proved. Then, if the constant ε in (4) is chosen so small that $|\varepsilon K_2| < k_1/2$, k_1 being as in (3), it will follow from (3) that

$$(\tau f, f) + K_1(f, f) + |K_2 K(\varepsilon)|(f, f) \geq \tfrac{1}{2} k_1 (f, f)_{(p)},$$

and, putting $K = K_1 + |K_2 K(\varepsilon)|(f, f) \geq k_1/2$, the present lemma will be proved.

All that remains then, to complete the proof of the present lemma, is the proof of (3). By the preceding lemma and by the Heine-Borel theorem, it is enough to show that each point x_0 in I_0 has a neighborhood I_1 sufficiently small so that for some $K_1 < \infty$ and $k_1 > 0$,

$$(5) \qquad (\tau_0 f, f) + K_1(f, f) \geq k_1(f, f)_{(p)}, \qquad f \in C_0^\infty(I_1).$$

Let

$$\sigma_0 = \sum_{|J_1| = |J_2| = p} \partial^{J_1} d_{J_1, J_2}(x_0) \partial^{J_2},$$

$$\sigma_1 = \sum_{|J_1| = |J_2| < p} \{\partial^{J_1} d_{J_1, J_2}(x) \partial^{J_2} - \partial^{J_1} d_{J_1, J_2}(x_0) \partial^{J_2}\}.$$

Then $\tau_0 = \sigma_0 + \sigma_1$. We shall show below that there exists a neighborhood I_1 of x_0 which is sufficiently small so that for some $K_2 < \infty$ and $k_2 > 0$ we have

$$(6) \qquad (\sigma_0 f, f) + K_2(f, f)_{(p)} \geq k_2(f, f)_{(p)}, \qquad f \in C_0^\infty(I_1).$$

Since $\tau_0 = \sigma_0 + \sigma_1$, it follows on integration by parts and from Schwarz' inequality that

$$(7) \qquad \begin{aligned} &|(\tau_0 f, f) - (\sigma_0 f, f)| \\ &\quad \leq \sum_{|J_1|=|J_2|=p} \int_{I_1} |d_{J_1,J_2}(x) - d_{J_1,J_2}(x_0)||\partial^{J_1} f(x)||\partial^{J_2} f(x)|dx \\ &\quad < \sup_{\substack{x \in I_1 \\ |J_1|=|J_2|=p}} |d_{J_1,J_2}(x) - d_{J_1,J_2}(x_0)| \sum_{|J_1|=|J_2|=p} |\partial^{J_1} f||\partial^{J_2} f| \\ &\quad \leq n^{2p} \cdot \delta |f|_{(p)}^2, \end{aligned}$$

for f in $C_0^\infty(I)$, if I_1 is chosen so small that

$$(8) \qquad \sup_{\substack{x \in I_1 \\ |J_1|=|J_2|=p}} |d_{J_1,J_2}(x) - d_{J_1,J_2}(x_0)| < \delta.$$

Thus it will follow from (6), if we choose δ so small that $n^{2p} \cdot \delta < k_2/2$, and I_1 so small that (8) is satisfied, that

$$(\tau_0 f, f) + K_2(f, f) \geqq \tfrac{1}{2} k_2(f, f)_{(p)}, \qquad f \in C_0^\infty(I),$$

from which (5) follows immediately. Therefore, it suffices to establish (6).

Making a translation of coordinates, we may evidently suppose without loss of generality that $x_0 = 0$, and may then choose I_1 sufficiently small so that it is contained in the cube

$$C = \{x \in E^n \mid |x_i| \leq \pi,\ i = 1, \ldots, n\}.$$

Let

$$f(x) = \sum_{|L|=n} f_L e^{iL \cdot x}, \qquad x \in C,$$

be the Fourier expansion of f in the cube C. (Cf. Lemma 3.39.) Then, by Lemma 3.40

$$(\sigma_0 f)(x) = \sum_{|L|=n} P(L) C_L(f) e^{iL \cdot x}, \qquad x \in C,$$

where

$$P(\xi) = (-1)^p \sum_{|J_1|=p,\ |J_2|=p} d_{J_1,J_2}(0)\xi^{J_1}\xi^{J_2},$$

is the Fourier expansion of $\sigma_0 f$ in the cube C. Thus (cf. IV.4.13)

$$(9)\quad (\sigma_0 f, f) + (Kf, f) = \sum_{|L|=n} \big(P(L)+K\big)|C_L(f)|^2, \qquad f \in C_0^\infty(I_1).$$

Similarly,

$$(10)\qquad (f, f)_{(p)} = \sum_{|L|=n} Q(L)|C_L(f)|^2, \qquad f \in C_0^\infty(I),$$

where

$$Q(\xi) = \sum_{|J|\leq p} (\xi^J)^2.$$

Since $Q(\xi)$ is a polynomial in ξ of order $2p$, there exists a finite positive constant A such that

$$(11)\qquad |Q(\xi)| \leq A(|\xi|^{2p}+1), \qquad \xi \in E^n.$$

Since $P(\xi)$ is by hypothesis a non-negative homogeneous polynomial in ξ of order $2p$, there exists a finite positive constant B such that

$$\liminf_{|\xi|\to\infty} (1+|\xi|^{2p})^{-1} P(\xi) > B.$$

Thus, for K_2 sufficiently large, it follows that

$$(12)\qquad B(1+|\xi|^{2p}) \leqq P(\xi) + K_2, \qquad \xi \in E^n.$$

Thus, by (9), (10), (11), and (12)

$$BA^{-1}(f, f)_p \leq (\sigma_0 f, f) + K_2(f, f)_h, \qquad f \in C_0^\infty(I),$$

so that by placing $k_L = BA^{-1}$, (6) is proved, and with it the present lemma. Q.E.D.

11 Corollary. *Let τ be an elliptic formal partial differential operator of even order $2p$, satisfying the hypothesis of Lemma 10, and defined in a bounded domain in E^n satisfying the hypotheses of that lemma. Let $T = T(\tau)$ be the operator in the Hilbert space $L_2(I)$ defined by the equation*

$$\mathfrak{D}(T(\tau)) = \mathfrak{D}(T) = \mathfrak{D}\big(T_1(\tau)\big) \cap H_0^p(I),$$
$$Tf = T_1(\tau)f, \qquad f \in \mathfrak{D}(T).$$

Then $\sigma(T)$ is a countable discrete set of points with no finite limit points, and for $\lambda \notin \sigma(T)$, $R(\lambda; T)$ is a compact operator.

Proof. It follows just as in the proof of the preceding lemma that $(T_1(\tau)f, g)$ can be written in the form

$$(1) \quad (T_1(\tau)f, g) = \sum_{|J_1| \leq p, |J_2| \leq p} (-1)^{|J_1|} \int_I d_{J_1, J_2}(x) \partial^{J_1} f(x) \partial^{J_2} \overline{g(x)} dx, \qquad f, g \in C_0^\infty(I),$$

where the coefficients d_{J_1, J_2} belong to $C_0^\infty(I_0)$, and, in particular, are uniformly bounded on I. Thus, by (1), by Schwarz' inequality, and by the preceding lemma, we may find a real number λ so large that there exist two constants K_1 and k_1 such that

$$(2) \qquad |(T_1(\tau+\lambda)f, g)| \leq K_1 |f|_{(p)} |g|_{(p)}, \qquad f, g \in C_0^\infty(I),$$

and

$$(3) \qquad \mathscr{R}(T_1(\tau+\lambda)f, f) \geqq k_1 |f|_{(p)}^2, \qquad f, g \in C_0^\infty(I).$$

By (2) the expression $(T_1(\tau+\lambda)f, g)$ can be extended to a continuous bilinear form $[f, g]$ defined on the closure $H_0^{(p)}(I)$ of $C_0^\infty(I)$ in $H_0^{(p)}(I)$ (cf. I.6.17). Since $H_0^{(p)}(I)$ is a Hilbert space (cf. 3.16), it follows that $[f, g] = (Af, g)_{(p)}$ (cf. IV.4.5) for some vector Af in $H_0^{(p)}(I)$ for each f in $H_0^{(p)}(I)$ and g in $H_0^{(p)}(I)$. Since $[f, g]$ is bounded and bilinear, A is a bounded linear mapping of $H_0^{(p)}(I)$ into itself. By (3) and Schwarz' inequality

$$(4) \qquad |Af|_{(p)} |f|_{(p)} \geqq |(Af, f)_{(p)}| \geqq k_1 |f|_{(p)}^2, \qquad f \in C_0^\infty(I),$$

so that by continuity these inequalities hold for all f in $H_0^{(p)}(I)$. This shows that $|Af|_{(p)} \geqq k_1 |f|_{(p)}$, so that A^{-1} is defined on $AH_0^{(p)}(I)$, single valued, and bounded. If $AH_0^{(p)}(I)$ is not dense in $H_0^{(p)}(I)$, then (cf. IV.4.4) there exists an element f in $H_0^{(p)}(I)$ such that $f \neq 0$ and $(AH_0^{(p)}(I), f)_{(p)} = 0$. But then $(Af, f)_{(p)} = 0$, contradicting (4). This shows that $AH_0^{(p)}(I)$ is dense in $H_0^{(p)}(I)$. Let f be an arbitrary element of $H_0^{(p)}(I)$, and let f_n in $H_0^{(p)}(I)$ be such that $|Af_n - f|_{(p)} \to 0$. Then, since A^{-1} is bounded, $\{f_n\}$ is a Cauchy sequence, and hence a convergent sequence with a limit g in $H_0^{(p)}(I)$. We have $Ag = f$. This shows that $AH_0^{(p)}(I) = H_0^{(p)}(I)$, so that A^{-1} is a bounded, everywhere defined operator in $H_0^{(p)}(I)$.

Let f be an arbitrary element of $H_0^{(p)}(I)$, and let g be in $C_0^\infty(I)$. Let f_m be in $C_0^\infty(I)$, and $|f_m - f|_{(p)} \to 0$ as $m \to \infty$. Then $f_m \to f$ in the topology of $L_2(I)$, so that by Definition 3.26 and Lemma 3.27, $f_m \to f$ in the topology of $D(I)$, and $(\tau+\lambda)f_m \to (\tau+\lambda)f$ in the topology of $D(I)$. Hence

$$\begin{aligned}(5)\qquad \int_I (\tau+\lambda)f(x)\overline{g(x)}dx &= \lim_{n\to\infty}\int_I (\tau+\lambda)f_n(x)\overline{g(x)}dx \\ &= \lim_{n\to\infty}(Af_n, g)_{(p)} = (Af, g)_{(p)}\end{aligned}$$

for each f in $H_0^{(p)}(I)$ and g in $C_0^\infty(I)$, and hence, *a fortiori*

$$(6)\qquad \big((T+\lambda I)f, g\big) = (Af, g)_{(p)}$$

for each f in $H_0^{(p)}(I)$ and g in $C_0^\infty(I)$. It follows immediately from continuity that (6) is valid for all $f \in \mathfrak{D}(T)$ and g in $H_0^{(p)}(I)$.

By (4), (6), and Schwarz' inequality,

$$|(T+\lambda I)f|\,|f| \geq \big|\big((T+\lambda I)f, f\big)\big| \geqq k_1|f|_{(p)}|f|_{(p)} \geq k_1|f|_{(p)}|f|,$$

for f in $\mathfrak{D}(T)$, proving that

$$(7)\qquad |(T+\lambda I)f| \geqq k_1|f|_{(p)}, \qquad f \in \mathfrak{D}(T).$$

Thus $(T+\lambda I)f$ is one-to-one and $(T+\lambda I)^{-1}$ is defined on $(T+\lambda I)\mathfrak{D}(T)$ and bounded.

If g is an arbitrarily prescribed element in $L_2(I)$, then $\varphi(f) = (g, f)$ is evidently a conjugate linear functional continuous on $H_0^{(p)}(I)$. Thus there exists an h in $H_0^{(p)}(I)$ such that

$$\begin{aligned}\int_I g(x)\overline{f(x)}dx = (g, f) = (h, f)_{(p)} &= (AA^{-1}h, f)_{(p)} \\ &= \int_I (\tau+\lambda)(A^{-1}h)(x)\overline{f(x)}dx,\end{aligned}$$

for each f in $C_0^\infty(I)$, the final equality following from (6). This shows that $(\tau+\lambda)A^{-1}h = g$, so that $A^{-1}h$ is in $\mathfrak{D}(T)$, establishing that $(T+\lambda I)^{-1}$ is bounded and everywhere defined.

By (7),

$$|(T+\lambda I)^{-1}f|_{(p)} \leq k_1^{-1}|f|, \qquad f \in L_2(I),$$

which shows, by Corollary 4.11, that $R(\mu_0; T_0)$ is compact for any particular μ_0 chosen to be sufficiently large in absolute value and negative. Thus, by Lemma VII.9.2 and Theorem VII.4.5, $\sigma(T)$ is a

countable discrete set of points with no finite limit points. If $\lambda \notin \sigma(T)$, then

$$R(\lambda; T) = R(\mu_0; T_0) + (\mu_0 - \lambda)R(\mu_0; T)R(\lambda; T),$$

(cf. Definition VII.9.3 and Theorem VII.9.5) so that $R(\lambda; T)$ is compact by Theorem VI.5.4. Q.E.D.

12 COROLLARY. *Let the hypotheses of Corollary* 11 *be satisfied. Then there exists a constant* $K < \infty$ *and a constant* $k > 0$ *such that*

$$\mathscr{R}(Tf, f) + K(f, f) \geqq k|f|^2_{(p)}, \qquad f \in \mathfrak{D}(T).$$

PROOF. This will follow immediately from formula (6) of the proof of the preceding corollary once it is shown that there exists a constant $k > 0$ such that

$$\mathscr{R}(Af, f)_{(p)} \geqq k(f, f)_{(p)}, \qquad f \in H_0^{(p)}(I).$$

From (3), this holds for all f in $C_0^\infty(I)$. But $C_0^\infty(I)$ is dense in $H_0^{(p)}(I)$; thus it holds for all f in $H_0^{(p)}(I)$ by continuity. Q.E.D.

13 LEMMA. *Let A_1 and A_2 be densely defined linear operators in Hilbert space. Suppose that there exists a $\lambda \notin \sigma(A_1)$ such that $\bar{\lambda} \notin \sigma(A_2)$, $A_1 \subseteq A_2^*$, and $A_2 \subseteq A_1^*$. Then $A_1 = A_2^*$, $A_2 = A_1^*$.*

PROOF. Let $B_1 = (\lambda I - A_1)^{-1}$, $B_2 = (\bar{\lambda} I - A_2)^{-1}$. Then if x and y are vectors in Hilbert space,

$$\begin{aligned}(B_1 x, y) = (B_1 x, (\bar{\lambda} I - A_2)B_2 y) &= \big((\lambda I - A_1)B_1 x, B_2 y\big) \\ &= (x, B_2 y).\end{aligned}$$

Hence $B_1 = B_2^*$, $B_2 = B_1^*$. It follows from Lemma XII.1.6 that $(\lambda I - A_1) = (\bar{\lambda} I - A_2)^*$, $(\bar{\lambda} I - A_2) = (\lambda I - A_1)^*$; then by Lemma XII.1.6 again, $A_1 = A_2^*$, $A_2 = A_1^*$. Q.E.D.

14 COROLLARY. *Let τ be an elliptic operator of even order $2p$, defined in a bounded domain I. Suppose that the hypotheses of Lemma* 10 *are satisfied. Let T and S be the operators in the Hilbert space $L_2(I)$ defined by*

$$\mathfrak{D}(T) = \mathfrak{D}(T_1(\tau)) \cap H_0^{(p)}(I); \quad \mathfrak{D}(S) = \mathfrak{D}(T_1(\tau^*)) \cap H_0^{(p)}(I),$$
$$Tf = T_1(\tau)f, \quad f \in \mathfrak{D}(T); \quad Sf = T_1(\tau^*)f, \quad f \in \mathfrak{D}(S).$$

Then $T = S^*$ and $S = T^*$.

PROOF. By the preceding lemma and by Corollary 11 it suffices to show that $(Tf, g) = (f, Sg)$ for f in $\mathfrak{D}(T)$ and g in $\mathfrak{D}(S)$. By Green's formula, proved in the last paragraph of Section 2, this equation is valid if f and g are in $C_0^\infty(I)$. It follows as in the proof of formula (6) of Corollary 11 that there exist bounded operators A and B mapping $H_0^{(p)}(I)$ into itself, such that

$$(Tf, g) = (Af, g)_{(p)}; \quad (f, Sg) = (Bf, g)_{(p)}, \quad f \in \mathfrak{D}(T), \quad g \in \mathfrak{D}(S).$$

We have seen that $(Af, g) = (Bf, g)$ for all f and g in $C_0^\infty(I)$, and $C_0^\infty(I)$ is dense in $H_0^{(p)}(I)$ by the definition of $H_0^{(p)}(I)$; it follows from continuity that $(Af, g) = (Bf, g)$ for all f and g in $H_0^{(p)}(I)$. Thus $(Tf, g) = (f, Sg)$ for all f in $\mathfrak{D}(T)$ and g in $\mathfrak{D}(S)$. Q.E.D.

Now we turn to an analysis of the problem of "differentiability up to the boundary." We shall prove a variant of Theorem 2, valid up to the boundary of a domain with smooth boundary. The method to be used is a close analogue of the method of proof of Theorem 2, complicated, however, by the presence of the boundary. Our idea for overcoming this obstacle is as follows. Assuming the boundary to be smooth, it follows that it may as well be assumed to be plane. This being assumed, the proof of Theorem 2 may be modified as follows. The general process of reducing the proof to the special case in which the partial differential operator to be considered has constant coefficients works just as before. To handle this special case, instead of expanding in a Fourier series as in the proof of Theorem 2, which is forbidden by the presence of the plane boundary, we make use of the convenient process of translation parallel to the boundary, of Gårding's inequality, and of an important lemma of J. L. Lions, which together enable us to handle the constant coefficient case using a simple inductive proof.

The following lemma is merely preliminary.

15 LEMMA. *Let I be a cube in E^n, and let p be a positive integer. Let φ be in $C^p(E^n)$, and suppose that every partial derivative of order at most p of the function φ vanishes on the boundary of I. Then there exists a sequence $\{\psi_n\}$ of functions in $C_0^\infty(I)$ such that $\psi_n \to \varphi$ in the norm of $C^p(\bar{I})$ as $n \to \infty$.*

PROOF. We may evidently suppose without loss of generality that

$$I = \{x \in E^n \mid |x_i| \leq 1,\ i = 1, \ldots, n\}.$$

If we place $\hat{\varphi}(x) = \varphi(x)$, $x \in I$, and $\hat{\varphi}(x) = 0$, $x \notin I$, then $\hat{\varphi}$ evidently satisfies the same hypotheses as φ. Thus, we may suppose that $\varphi(x) = 0$ for $x \notin I$. Since the function φ_ε defined by $\varphi_\varepsilon(x) = \varphi\big((1-\varepsilon)x\big)$ converges to φ in the norm of $C^p(\bar{I})$ as $\varepsilon \to 0$ by Lemma 2.5, it is clear that we may assume without loss of generality that φ is in $C_0^p(I)$. This will be assumed in what follows.

Let K be a compact subset of the interior of I outside of which the function φ vanishes. Let $\varepsilon_1 > 0$ be sufficiently small so that every point within a distance $2\varepsilon_1$ of some point of K is interior to I. For each ε with $0 < \varepsilon < \varepsilon_1$, Lemma 2.1 may be used to find a function $\eta = \eta_\varepsilon$ such that $\eta(x) = 0$ for $|x| > \varepsilon$ and

$$\int_{E^n} \eta(x)dx = 1. \tag{1}$$

Put

$$\begin{aligned} \psi(x) &= \int_{E^n} \eta(x-y)\varphi(y)dy \\ &= \int_{E^n} \varphi(x-y)\eta(y)dy. \end{aligned} \tag{2}$$

Since η is in $C_0^\infty(E^n)$, the first of these integrals may be differentiated arbitrarily often under the sign of integration, so that ψ is in $C^\infty(E^n)$. If $\psi(x) \neq 0$, there must exist some point y such that $\eta(x-y) \neq 0$ and $\varphi(y) \neq 0$. Thus $|x-y| < \varepsilon_1$ and $y \in K$, so that x is interior to I, and at a distance of at least ε_1 from the boundary of I. Thus ψ is in $C_0^\infty(I)$. Now, since φ is in $C_0^p(E^n)$, the second integral in (2) may be differentiated up to p times under the sign of integration. Thus

$$\partial^J \psi(x) = \int_{E^n} (\partial^J \varphi)(x-y)\eta(y)dy, \qquad |J| \leq p.$$

It follows, using (1), that

$$\begin{aligned} |\partial^J \psi(x) - \partial^J \varphi(x)| &= \left| \int_{E^n} \{(\partial^J \varphi)(x-y) - \partial^J \varphi(x)\}\eta(y)dy \right| \\ &\leq \max_{|y| \leq \varepsilon} |(\partial^J \varphi)(x-y) - \partial^J \varphi(x)|. \end{aligned} \tag{3}$$

Since φ is in $C_0^p(E^n)$ it is clear that if $\delta > 0$, by choosing $\varepsilon = \varepsilon(\delta)$ sufficiently small, we can assure the inequality

$$\max_{|y|\leq\varepsilon} |(\partial^J\varphi)(x-y)-\partial^J\varphi(x)| < \delta, \qquad |J| \leq p.$$

This being done, the lemma follows immediately. Q.E.D.

The first main step in our analysis is to prove the following lemma, due to J. L. Lions.

16 LEMMA. *Let I denote the cube $\{x \in E^n | 0 < x_j < 1, j = 1, \ldots, n\}$. Let F be in $D(I)$, and let $p > 0$ and q be integers, with $p+q > 0$. Then, if F is in $L_2(I)$ and $\partial_j^p F$ is in $H^{(q)}(I)$, $j = 1, \ldots, n$, it follows that F is in $H^{(p+q)}(I)$.*

PROOF. The proof is based on a very elementary "reflection principle" which we have only to set up in the proper form. Let k be a positive integer, to be specified below. Let $\alpha_{-k}, \ldots, \alpha_k$ be a set of distinct positive numbers greater than one. Let $c_{-k}, \ldots, c_k$ be the set of solutions of the system

$$\sum_{j=-k}^{k} (-1)^j \alpha_j^l c_j - 1 = 0, \qquad l = -k, \ldots, +k \tag{1}$$

of linear equations. Since the Vandermonde determinant of this system of linear equations does not vanish, the real numbers c_j exist and are unique. For each φ in $C_0^\infty(E^n)$, write

$$(R_j^{(l)}\varphi)(x_1, \ldots, x_n) = \varphi(x_1, \ldots, x_{l-1}, -\alpha_j x_l, x_{l+1}, \ldots, x_n), \tag{2}$$

so that $R_j^{(l)}$ defines a mapping of $C_0^\infty(E^n)$ into itself for each pair of integers satisfying the inequalities $1 \leq l \leq n$ and $-k \leq j \leq +k$. All the operators $R_j^{(l)}$ evidently commute with each other. Put

$$S_j^{(l)}\varphi = \varphi - \sum_{i=-k}^{k} (-\alpha_i)^j c_i R_i^{(l)}\varphi, \quad 1 \leq l \leq n, \quad -\infty \leq j \leq +\infty, \tag{3}$$

so that all the operators $S_j^{(l)}$ commute with one another, and

$$\begin{aligned} &\partial_m S_j^{(l)} = S_j^{(l)}\partial_m, \quad l \neq m, \quad 1 \leq l, m \leq n, \quad -\infty < j < +\infty, \\ &\partial_l S_j^{(l)} = S_{j+1}^{(l)}\partial_l, \quad 1 \leq l < n, \quad \infty < j < +\infty. \end{aligned} \tag{4}$$

Put

$$S_L = S_{l_1}^{(1)} \ldots S_{l_n}^{(n)} \tag{5}$$

for each index L such that $|L| = n$. Then it follows from (4) that

$$\partial_m S_L = S_{L'} \partial_m, \tag{6}$$

where $L' = [l_1, \ldots, l_{m-1}, l_m+1, l_{m+1}, \ldots, l_n]$ and $L = [l_1, \ldots, l_n]$.

By (1) and by the definitions (2), (3), and (5) of S_L, it follows that

$$(S_L\varphi)(x) = 0, \varphi \in C(E^n), \qquad k \leqq \min(L) \leqq \max(L) \leqq k, \tag{7}$$

if one of $x_1, \ldots, x_n$ *is zero.* Suppose that we let I_0 denote the cube $I_0 = \{x \in E^n \mid |x_i| \leqq 1, \ i = 1, \ldots, n\}$. Then for $-k \leqq \min(L) \leqq \max(L) \leqq k$, S_L is a mapping of $C_0^\infty(I_0)$ into itself, which maps all of $C_0^\infty(I_0)$ into the set of functions vanishing on the boundary of the cube I. Next observe that by (6), $\partial_m S_L \varphi = S_{L'} \partial_m \varphi$, so that by (7), $S_L\varphi$ vanishes together with all its first derivatives if one of $x_1, \ldots, x_n$ is zero and if $-k \leqq \min(L) \leqq \max(L) \leqq k-1$. In the same way we see, using (6) and (7), that $S_L\varphi$ vanishes together with all its derivatives of order at most j if one of $x_1, \ldots, x_n$ is zero and $-k \leqq \min(L) \leqq \max(L) \leqq k-j$. It follows that if $T_L\varphi$ is defined by $T_L\varphi = S_L\varphi | I$, then T_L maps $C_0^\infty(I_0)$ into $C^\infty(\bar{I})$, and all the first j derivatives of $T_L\varphi$ vanish on the boundary of I for $-k \leqq \min(L) \leqq \max(L) \leqq k-j$. Hence, by the preceding lemma, T_L maps $C_0^\infty(I_0)$ into $H_0^{(j)}(I)$ for $-k \leqq \min(L) \leqq \max(L) \leqq k-j$. It is clear from Definition 3.15 that this mapping is continuous.

Suppose now that $q \leqq 0$. Note that since F is in $L_2(I)$, the formula $F(\varphi) = \int F(x)\varphi(x)dx$ defines an extension of the linear functional F to a continuous linear functional on $L_2(I)$. Let G in $D(I_0)$ be defined by the equation $G(\varphi) = F(T_0\varphi)$, where 0 denotes the index all of whose entries are zero, and where in all that follows we suppose k to be chosen so that $k \geqq 2 \max(-q, p)$. Then, by (6) and the definition of T_L,

$$G(\partial_j^p \varphi) = F(T_0 \partial_j^p \varphi) = F(\partial_j^p T_{L_j^p} \varphi), \qquad \varphi \in C_0^\infty(I_0), \tag{8}$$

where L_j^p is the n-tuple with zero in each place but the j th, and $-p$ in the j th place. Now, by hypothesis and by Definition 3.17, there exists a finite constant A such that

$$|F(\partial_j^p \psi)| \leqq A|\psi|_{(-q)}, \qquad \psi \in C_0^\infty(I), \qquad 1 \leqq j \leqq n.$$

It follows from the continuity of the functional $F(\tilde{\psi})$ ($\tilde{\psi}$ being taken with the norm $|\psi|_{(0)}$) and from the continuity of the map $\psi \to \partial_j^p \psi$

of $H^{(p)}(I)$ into $H^{(0)}(I)$ that this same inequality holds for φ in $H_0^{(r)}(I)$ and $r = \max(-q, p)$. Since $k \geqq 2\max(-q, p)$, so that $r \leqq k - p$ and hence $T_{L_j^p}\varphi$ is in $H_0^{(r)}(I)$, it follows from (8) that

$$|(\partial_j^p G)(\varphi)| \leqq A|T_{L_j^p}\varphi|_{(-q)}, \qquad 1 \leqq j \leqq n. \tag{9}$$

It follows immediately from the definition of T_L and from Definition 3.15 that there exists a finite constant B such that $|T_L\varphi|_{(-q)} \leqq B\,|\varphi|_{(-q)}$ for φ in $C_0^\infty(I_0)$. Thus, by (9), $\partial_j^p G$ is in $H^{(q)}(I_0)$ for $1 \leq j \leq n$. Hence, if we apply Theorem 2 to the elliptic operator

$$\tau = \partial_1^p + \cdots + \partial_n^p,$$

we find that G is in $H^{(p+q)}(I_0)$. Now, if φ is in $C_0^\infty(I)$, it is clear that $T_0\varphi = \varphi$. Hence, since $G(\varphi) = F(T_0\varphi)$, $G|I = F$. Together with Definitions 3.15 and 3.17, this shows that $F|I_1$ is in $H^{(p+q)}(I_1)$ for any open subset I_1 of I whose closure does not touch any face of the cube I not containing the corner of I which is at the origin. Since any corner of the cube I may be treated in exactly the same way, the case $q \leq 0$ of the present lemma follows from Lemma 3.21.

Next suppose that $q \geqq 0$. Then $\partial_j^p \partial^J F$ is in $L_2(I)$ for $1 \leqq j \leqq n$ and $|J| = q$. Hence $\partial^J F$ is in $H^{(p)}(I)$ by what has just been proved, so that F is in $H^{(p+q)}(I)$ by Definition 3.15. Q.E.D.

Throughout the next few proofs, we shall work within the cube

$$C = \{x \in E^n | 0 \leqq x_1 \leq 2\pi,\ |x_j| \leq \pi,\ j = 2, \ldots, n\}$$

of Euclidean n-space E^n. We shall constantly suppose E^n to be written as the Cartesian product of E^1 and E^{n-1}, so that $E^n = E^1 \oplus E^{n-1}$, and correspondingly, shall write each x in E^n as $x = [x_1, y]$, where $y = [x_2, \ldots, x_n]$ is in E^{n-1}. Then C may be written as the Cartesian product $C = [0, 2\pi] \times C_1$ of the cube

$$C_1 = \{y \in E^{n-1} |\ 0 \leqq y_i \leqq 2\pi,\ i = 1, \ldots, n-1\}$$

and of the interval $[0, 2\pi]$. We wish to fix our attention on the two faces $F_- = \{[0, y] | y \in C_1\}$ and $F_+ = \{[2\pi, y] | y \in C_1\}$ of the cube C, to the exclusion of all its other faces. For this purpose, we shall always assume, unless the contrary is explicitly stated, that all functions of x to be considered are multiply periodic of period 2π in the variables $y = [y_1, \ldots, y_{n-1}]$. Correspondingly, we shall work with the space $F_{\pi_y}(C)$ of all functions of x which are multiply periodic of period 2π

in the variables $y = [y_1, \ldots, y_{n-1}]$ with the spaces

$$C^{\infty}_{\pi_y}(C) = \{f \in C^{\infty}(C) | f \in F_{\pi_y}(C)\},$$
$$C^{p}_{\pi_y}(C) = \{f \in C^{p}(C) | f \in F_{\pi_y}(C)\},$$

and

$$C^{\infty}_{\pi,0}(C) = \{f \in C^{\infty}_{\pi_y}(C) | f(x, y) = 0 \text{ for } x_1 \text{ near } 0 \text{ or } 2\pi\}.$$

It should be evident from this last formula that much as in the corresponding case of the space $C^{\infty}_{\pi}(C)$, we may regard any point $x = [x_1, y]$ for which $0 < x_1 < 2\pi$ as belonging, in a suitable sense, to the interior of C: that is, to argue at such a point as we would at an interior point, we have only to make use of the multiple periodicity in the variable $y = [y_1, \ldots, y_{n-1}]$ of all functions to be considered by introducing shifted coordinates in C in terms of which p is interior to $C = [0, 2\pi] \times C_1$. (Cf. the corresponding discussion in Section 3 preceding Definition 3.29).

We write $f_n \xrightarrow{\tau} f$ for f_n, f in $C^{\infty}_{\pi,0}(C)$ if $f_n \to f$ in the topology of $C^{\infty}_{\pi_y}(C)$ and if all the functions f_n vanish outside a fixed set of the form $[\varepsilon, 2\pi - \varepsilon] \times C_1$, $\varepsilon > 0$, and we let $D_{\pi_y}(C)$ be the set of all linear functionals G on $C^{\infty}_{\pi_y,0}(C)$ which are continuous in the sense that $f_n \xrightarrow{\tau} f$ implies $G(f_n) \to G(f)$. All the other notions of the theory of distributions, e.g., the sum of two elements of D_{π_y}, the product of an element of $D_{\pi_y}(C_1)$ by an element of $C^{\infty}_{\pi_y}(C_1)$, the partial derivatives of an element of $D_{\pi_y}(C)$, may then be introduced much as they were in Section 3. (Cf. the corresponding discussion in Section 3 following Definition 3.28). Letting J denote an arbitrary index in E^n, we may then introduce the Hilbert space

$$H^{(p)}_{\pi_y}(C) = \{f \in D_{\pi_y}(C) | f \in L_2(C), \partial^J f \in L_2(C), |J| \leq p\}$$

with corresponding inner products

$$(f, g)_{(p)} = \sum_{|J| \leq p} \int_C \partial^J f(y) \overline{\partial^J g(y)} dy.$$

By $H^{(p)}_{\pi_y,0}(C)$ we shall denote the closure in the topology of $H^{(p)}_{\pi_y}(C)$ of $C^{\infty}_{\pi_y,0}(C)$. It should be noted that since functions in $L_2(C)$, $L_2(C_1)$, etc., are only defined almost everywhere, they may always be taken to be multiply periodic of period 2π in all their variables. Hence, it is

pointless to introduce a space such as $H^{(0)}_{\pi_y}(C)$, since this would be a linear space identical to $L_2(C)$.

For convenience in stating the next lemma, we introduce the rectangle $R = [-\pi, 3\pi] \times C_1$, and the space $C^\infty_{\pi_y}(R)$ of all functions in $C^\infty(R)$ which are multiply periodic of period 2π in the variables $y = [y_1, \ldots, y_{n-1}]$.

For partial differential operators in R with coefficients belonging to $C^\infty_{\pi_y}(R)$, we may state the following analogue of the Gårding inequality, Lemma 10. As to its proof, we need only remark that since, as has been pointed out above, only the points $\{0\} \times C_1$ and $\{2\pi\} \times C_1$ count as boundary points of C, its proof may be adapted immediately from that of the Gårding inequality. We leave the details involved in this to the reader.

17 LEMMA. *Let τ be an elliptic formal partial differential operator of even order $2p$ defined in R. Suppose that τ is of the form*

$$\tau = \sum_{|J| \leq 2p} a_J(y)\partial^J,$$

where all the coefficients a_J of τ belong to $C^\infty_{\pi_y}(R)$, and that

$$\mathscr{R}(-1)^p \sum_{|J|=2p} a_J(y)\xi^J > 0, \qquad \xi \neq 0, \qquad y \in R.$$

Then there exist constants $K < \infty$ and $k > 0$, such that

$$\mathscr{R}((\tau + K)f, f) \geqq k|f|_{(p)}, \qquad f \in C^\infty_{\pi_y,0}(C).$$

Moreover, there exists a constant $A < \infty$ such that

$$|(\tau f, g)| \leqq A|f|_{(p)}|g|_{(p)}, \qquad f, g \in C^\infty_{\pi,0}(C).$$

Now we shall prove an important lemma on elliptic partial differential equations with constant coefficients.

18 LEMMA. *Let σ be a formal partial differential operator of even order $2p$ with constant coefficients, having the form*

$$\sigma = \sum_{|J|=2p} a_J \partial^J,$$

where

$$(*) \qquad \mathscr{R}(-1)^p \sum_{|J|=2p} a_J \xi^J > 0, \qquad \xi \neq 0.$$

Then there exists a constant K such that for each $k \geqq p$, $\sigma + K$ is a one-to-one mapping with a bounded inverse of $H^{(p)}_{\pi_v,0}(C) \cap H^{(k)}_{\pi_v}(C)$ onto $H^{(k-2p)}_{\pi_v}(C)$.

PROOF. Using Lemma 17, choose K, A, and $a > 0$ such that

$$\mathscr{R}\big((\sigma^* + K)f, f\big) \geqq a|f|^2_{(p)}, \qquad f \in C^{\infty}_{\pi_v,0}(C), \tag{1}$$

and

$$\big|\big((\sigma^* + K)f, g\big)\big| \leqq A|f|_{(p)}|g|_{(p)}; \qquad f, g \in C^{\infty}_{\pi_v,0}(C). \tag{2}$$

It follows from (2) that the Hermitian-bilinear form $\big((\sigma^* + K)f, g\big)$ can be extended uniquely from $C^{\infty}_{\pi_v,0}(C)$ to a continuous Hermitian-bilinear form defined on $H^{(p)}_{\pi_v,0}(C)$, and hence by Lemma X.2.2 that there exists a bounded mapping Λ of $H^{\infty}_{\pi_v,0}(C)$ into itself such that

$$(\Lambda f, g)_{(p)} = \big((\sigma^* + K)f, g\big), \qquad f, g \in C^{\infty}_{\pi_v,0}(C). \tag{3}$$

By continuity, this equation holds also for $f \in C^{\infty}_{\pi_v,0}(C)$ and $g \in H^{(\infty)}_{\pi_v,0}(C)$. By (1) we have $\mathscr{R}(\Lambda f, f)_{(p)} \geqq a|f|^2_{(p)}$ for f in $C^{\infty}_{\pi_v,0}(C)$. This inequality must hold by continuity for all f in $H^{(p)}_{\pi_v,0}(C)$. Thus $|\Lambda f|_{(p)}|f|_{(p)} \geqq a|f|^2_{(p)}$, so that $|\Lambda f|_{(p)} \geqq a|f|_{(p)}$, and Λ is one-to-one. We may continue in the same way, as in the first paragraph of the proof of Corollary 11, to show that Λ^{-1} is defined everywhere in $H^{(p)}_{\pi_v,0}(C)$.

Let g be in $H^{(-p)}_{\pi_v}(C)$. Then, by Definition 3.17, the mapping $\varphi \to (\varphi, g)$ defined on $C^{\infty}_{\pi_v,0}(C)$ has a unique extension to a continuous linear functional defined on the space $H^{(p)}_{\pi_v,0}(C)$, in which $C^{\infty}_{\pi_v,0}$ is dense by definition. Hence, by Lemma IV.4.5, we may write

$$(\varphi, g) = (\varphi, h)_{(p)}, \qquad \varphi \in C^{\infty}_{\pi_v,0}(C), \tag{4}$$

h being some suitably chosen element of $H^{(p)}_{\pi_v,0}(C)$. Let $f = (\Lambda^*)^{-1}h$. Then by (4) and (3)

$$(\varphi, g) = (\Lambda\varphi, (\Lambda^*)^{-1}h)_{(p)} = \big((\sigma^* + K)\varphi, f\big), \qquad \varphi \in C^{\infty}_{\pi_v,0}(C).$$

Thus $(\sigma + K)f = g$, proving that $\sigma + K$ maps $H^{(p)}_{\pi_v,0}(C)$ onto $H^{(-p)}_{\pi_v}(C)$. If $(\sigma + K)f = 0$ and if f is in $H^{(p)}_{\pi_v,0}(C)$, we have

$$\big((\sigma^* + K)\varphi, f\big) = (\Lambda\varphi, f)_{(p)} = (\varphi, \Lambda^* f)_{(p)} = 0, \qquad \varphi \in C^{\infty}_{\pi_v,0}(C),$$

by (3). Thus, by continuity, $(g, \Lambda^* f)_{(p)} = 0$ for all g in $H^{(p)}_{\pi_v,0}(C)$, so that

$\Lambda^* f = 0$, and hence $f = 0$. This proves that $\sigma + K$ is a one-to-one mapping of $H^{(p)}_{\pi_x,0}(C)$ onto $H^{(-p)}_{\pi_x}(C)$; taken together with the closed graph theorem (II.2.4), this proves our lemma in the special case $k = p$.

The mapping $(\sigma+K)H^{(k)}_{\pi_y}(C) \to H^{(k-2p)}_{\pi_y}(C)$ is continuous by Lemma 3.22. Moreover, $(\sigma+K)\big(H^{(p)}_{\pi_y,0}(C) \cap H^{(k)}_{\pi_y}(C)\big) \subseteq H^{(k-2p)}_{\pi_y}(C)$ by Lemma 3.22, and $(\sigma+K)H^{(p)}_{\pi_y,0}(C) \supseteq H^{(k-2p)}_{\pi_y}(C)$ by what has just been proved. Hence, to establish the present lemma, we have only to show that $\{(\sigma+K)^{-1}H^{(k-2p)}_{\pi_y}(C)\} \cap H^{(p)}_{\pi_y,0}(C) \subseteq H^{(k)}_{\pi_y}(C)$, since once this is shown we shall know that $\sigma+K$ is a continuous one-to-one mapping of the closed subspace $H^{(k)}_{\pi_y}(C) \cap H^{(p)}_{\pi_y,0}(C)$ of $H^{(k)}_{\pi_y}(C)$ onto $H^{(k-2p)}_{\pi_y}(C)$. The continuity of its inverse will then follow immediately from the closed graph theorem (II.2.4).

That is, it suffices to show that $k \geqq p$, f in $H^{(p)}_0(C)$, and $(\sigma+K)f$ in $H^{(k-2p)}_{\pi}(C)$ together imply that f is in $H^{(k)}_{\pi_y}(C)$. This will be shown by induction on k. Suppose that we know it to be true for all $k < k_0$, where $k_0 \geqq p$. Let f be in $H^{(p)}_{\pi_y,0}(C)$, and $g = (\sigma+K)f$ in $H^{(k_0+1-2p)}_{\pi_y}(C)$. We shall show that $\partial^J \partial_j f$ is in $H^{(k_0-2p)}_{\pi_y}(C)$ for each $1 \leqq j \leqq n$, and each $|J| = 2p$. From this the desired result will follow immediately from Lemma 16 (as generalized from $D(C)$ to $D_{\pi_y}(C)$) (cf. the paragraphs preceding 3.38 and following 3.27). Since hypothesis (∗) implies that the coefficient of ∂_1^{2p} in $\sigma+K$ is non-zero, it is evidently sufficient, in virtue of Lemma 3.22 and the fact that $(\sigma+K)f = g$ is in $H^{(k_0+1-2p)}_{\pi_y}(C)$, to show that $\partial_j \partial^J f$ is in $H^{(k_0+1-2p)}_{\pi_y}(C)$ for each j and J such that $2 \leqq j \leqq n$ and $|J| = 2p-1$. For definiteness in notation, we shall suppose that $j = 2$, so that we must show that $\partial_2 \partial^J f$ is in $H^{(k_0+1-2p)}(C)$ for each J such that $|J| = 2p-1$. Thus, by Lemma 3.22, the present lemma will be completely proved if we can show that $\partial_2 f$ is in $H^{k_0}_{\pi_y}(C)$.

For each $\Delta > 0$, let S_Δ be the continuous mapping of C into itself defined by the equations

$$S_\Delta[x_1, x_2, \ldots, x_n] = [x_1, x_2+\Delta, x_3, \ldots, x_n], \qquad x \in C, \qquad x_2+\Delta \leqq \pi,$$

$$S_\Delta[x_1, x_2, \ldots, x_n] = [x_1, x_2+\Delta-2\pi, x_3, \ldots, x_n], \quad x \in C, \quad x_2+\Delta > \pi.$$

Then, from Lemma 3.50 (as generalized from $D_\pi(C)$ to $D_{\pi_y}(C)$), in order to show that $\partial_2 f$ is in $H^{k_0}_{\pi_y}(C)$, we have only to show that $|\Delta^{-1}(f \circ S_\Delta^{-1} - f)|_{(k_0)}$ is uniformly bounded in Δ. By Lemma 3.47, $\Delta^{-1}(f \circ S_\Delta^{-1} - f)$ is in $H^{(p)}_{\pi_y,0}(C)$, and by Lemmas 3.47 and 3.50,

$$|(\sigma+K)\Delta^{-1}(f\circ S_\Delta^{-1} - f)|_{(k_0-2p)} = |\Delta^{-1}(g\circ S_\Delta^{-1} - g)|_{(k_0-2p)}$$

is uniformly bounded in Δ.

Since, by the inductive hypothesis, $(\sigma+K)^{-1}$ is a uniformly bounded mapping of $H^{(k_0-2p)}_{\pi_\nu}(C)$ onto $H^{(k_0)}_{\pi_\nu}(C) \cap H^{(p)}_{\pi_\nu,0}(C)$, it follows that $|\Delta^{-1}(f\circ S_\Delta^{-1} - f)|_{(k_0)}$ is uniformly bounded in Δ, from which the present lemma follows, as has been shown above. Q.E.D.

Lemma 18 enables us to use the method of proof of Theorem 2 in the neighborhood of the boundary of a domain with smooth boundary. This is carried out in the next two lemmas.

19 Lemma. *Let σ be an elliptic formal partial differential operator of even order $2p$, defined in a domain I_0 of Euclidean n space E^n, and having the form*

$$\sigma = \sum_{|J|\leq 2p} a_J(x)\partial^J,$$

where

$$\mathscr{R}(-1)^p \sum_{|J|=2p} a_J(x)\xi^J > 0, \qquad x\in I_0, \qquad \xi \neq 0.$$

Let I be a subdomain of I_0 whose boundary is β. Suppose that β contains a smooth surface Σ which is contained in I_0, and that no point in Σ is in the closure of $\beta - \Sigma$ or in the interior of $I \cup \beta$. Let Σ_0 be a compact subset of Σ, and let $m > -p$ be an integer. Then, if f is in $H_0^{(p)}(I)$ and σf is in $H^{(m)}(I)$, there exists a neighborhood V of Σ_0 such that the restriction of f to VI belongs to $H^{(2p+m)}(VI)$.

This lemma will be deduced from the following lemma:

20 Lemma. *Let the hypotheses of Lemma 19 be satisfied, and let k be an integer with $p \leq k < 2p+m$. Then, if there exists a neighborhood V_1 of Σ_0 such that $f|V_1I$ is in $H^{(k)}(V_1I)$, there also exists a neighborhood V_2 of Σ_0 such that $f|V_2I$ is in $H^{(k+1)}(V_2I)$.*

Proof that Lemma 20 implies Lemma 19. By the hypothesis of Lemma 19, we know that f is in $H^{(p)}(I)$. It then follows by Lemma 20 that there exists a neighborhood V_1 of Σ_0 such that $f|V_1I$ is in $H^{(p+1)}(V_1I)$, and, inductively, that there exists a neighborhood V_j of Σ_0 such that $f|V_jI$ is in $H^{(p+j)}(V_jI)$ for each $j \leq p+m$. Putting $j = p+m$, we have Lemma 19. Q.E.D.

Proof (*of Lemma* 20). By Lemma 8.24, it is sufficient to show that each point in q in Σ_0 has a neighborhood U such that $f|UI$ is in

$H^{(k+1)}(UI)$. Let $U_1 \subseteq I_0$ be a bounded neighborhood of q chosen so small that $\beta U_1 \subseteq \Sigma$, and so that there exists a mapping φ of U_1 onto the unit spherical neighborhood V of the origin such that

(i) φ is one-to-one, is infinitely often differentiable, and φ^{-1} is infinitely often differentiable;

(ii) $\varphi(\Sigma \cap U_1) = V \cap \{x \in E^n | x_1 = 0\}$;

(iii) $\varphi(q) = 0$.

By hypothesis, no point in V but the points $\varphi(\Sigma_0 \cap U_1)$ belong to the boundary of $\varphi(I \cap U_1)$, and no point in Σ_0 is interior to the closure of I. It follows that $\varphi(IU_1)$ must consist of one or another of the hemispheres $V_+ = \{x \in V | x_1 > 0\}$ or $V_- = \{x \in V | x_1 < 0\}$. For the sake of definiteness, we shall suppose that $\varphi(IU_1) = V_+$; the other case is equivalent to this by a change of variables.

Let U be a neighborhood of q whose closure is contained in U_1. Using Lemma 2.1, let ζ in $C_0^\infty(U_1)$ be a function which is identically equal to one in U. In order to show, as we must, that $f|UI$ is in $H^{(k+1)}(UI)$, it is sufficient to show that $\zeta f|U_1 I$ is in $H^{(k+1)}(U_1 I)$. The mapping $g \to \zeta g|U_1 I$ is, by Lemmas 3.22 and 3.23, a continuous mapping of $H^{(j)}(I)$ into $H^{(j)}(U_1 I)$ for each j. Since it sends $C_0^\infty(I)$ into $C_0^\infty(U_1 I)$, it is a continuous mapping of $H_0^{(j)}(I)$ into $H_0^{(j)}(U_1 I)$ for every $j > 0$ by Definition 3.15. Thus, $\zeta f|U_1 I$ is in $H_0^{(p)}(U_1 I)$. By hypothesis and by Lemma 3.10, there exists a neighborhood $V_3 \subseteq U$ of $\Sigma \cap U_1$ such that $\zeta f|V_3 IU_1$ is in $H^{(k)}(V_3 IU_1)$. By Leibniz' rule, we may write $\sigma\zeta f = \zeta\sigma f + \sigma_\zeta f$, where σ_ζ is a partial differential operator of order $2p-1$ at most. Using Lemma 3.22 and $k < 2p+m$, $\sigma\zeta f$ is in $H^{(k-2p+1)}(I)$. Therefore, from Lemma 3.23, $\sigma(\zeta f|U_1 I)$ is in $H^{(k-2p+1)}(U_1 I)$. We have thus verified that the elements $\zeta f|U_1 I$, $U_1 I$, $U_1 I_0$, and $m_1 = k-2p+1$ satisfy all the hypotheses placed on the elements f, I, I_0, and m in Lemma 20. Since we have only to show that ζf is in $H^{(k+1)}(UI)$ for some neighborhood $U \subseteq U_1$ of p, it is clear that we may assume without loss of generality that $U_1 = U_1 I_0 = I_0$. This will be assumed in what follows. Making use of the properties (i) and (ii) of the mapping φ, and of Lemmas 3.47 and 3.48, we see that we may assume without loss of generality that $I_0 = V$, $I = V_+$, and $q = 0$. All this will also be assumed in the following arguments.

Let σ_0 denote the partial differential operator

$$\sigma_0 = \sum_{|J|=2p} a(0)\partial^J.$$

Let C denote the cube

$$C = \{x \in E^n \mid 0 < x_1 < 2\pi, \qquad |x_j| < \pi, \qquad j = 2, \ldots, n\}$$

in E^n. By Lemma 18, there exists a constant K such that $\sigma_0 + K$ is a continuous one-to-one mapping of $H^{(p)}_{\pi_v, 0}(C) \cap H^{(k)}_{\pi_v}(C)$ onto $H^{(k-2p)}_{\pi_v}(C)$ and a continuous one-to-one mapping of $H^{(p)}_0(C) \cap H^{(k+1)}_{\pi_v}(C)$ onto $H^{(k+1-2p)}_{\pi_v}(C)$, having in each case a continuous inverse.

Let $\sigma f = g$. For each $\varepsilon > 0$, let S_ε be the map of E^n into itself defined by $S_\varepsilon x = \varepsilon x$. By Lemma 3.47, $f \circ S_\varepsilon^{-1}$ is a solution of the partial differential equation

$$(1) \qquad \sigma_\varepsilon(f \circ S_\varepsilon^{-1}) = \sum_{|J| \leqq 2p} a_J(\varepsilon x)\varepsilon^{2p-|J|}\partial^J(f \circ S_\varepsilon^{-1}) = \varepsilon^{2p}(g \circ S_\varepsilon^{-1})$$

in the domain $\varepsilon^{-1}I$. Moreover, by Lemma 3.48, $f \circ S_\varepsilon^{-1}$ is in $H^{(p)}_0(\varepsilon^{-1}I)$ and $f \circ S_\varepsilon^{-1}$ is in $H^{(k)}(\varepsilon^{-1}I)$, while $g \circ S_\varepsilon^{-1}$ is in $H^{(k-2p+1)}(\varepsilon^{-1}I)$. Let ε be so small that the domain $\varepsilon^{-1}I$ contains the cube C, and let ξ in $C_0^\infty(E^n)$ be identically equal to 1 in a neighborhood of $p = 0$ and identically equal to zero outside the unit sphere in E^n. Let ξ in $C_0^\infty(E^n)$ be identically equal to 1 in a neighborhood of the unit closed sphere in E^n and identically zero outside the sphere of radius 2 in E^n.

We wish to show that $f|U$ is in $H^{(k+1)}(UI)$ for some neighborhood U of the origin. We see from Lemmas 3.48 and 3.23 that it is sufficient to show that $(\xi(f \circ S_\varepsilon^{-1})|C)$ is in $H^{(k+1)}(C)$ for some sufficiently small ε. Let τ_ε denote the formal partial differential operator

$$\tau_\varepsilon = \sum_{|J|=2p} \xi(x)\big(a_J(\varepsilon x) - a_J(0)\big)\partial^J.$$

By Leibniz' rule it follows that for each $\varepsilon > 0$ and $h \in D(C)$ we may write

$$\sigma_\varepsilon \xi h = \xi\sigma_\varepsilon h + \partial_{\varepsilon,\xi} h,$$

where $\partial_{\varepsilon,\xi}$ is a partial differential operator of order at most $2p - 1$. $f \circ S_\varepsilon^{-1}|C$ is in $H^{(k)}(C)$ by hypothesis. Using Lemma 3.23, (1), and Lemma 3.22 we see that $f_\varepsilon = \xi(f \circ S_\varepsilon^{-1})|C$ satisfies a partial differential equation of the form

$$(2) \qquad \sum_{|J|=2p} a_J(\varepsilon x)\partial^J f_\varepsilon + K f_\varepsilon = g_\varepsilon,$$

where g_ε is in $H^{(k-2p+1)}(C)$. From Lemma 3.47 and our hypothesis, it follows that $f \circ S_\varepsilon^{-1}$ is in $H_0^{(p)}(\varepsilon^{-1}I)$. The mapping $g \to \xi g|C$ is a continuous mapping of $H^{(p)}(\varepsilon^{-1}I)$ into $H^{(p)}(C)$ by Lemmas 3.22 and 3.23, and evidently maps $C_0^\infty(I)$ into $C_0^\infty(C)$. It follows from Definition 3.15 that it maps $H_0^{(p)}(\varepsilon^{-1}I)$ into $H_0^{(p)}(C)$. Thus, $f_\varepsilon = \xi(f \circ S_\varepsilon^{-1})|C$ belongs to $H_0^{(p)}(C)$. By Lemma 3.13, f_ε vanishes outside the unit sphere in E^n. From Lemmas 3.10 and 3.9 equation (2) may be written as

$$(\sigma_0+K)f_\varepsilon+\tau_\varepsilon f = g_\varepsilon. \tag{3}$$

Since the coefficients of the operators σ_0+K and τ_ε are all periodic of period 2π in the variable $y = [x_2, \ldots, x_n]$, and since the carrier of the distributions f_ε and (by Lemma 3.13) g_ε both are contained in the unit sphere of E^n, it follows from Lemmas 3.33 and 3.24 (as generalized to $D_{\pi_y}(C)$) that we can extend f_ε and g_ε to elements $\hat{f}_\varepsilon$ and $\hat{g}_\varepsilon$ in $H_{\pi_y}^{(k)}(C)$ and $H_{\pi_y}^{(k-2p+1)}(C)$, respectively, having carriers, respectively, equal to the carriers of f_ε and g_ε, and that we then have

$$(\sigma_0+K)\hat{f}_\varepsilon+\tau_\varepsilon\hat{f}_\varepsilon = \hat{g}_\varepsilon. \tag{4}$$

Moreover, by Lemma 3.24 (as generalized to $D_{\pi_y}(C)$), $\hat{f}_\varepsilon$ is in $H_{\pi_y,0}^{(p)}(C)$. It is clear that all the coefficients of the operator τ_ε converge to zero uniformly in the topology of $C^\infty(E^n)$ as $\varepsilon \to 0$. For each $j > p$, let ν_j and $\hat{\nu}_j$ be the norm of the map

$$(\sigma_0+K)(H_{\pi_y}^{(j)}(C) \cap H_{\pi_y,0}^{(p)}(C)) \to H_{\pi_y}^{(j-2p)}(C)$$

and of its inverse, respectively. Next, using Lemma 3.28 (as generalized to $D_{\pi_y}(C)$), let ε be so small that the norm of τ_ε, both as a mapping of $H_{\pi_y}^{(k)}(C)$ into $H_{\pi_y}^{(k-2p)}(C)$ and of $H_{\pi_y}^{(k+1)}(C)$ into $H_{\pi_y}^{(k-2p+1)}(C)$, is less than $\min(\hat{\nu}_k, \hat{\nu}_{k+1})$. Then, by Lemma VII.3.4, the mapping

$$(I+\tau_\varepsilon(\sigma_0+K)^{-1}),$$

regarded either as a mapping of $H_{\pi_y}^{(k)}(C)$ or of $H_{\pi_y}^{(k+1)}(C)$ into itself, has a bounded, everywhere-defined inverse. We then have

$$\begin{aligned}\big((\sigma_0+K)+\tau_\varepsilon\big)&(\sigma_0+K)^{-1}(I+\tau_\varepsilon(\sigma_0+K)^{-1})^{-1} \\ &= (I+\tau_\varepsilon(\sigma_0+K)^{-1})(I+\tau_\varepsilon(\sigma_0+K)^{-1})^{-1} = I\end{aligned}$$

and

$$\begin{aligned}(\sigma_0+K)^{-1}\big(I+\tau_\varepsilon(\sigma_0+K)^{-1}\big)^{-1}(\sigma_0+K+\tau_\varepsilon)&\\ =(\sigma_0+K)^{-1}\big(I+\tau_\varepsilon(\sigma_0+K)^{-1}\big)^{-1}\big(I+\tau_\varepsilon(\sigma_0+K)^{-1}\big)(\sigma_0+K)&\\ =(\sigma_0+K)^{-1}(\sigma_0+K)=I,&\end{aligned}$$

whether σ_0+K and τ_ε are regarded as mappings of $H^{(p)}_{\pi_\nu,0}(C)\cap H^{(k)}_{\pi_\nu}(C)$ into $H^{(k-2p)}_{\pi_\nu}(C)$ or as mappings of $H^{(p)}_{\pi_\nu,0}(C)\cap H^{(k+1)}_{\pi_\nu}(C)$ into $H^{(k-2p+1)}_{\pi_\nu}(C)$. Thus $\sigma_0+K+\tau_\varepsilon$ is a one-to-one mapping both of $H^{(p)}_{\pi_\nu,0}(C)\cap H^{(k)}_{\pi_\nu}(C)$ onto $H^{(k-2p)}_{\pi_\nu}(C)$ and of $H^{(p)}_{\pi_\nu,0}(C)\cap H^{(k+1)}_{\pi_\nu}(C)$ onto $H^{(k-2p+1)}_{\pi_\nu}(C)$. Since $\hat{g}_\varepsilon$ is in $H^{(k-2p+1)}_{\pi_\nu}(C)$, it follows that there exists an element $\tilde{f}_\varepsilon$ in $H^{(p)}_{\pi_\nu,0}(C)\cap H^{(k+1)}_{\pi_\nu}(C)$ such that

$$\big((\sigma_0+K)+\tau_\varepsilon\big)\tilde{f}_\varepsilon=\hat{g}_\varepsilon. \tag{5}$$

Since by Lemma 3.18 we also have $\tilde{f}_\varepsilon$ in $H^{(p)}_{\pi_\nu,0}(C)\cap H^{(k)}_{\pi_\nu}(C)$, and since $(\sigma_0+K)+\tau_\varepsilon$ has just been seen to be a one-to-one mapping of this last space, it follows from (4) and (5) that we must have $\hat{f}_\varepsilon=\tilde{f}_\varepsilon$, so that $\hat{f}_\varepsilon$ is in $H^{(k+1)}_{\pi_\nu}(C)$. Therefore, by Lemma 3.28, f_ε is in $H^{(k+1)}(C)$, which proves the present lemma. Q.E.D.

The following elementary lemma provides a useful and interesting sidelight on the space $H^{(p)}_0(I)$.

21 LEMMA. *Let I be a domain in E^n whose boundary β contains a smooth surface Σ, and suppose that no point in Σ is in the closure of $\beta-\Sigma$ or in the interior of $I\cup\beta$. Let f be in $C^p(\bar{I})\cap H^p_0(I)$. Then $\big((\partial_\nu(\Sigma))^k f\big)(x)=0$ for all x in Σ and all k such that $0\le k\le p-1$.*

PROOF. By Definition 5.1 we have only to show that each point p in Σ has a neighborhood U such that $\{(\partial_\nu(\Sigma))^k f|UI\}(x)=0$ for $x\in\Sigma U$ and $0\le k\le p-1$. Let U be a neighborhood of p chosen so small that $\beta U\subseteq\Sigma$ and so that there exists a mapping η of U onto the unit spherical neighborhood V of the origin such that

(i) η is one-to-one, is infinitely often differentiable, and η^{-1} is infinitely often differentiable;

(ii) $\eta(\Sigma U)=V\cap\{x\in E^n|x_1=0\}$.

Since, by hypothesis, no point in V but the points $\eta(\Sigma U)$ belong to the boundary of $\eta(IU)$, and no point of $\eta(\Sigma U)$ is interior to the closure of $\eta(IU)$, it follows that $\eta(IU)$ must consist of one or the other of the hemispheres $V_+=\{x\in V|x_1>0\}$ and $V_-=\{x\in V|x_1<0\}$. For the sake of definiteness, we shall suppose that $\eta(\Sigma U)=V_+$. It now

follows from Lemmas 3.22, 3.28, and 5.4 that we may (and shall) assume without loss of generality that $I = V_+$ and $\Sigma = V_0 = \{x \in V | x_1 = 0\}$. Let φ be any function of the $n-1$ variables $y = [y_1, \ldots, y_{n-1}]$ belonging to $C_0^\infty(E^{n-1})$ and vanishing outside the unit sphere V_0 of E^{n-1}, and let ζ_φ be any function of the variable x_1 such that $\zeta_\varphi \in C_0^\infty(E^1)$, the function ψ_φ defined by $\psi_\varphi(x) = \zeta_\varphi(x_1)\varphi(x_2, \ldots, x_n)$ belongs to $C_0^\infty(V)$, and $\zeta_\varphi(x_1) = x_1^{p-1}$ for all sufficiently small x_1. Plainly $\psi_\varphi | V_+$ is in $C^p(V_+)$.

It follows on integrating by parts p times with respect to x_1 that

$$(*) \qquad \int_{V_+} h(x)\partial_1^p g(x)dx = (-1)^p \int_{V_+} \partial_1^p h(x)g(x)dx$$

for all g in $C_0^\infty(V_+)$ and h in $C^p(\bar{V}_+)$, and it follows by continuity, since $C_0^\infty(V_+)$ is by definition of $H_0^p(V_+)$ dense in $H_0^p(V_+)$, that this identity must hold for all g in $H_0^{(p)}(V_+)$ and h in $C^p(\bar{V}_+)$. Thus, putting $f = g$ and $h = \psi_\varphi$ in $(*)$, we have

$$\int_{V_+} f(x)\partial_1^p \psi_\varphi(x)dx = (-1)^p \int_{V_+} \partial_1^p f(x)\psi_\varphi(x)dx, \qquad \varphi \in C_0^\infty(V_0).$$

On the other hand, it follows on integrating by parts p times with respect to x_1 that

$$\int_{V_+} f(x)\partial_1^p \psi_\varphi(x)dx = (-1)^p \int_{V_+} \partial_1^p f(x)\psi_\varphi(x)dx$$
$$+(-1)^p(p-1)! \int_{V_0} f(0, y)\varphi(y)dy$$

for each φ in $C_0^\infty(V_0)$. Thus

$$\int_{V_0} \varphi(y)f(0, y)dy = 0, \qquad \varphi \in C_0^\infty(V_0),$$

so that by Lemma 2.2, $f(0, y) = 0$ for $y \in V_0$.

Next let $\hat{\zeta}_\varphi$ in $C_0^\infty(E^1)$ be such that the function $\hat{\psi}_\varphi$ defined by the equation $\hat{\psi}_\varphi(x) = \hat{\zeta}_\varphi(x_1)\varphi(x_2, \ldots, x_n)$ belongs to $C_0^\infty(V)$, and such that $\hat{\zeta}_\varphi(x_1) = x_1^{p-2}$ for all sufficiently small x_1. Then, integrating the left-hand side of the following equation by parts p times with respect to x_1, and using the fact that $f(0, y) = 0$, which has already been established, it follows that

$$\int_{V_+} f(x)\partial_1^p \hat{\psi}_\varphi(x)dx = (-1)^p \int_{V_+} \partial_1^p f(x)\hat{\psi}_\varphi(x)$$
$$+(-1)^{p-1}(p-2)! \int_{V_0} (\partial_1 f)(0, y)\varphi(y)dy$$

for each φ in $C_0^\infty(V_0)$. On the other hand, if we put $f = g$ and $h = \hat{\varphi}_\varphi$ in the equation $(*)$, we find that

$$\int_{V_+} f(x)\partial_1^p \hat{\varphi}_\varphi(x)dx = (-1)^p \int_{V_+} \partial_1^p f(x)\hat{\varphi}_\varphi(x)dx,$$

so that

$$\int_{V_0} (\partial_1 f)(0, y)\varphi(y)dy = 0, \qquad \varphi \in C_0^\infty(V_0).$$

Hence, by Lemma 2.2, $(\partial_1 f)(0, y) = 0$ for y in V_0. Proceeding inductively in this way, we find that $(\partial_1^k f)(0, y) = 0$ for $0 \leqq k \leqq p-1$. Q.E.D.

22 Corollary. *Let $j \geqq 0$ be an integer. Let the hypothesis of Lemma 19 be satisfied, let $f \in H_0^{(p)}(I) \cap H^{(m)}(I)$. Let $2p+m-[n/2] \geqq j+1$. Then there exists a neighborhood V of Σ_0 such that the restriction of f to VI has a continuous extension $\hat{f}$ to the closure $\overline{VI}$ of VI with $\hat{f}$ in $C^j(\overline{VI})$, and*

$$\{(\partial_\nu(\Sigma_0))^k f\}(x) = 0, \qquad x \in \Sigma_0, \qquad 0 \leqq k \leqq \min(j, p).$$

Proof. The first assertion follows from Lemma 19 and Sobolev's theorem (4.5), and the second from the first and from the preceding lemma. Q.E.D.

In the statement and proof of the next theorem, we shall use the notation $T(\tau)$ for the operator in $L_2(I)$ defined by the equations

$$\mathfrak{D}(T(\tau)) = H_0^{(p)}(I) \cap H^{(2p)}(I),$$
$$T(\tau)f = \tau f, \qquad f \in \mathfrak{D}(T(\tau)),$$

τ being a formal partial differential operator defined in a domain I of E^n.

The following theorem, of which Lemma 21 now enables us to give a short proof, amounts to a solution of the classical Dirichlet problem in a very general setting.

23 Theorem. *Let τ be an elliptic formal partial differential operator of even order $2p$ defined in a domain I_0 in E^n. Suppose that τ is of the form*

$$\tau = \sum_{|J| \leq 2p} a_J(x)\partial^J,$$

and that

$$(-1)^p \mathscr{R} \sum_{|J|=2p} a_J(x)\xi^J > 0, \qquad x \in I_0, \qquad \xi \in E^n, \qquad \xi \neq 0.$$

Let I be a bounded subdomain whose closure is contained in I_0. Suppose that the boundary of I is a smooth surface S, and that no point in S is interior to the closure of I. Let T and $\hat{T}$ be the operators in the Hilbert space $L_2(I)$ defined by the equations

$$[*] \quad \mathfrak{D}(T) = \mathfrak{D}(\hat{T}) = \{F \in C^\infty(\bar{I}) | f(x) = \partial_\nu(S)f(x) = \ldots = \partial_\nu^{p-1}(S)f(x) = 0, \quad x \in S\}$$

$$Tf = \tau f, \qquad \hat{T}f = \tau^* f. \qquad f \in \mathfrak{D}(T) = \mathfrak{D}(\hat{T}).$$

Let V and $\hat{V}$ denote the operators whose graphs are the closures of the graphs of T and of $\hat{T}$, respectively. Then

(i) $V^* = \hat{V}$. $\quad \hat{V}^* = V$.

(ii) $\sigma(V)$ *is a countable discrete set of points with no finite limit point.*

(iii) *If $\lambda \notin \sigma(V)$, $R(\lambda; V)$ is a compact operator.*

(iv) *If $\lambda \notin \sigma(V)$, $R(\lambda; V)$ is a continuous mapping of $H^{(m)}(I)$ into $H^{(m+2p)}(I)$ for every $m \geqq 0$.*

(v) *If Vf is in $H^{(m)}(I)$, where $m \geqq [n/2]-2p$, then f is in $C^{p-1}(\bar{I})$, and f satisfies the boundary conditions defining $\mathfrak{D}(T)$ stated in formula* [*].

PROOF. We shall show that $V = T(\tau)$ and $\hat{V} = T(\tau^*)$. Then parts (i), (ii), and (iii) of this theorem will follow immediately from Corollaries 14 and 11, while part (v) will follow from Corollary 22.

Assume for the moment that $V = T(\tau)$. Then to prove (iv), we may reason as follows. Let $m \geqq 0$ and $\lambda \notin \sigma(V) = \sigma(T(\tau))$. By Lemma 19, Theorem 2, and Lemma 3.24, $R(\lambda; V)$ maps $H^{(m)}(I)$ into $H^{(m+2p)}(I)$. For $n \geq 1$ let f_n be in $H^{(m)}(I)$; let $g_n = R(\lambda; V)f_n$; let $|f_n - f|_{(m)} \to 0$ and $|g_n - g|_{(m+2p)} \to 0$ as $n \to \infty$. Then, since $|g_n - g|_{(p)} \to 0$ as $n \to \infty$, and since $H_0^{(p)}(I)$ is a closed subspace of $H^{(p)}(I)$ (cf. Definition 3.15 (i) and (ii)), f is in $H_0^{(p)}(I)$. By Lemma 3.22, $(\lambda - \tau)f = g$. Thus f is in $\mathfrak{D}(T(\tau)) = \mathfrak{D}(V)$ and $(\lambda - T(\tau))f = g$. Hence, $R(\lambda; V)g = f$, proving that $R(\lambda; V)$ is a closed mapping of $H^{(m)}(I)$ into $H^{(m+2p)}(I)$. Statement (iv) now follows immediately from the closed graph theorem (II.2.4).

All that remains to complete the proof of the present theorem is the proof that $V = T(\tau)$ and $\hat{V} = T(\tau^*)$.

The proofs of the two statements $V = T(\tau)$ and $\hat{V} = T(\tau^*)$ are exactly parallel, so that we shall only consider the proof of $V = T(\tau)$. We shall show below that $\mathfrak{D}(T) \subseteq \mathfrak{D}(T(\tau))$, so that $T \subseteq T(\tau)$. Since $T(\tau)$ is closed by Corollary 14 and Lemma XII.1.6(a), it follows that $V \subseteq T(\tau)$. (In particular, it follows from this equation that the graph of V, i.e., the closure of the graph of T, is really the graph of a well-defined, that is, single-valued operator.) Next, using Corollary 11, choose some $\lambda \notin \sigma(T(\tau))$. Let f be in $\mathfrak{D}(T(\tau))$ and $(\lambda I - T(\tau))f = g$. Using Lemma 2.2, find a sequence $\{g_n\}$ of functions in $C^\infty(\bar{I})$ such that $g_n \to g$ in the topology of $L_2(I)$ as $n \to \infty$. Then, if $f_n = (\lambda I - T(\tau))^{-1} g_n$ we have $f_n \to f$ in the topology of $L_2(I)$ as $n \to \infty$. By Corollary 22, Lemma 19, and Sobolev's theorem (4.5), f_n is in $\mathfrak{D}(T)$. Hence f is in $\mathfrak{D}(V)$, proving that $T(\tau) \subseteq V$.

Thus to complete the proof of the present theorem it is sufficient to show that $\mathfrak{D}(T) \subseteq \mathfrak{D}(T(\tau))$. According to the definition of $T(\tau)$, this amounts to showing that $\mathfrak{D}(T) \subseteq H_0^{(p)}(I)$.

For each point p in S, choose a neighborhood U_p so small that there exists a mapping φ_p of U_p onto the open unit sphere Σ in E^n, having the following two properties:

(a) φ_p is one-to-one, φ_p is infinitely often differentiable, and φ_p^{-1} is infinitely often differentiable.

(b) $\varphi_p(p) = 0$, and $\varphi_p(S \cap U_p) = V \cap \{x \in E^n | x_1 = 0\}$.

Using the compactness of S, choose a finite collection $\{U_{p_i}\}$, $i = 1, \ldots, s$, of the neighborhoods U_p such that $\bigcup_{i=1}^s U_{p_i} \supseteq S$. Using Lemma 2.3, let $\{\zeta_i\}$, $i = 1, \ldots, s$, be a finite collection of functions in $C^\infty(E^n)$ such that each function ζ_i vanishes outside a compact subset C_i of some one of the neighborhoods $U_{p_i} = U_i$, and such that $\sum_{i=1}^s \zeta_i(x) = 1$ for each x in a neighborhood of S. Let f be in $\mathfrak{D}(T)$. Then, since $f - \sum_{i=1}^s \zeta_i f$ is in $C_0^\infty(I)$, it will follow that f is in $H_0^{(p)}(I)$ if we show that $\zeta_i f$ is in $H_0^{(p)}(I)$ for each $i = 1, \ldots, s$. That is, we must construct a sequence $\{g_n\}$ of elements of $C_0^\infty(I)$ such that $|g_n - \zeta_i f|_{(p)} \to 0$ as $n \to \infty$. The mapping $\varphi_i = \varphi_{p_i}$ maps the set $U_i I$ onto a set $\varphi_i(U_i I)$ whose boundary in Σ contains no points but $\Sigma \cap \{x \in E^n | x_1 = 0\}$. Since no point in the boundary of I is in the interior of the closure of I, it follows that $\varphi_i(U_i I)$ must consist of one or another of the hemispheres $\Sigma_+ = \{x \in \Sigma | x_1 > 0\}$ of $\Sigma_- = \{x \in \Sigma | x_1 < 0\}$. For the sake of definiteness, we shall take $\varphi_i(U_i I) = \Sigma_+$. Then, using Lemmas 3.22

(ii), 3.47, and 3.48, we see that to complete the proof of the present theorem we need only construct a sequence of functions $\hat{g}_n$ in $C_0^\infty(\Sigma_+)$, all vanishing outside a fixed compact subset of Σ, such that $|\hat{g}_n(\cdot)-\zeta_i(\varphi_{p_i}^{-1}(\cdot))f(\varphi_{p_i}^{-1}(\cdot))|_{(p)} \to 0$ as $n \to \infty$. By Lemma 15 and Lemma 5.4, there exists a sequence of functions $\tilde{g}_n$ in $C_0^\infty(\{x \in E^n | x_1 > 0\})$ such that $|\tilde{g}_n(\cdot)-\zeta_i(\varphi_{p_i}^{-1}(\cdot))f(\varphi_{p_i}^{-1}(\cdot))|_{(p)} \to 0$ as $n \to \infty$. Let ψ_i be a function in $C^\infty(E^n)$ equal to 1 for x in $\varphi_i(\bar{U}_i)$ and equal to zero for x outside some compact subset Σ_i of Σ_+ and put $\hat{g}_n = \tilde{g}_n \psi_i$. Lemma 3.22(ii) now gives the desired properties of $\hat{g}_n$. Q.E.D.

Remark. In the remaining theorems of the present section, we use the notation $T(\tau)$ for the operator defined by the equations

$$\mathfrak{D}(T(\tau)) = \mathfrak{D}(T_1(\tau)) \cap H_0^p(I); \qquad T(\tau)f = \tau f, \qquad f \in \mathfrak{D}(T(\tau)),$$

where τ is a formal partial differential operator of order $2p$ defined in a domain I. If the operator τ, the domain I, surface S, etc., satisfy the hypotheses of Theorem 23 (as they shall), then as shown in the preceding proof, $T(\tau)$ is identical with the operator V, defined as the closure of the operator V_0 specified by

$$\mathfrak{D}(V_0) = \{f \in C^\infty(\bar{I}) | f(x) = \partial_\nu(S)f(x) = \ldots = \partial_\nu^{p-1}(S)f(x) = 0, \quad x \in S\}$$
$$V_0 f = \tau f, \qquad f \in \mathfrak{D}(V_0).$$

The equation $T(\tau) = V$ will frequently be used in what follows.

24 Corollary *Let the hypotheses of the preceding theorem be satisfied. Let k be a positive integer with $2pk \geq [n/2]+1$. Then, if λ is in $\sigma(V)$, the operator $R_k = \{R(\lambda; V)\}^k$ is of Hilbert Schmidt class. If $2pk \geq [n/2]+s+1$, R_k is a continuous mapping of $L_2(I)$ into $C^s(\bar{I})$.*

Proof. By the preceding theorem and by Sobolev's theorem (4.5), R_k is a continuous mapping of $L_2(I)$ into $C^s(\bar{I})$, proving the second assertion. To prove the first, we may show that each continuous mapping of $L_2(I)$ into $C(\bar{I})$ is of Hilbert Schmidt class. Now, by Theorem IV.4.5 we may write $(K_k f)(x) = (f, \psi_x)$ for each x in $\bar{I}$, where ψ_x is in $L_2(I)$ and where, by the uniform boundedness theorem (II.3.20), there exists a finite constant M such that $|\psi_x| \leq M$ for each x in $\bar{I}$. Thus, if $\{\varphi_n\}$ is a complete orthonormal set in $L_2(I)$, we have

$$\begin{aligned}\sum_{n=1}^{\infty}|K_k\varphi_n|^2 &= \sum_{n=1}^{\infty}\int_I|(\varphi_n, \psi_x)|^2dx \\ &= \int_I\left\{\sum_{n=1}^{\infty}|(\varphi_n, \psi_x)|^2\right\}dx \\ &= \int_I|\psi_x|^2dx \leq \int_I Mdx < \infty\end{aligned}$$

by Theorem IV.4.13. Q.E.D.

The reader will observe that the somewhat complex analysis leading up to Theorem 23 has enabled us to treat only a very special set of boundary conditions for the formal elliptic operator τ. It is possible to extend the methods used to treat considerably more general sets of boundary conditions. However, in order not to extend the present discussion beyond all limits, we shall not enter into a discussion of this point.

It is worth stating the special form which Theorem 23 takes on if the formal partial differential operator is self adjoint.

25 THEOREM. *Let τ be an elliptic formal partial differential operator of even order $2p$ defined in a domain I_0 in E^n. Suppose that τ is formally self adjoint so that $\tau = \tau^*$. Let τ be of the form*

$$\tau = \sum_{|J| \leq 2p} a_J(x)\partial^J$$

where

$$(-1)^p \sum_{|J| = 2p} a_J(x)\xi^J > 0, \qquad x \in I_0, \qquad \xi \in E^n, \qquad \xi \neq 0.$$

Let I be a bounded subdomain whose closure is contained in I_0. Suppose that the boundary of I is a smooth surface S, and that no point in S is interior to the closure of I. Let T be the operator in the Hilbert space $L_2(I)$ defined by the equations

$$[*] \quad \mathfrak{D}(T) = \{f \in C^\infty(\bar{I}) | f(x) = \partial_\nu(S)f(x) = \ldots \partial_\nu^{p-1}(S)f(x) = 0, \quad x \in S\}$$

$$Tf = \tau f, \qquad f \in \mathfrak{D}(T).$$

Let V be the closure of T. Then.

(i) *the operator V is self adjoint;*

(ii) *the spectrum $\sigma(V)$ is a sequence of points $\{\lambda_n\}$ tending to ∞, and for λ in the resolvent set $R(\lambda; V)$ is a compact operator;*

(iii) *the operator V has a complete countable set $\{\varphi_n\}$ of eigenfunctions. Each eigenfunction satisfies the partial differential equation $\tau\varphi_n = \lambda_n\varphi_n$ in I, has infinitely many continuous derivatives in the closure of the domain I, and satisfies the boundary conditions defining $\mathfrak{D}(T)$ in formula* [*].

PROOF. Parts (i) and (iii) follow immediately from Theorem 23. Part (ii) will also follow immediately from Theorem 23 once we show that $\sigma(V)$ (which we know to be a sequence of real numbers without a finite limit point) is bounded below. This, however, follows immediately from Corollary 12 (cf. XII.7.2). Q.E.D.

26 COROLLARY. *Let the hypotheses of the preceding theorem be satisfied. Let α be any angular sector in the complex plane including the positive real axis and the countable set of real numbers $\sigma(V)$. Then,*

(i) *there exists a real number K such that*

$$|R(\lambda; V)f|_{(2p)} \leq K|f|, \qquad f \in L_2(I), \qquad \lambda \notin \alpha;$$

(ii) *if f is in $L_2(I)$, then $|R(\lambda; V)f|_{(2p)} \to 0$ as $|\lambda| \to \infty$, λ remaining in the complement of α;*

(iii) *if for each $\lambda \notin \alpha$, $R(\lambda; V)$ is regarded as mapping from $L_2(I)$ into $H^{(2p-1)}(I)$, then its norm approaches zero as $|\lambda| \to \infty$, λ remaining in the complement of α.*

PROOF. Let λ_0 be outside $\sigma(V)$. Then

$$(\lambda_0 I - V)R(\lambda; V) = (\lambda_0 - \lambda)R(\lambda; V) - I,$$

so that by Theorem XII.2.9(a), $|(\lambda_0 I - V)R(\lambda; V)f|$ has a uniform bound of the form $K|f|$ for $\lambda \notin \alpha$. Thus, by Theorem 25, $|R(\lambda; V)f|_{(2p)} \leq K'|f|$ for some $K' < \infty$ and all $\lambda \notin \alpha$. Again by Theorem XII.2.6(c) and the Lebesgue dominated convergence theorem, $|(\lambda_0 I - V)R(\lambda; V)f| \to 0$ as $|\lambda| \to \infty$, λ remaining in the complement of α, so that by Theorem 25 $|R(\lambda; V)f|_{(2p)} \to 0$ as $|\lambda| \to \infty$, λ remaining in the complement of α, which proves (ii). If (iii) is false, there exists a sequence $\{\lambda_n\}$ in the complement of α and a sequence $\{f_n\}$ of elements of $L_2(I)$ such that $|f_n| = 1$ and such that $|R(\lambda_n; V)f_n|_{(2p-1)}$ does not converge to zero. Since $|R(\lambda_n; V)f_n|_{(2p)}$ is uniformly bounded by (i), it follows from Corollary 4.11 that we may suppose without loss of generality that there is an element g in

$H^{(2p-1)}(I)$ such that $|g_n - g|_{(2p-1)} \to 0$, where we have put $g_n = R(\lambda_n; V)f_n$. By Theorem XII.2.9(a), $|g_n| \to 0$. Hence we must have $g = 0$, contradicting the statement that $|g_n|_{(2p-1)}$ does not converge to zero. Q.E.D.

27 Corollary. *Let τ be an elliptic formal partial differential operator of even order $2p$, defined in a domain I_0 in E^n. Suppose that*

$$\tau = \sum_{|J| \leq 2p} a_J(x)\partial^J,$$

that $a_J(x)$ is real if $|J| = 2p$, and that

$$(-1)^p \sum_{|J| = 2p} a_J(x)\xi^J > 0, \qquad x \in I_0, \qquad \xi \in E^n, \qquad \xi \neq 0.$$

Let I be a bounded open set whose closure $\bar{I}$ is contained in I_0. In addition, let I and V be as in Theorem 25. Then,

(i) *if α is any open angle in the complex plane containing the positive real axis, all but a finite number of the points in $\sigma(V)$ belong to α;*

(ii) *if α is as in* (i), *and, in addition, is assumed to contain all of $\sigma(V)$, then $|\lambda R(\lambda; V)|$ is bounded for λ in the complement of α, and $|R(\lambda; V)f| \to 0$ as $|\lambda| \to \infty$, λ remaining in the complement of α.*

Proof. By the definition of τ^* and by Leibniz' formula, we have

$$\tau^* = \sum_{|J| \leq 2p} a_J^*(x)\partial^J,$$

where $a_J^* = a_J$ for $|J| = 2p$. Thus if we write $\sigma = (\tau + \tau^*)/2$ and $\zeta = (\tau - \tau^*)/2i$, we have $\tau = \sigma + i\zeta$, where σ is formally self adjoint and satisfies the hypotheses of the preceding corollary, and where ζ is a formal partial differential operator of order at most $2p - 1$. Let $T(\sigma)$ be defined by the equations

$$\mathfrak{D}(T(\sigma)) = \mathfrak{D}(T_1(\sigma)) \cap H_0^{(p)}(I); \qquad T(\sigma)f = \sigma f, \quad f \in \mathfrak{D}(T(\sigma)),$$

and let $T(\tau)$ be defined by the equations

$$\mathfrak{D}(T(\tau)) = \mathfrak{D}(T_1(\tau)) \cap H_0^{(p)}(I); \qquad T(\tau)f = \tau f, \quad f \in \mathfrak{D}(T(\tau)).$$

Using the preceding corollary, and the remark immediately following the proof of Theorem 23, let $\Lambda > 0$ be so large that $|\zeta R(\lambda; T(\sigma))| < \frac{1}{2}$ for λ in α and $|\lambda| > \Lambda$. Then, by Lemma VII.3.4, the operator $B_\lambda =$

$(I - i\zeta R(\lambda; T(\sigma)))^{-1}$ exists as an everywhere defined mapping in $L_2(I)$, and $|B_\lambda| \leq 2$ for $\lambda \notin \alpha$, $|\lambda| > \Lambda$. We have

$$(\lambda - (\sigma + i\zeta))R(\lambda; T(\sigma))B_\lambda f = (I - i\zeta R(\lambda; T(\sigma)))B_\lambda f = f$$

for each f in $L_2(I)$, and $R(\lambda; T(\sigma))B_\lambda f \in \mathfrak{D}(T(\sigma)) \cap H_0^{(p)}(I)$. Thus (cf. the definition of $T(\tau)$ given above, and the definition of $T_1(\tau)$ given in the next to last paragraph of Section 3) $R(\lambda; T(\sigma))B_\lambda f$ is in $\mathfrak{D}(T(\tau))$ for each f in $L_2(I)$, and

$$(\lambda I - T(\tau))R(\lambda; T(\sigma))B_\lambda f = f, \qquad f \in L_2(I), \qquad \lambda \notin \alpha, \qquad |\lambda| > \Lambda.$$

If f is in $\mathfrak{D}(T(\tau))$, then by Theorem 23, f is in $H^{(2p)}(I)$. Hence, since f is also in $H_0^{(p)}(I)$ by the definition of $T(\tau)$, we have f in $\mathfrak{D}(T(\sigma))$ and

$$\begin{aligned} R(\lambda; T(\sigma))B_\lambda(\lambda I - T(\tau))f &= R(\lambda; T(\sigma))B_\lambda(\lambda I - \sigma - i\zeta)f \\ &= R(\lambda; T(\sigma))B_\lambda(\lambda - \sigma - i\zeta)(\lambda I - T(\sigma))^{-1}(\lambda I - T(\sigma))f \\ &= R(\lambda; T(\sigma))B_\lambda(I - i\zeta R(\lambda; T(\sigma)))(\lambda I - T(\sigma))f \\ &= R(\lambda; T(\sigma))(\lambda I - T(\sigma))f = f, \qquad \lambda \in \alpha, \qquad |\lambda| > \Lambda. \end{aligned}$$

This shows that for $\lambda \notin \alpha$, $|\lambda| > \Lambda$, we have $\lambda \notin \sigma(T(\tau))$ and $R(\lambda; T(\tau)) = R(\lambda; T(\sigma))B_\lambda$. The present corollary follows immediately from Theorem 23, from the remark immediately following the proof of Theorem 23, and from Theorems XII.2.9(a) and 25. Q.E.D.

The following very interesting theorem gives a general completeness principle for the eigenfunctions of nonselfadjoint elliptic boundary value problems.

28 THEOREM. (Browder completeness theorem) *Let τ be an elliptic formal partial differential operator of even order $2p$, defined in a domain I_0 in E^n. Suppose that*

$$\tau = \sum_{|J| \leq 2p} a_J(x)\partial^J,$$

that $a_J(x)$ is real if $|J| = 2p$, and that

$$(-1)^p \sum_{|J|=2p} a_J(x)\xi^J > 0, \qquad x \in I_0, \qquad \xi \in E^n, \qquad \xi \neq 0.$$

Let I be a bounded open set whose closure $\bar{I}$ is contained in I_0. Suppose that I is bounded by a smooth surface Σ. Let V_0 be the extension of $T_0(\tau)$ defined by the equations

$$\mathfrak{D}(V_0) = \{f \in C^{2p}(I) \mid f(x) = \partial_\nu(\Sigma)f(x) = \dots = \partial_\nu^{p-1}(\Sigma)f(x) = 0, \quad x \in \Sigma\}$$

$$V_0 f = T_1(\tau)f, \qquad f \in \mathfrak{D}(V_0).$$

Let V be the closure of V_0. Then $\sigma(V)$ is a discrete set with no finite limit points, $R(\lambda; V)$ is a compact operator for λ in the resolvent set of V, and the set of functions f in $L_2(I)$ satisfying an equation

$$(V - \mu I)^k f = 0$$

for some integer $k \geqq 1$ is fundamental in $L_2(I)$. Each such function f belongs to the intersection of $C^\infty(\bar{I})$ and $\mathfrak{D}(V_0)$.

Proof. Passing without loss of generality from τ to $\tau + \lambda$, we may assume that $0 \notin \sigma(V)$. Let $V_1 = V^m$, where we choose m so large that $2pm \geqq [n/2]+1$. Then, by Corollary 24, V_1^{-1} is an operator of Hilbert-Schmidt class. Let $\omega_1, \dots, \omega_m$ be the m-th roots of unity. Then, if $\lambda\omega_i \in \sigma(V)$, $i = 1, \dots, m$, we have $\lambda^m \notin \sigma(V_1)$ and

$$R(\lambda^m, V_1) = R(\omega_1\lambda, V) \dots R(\omega_m\lambda, V),$$

by Definition VII.9.6 and Theorem VII.9.8. It follows from Corollary 27 that if $\lambda^n \to \infty$ along a ray $\{\mu = re^{i\theta}, r > 0\}$, where $\theta \neq 0$, then λ^n is eventually in $\sigma(V_1)$, while $R(\lambda^n; V_1)$ becomes and remains bounded. Thus, by applying Corollary XI.6.31 to V_1, we find that the set of functions f which satisfy an equation of the form

$$(1) \qquad (V-\mu\omega_1)^k \dots (V-\mu\omega_m)^k f = 0$$

for some complex μ and some integer $k \geq 1$ is fundamental in $L_2(I)$. If f satisfies (1), it is evident (by induction, cf. VII.9.6) that $f \in \bigcap_{l>1} \mathfrak{D}(V^l)$. Since the polynomials $p_j(z) = \prod_{i \neq j} (z - \mu\omega_i)^k$, $j = 1, \dots, m$, have no non trivial common factor, there exist polynomials q_j, $j = 1, \dots, m$, such that $\sum_{j=1}^m p_j(z)q_j(z) = 1$. Then

$$f = \sum_{j=1}^{m} q_j(V)p_j(V)f.$$

Let $f_j = q_j(V)p_j(V)f$. Then $f = \sum_{j=1}^m f_j$, and $(V - \mu\omega I)^k f = 0$ for $j = 1, \dots, m$. Hence the set of functions g which satisfy an equation $(V-\mu I)^k g = 0$ for some complex μ is fundamental in $L_2(I)$.

Any such function evidently (by induction, cf. VII.9.6) belongs to $\bigcap_{l \geqq 1} \mathfrak{D}(V^l)$. Hence, by Theorem 23, any such function belongs to $C^\infty(I)$. Q.E.D.

7. Linear Hyperbolic Equations and the Cauchy Problem

The present section is devoted to the proof (as simplified by P. Lax) of the interesting and important theorem of K. O. Friedrichs on *symmetric hyperbolic systems.* As an example of the way in which this theorem may be applied, consider the Cauchy problem:

$$\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\right) f(x_1, x_2) = 0, \tag{1}$$

$$f(x_1, 0) = g(x_1), \qquad \frac{\partial}{\partial x_2} f(x_1, 0) = h(x_1),$$

for the (hyperbolic) formal partial differential operator

$$L_1 = \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}. \tag{2}$$

If we introduce the auxiliary function

$$\hat{f}(x_1, x_2) = \left[\left(\frac{\partial}{\partial x_1}\right) + \left(\frac{\partial}{\partial x_2}\right)\right] f(x_1, x_2),$$

the Cauchy problem (1) may be written in the form

$$\begin{aligned} &\frac{\partial}{\partial x_1} f(x_1, x_2) = -\frac{\partial}{\partial x_2} f(x_1, x_2) + \hat{f}(x_1, x_2) \\ &\frac{\partial}{\partial x_1} \hat{f}(x_1, x_2) = \frac{\partial}{\partial x_2} \hat{f}(x_1, x_2) \\ &f(x_1, 0) = g(x_1), \quad \hat{f}(x_1, 0) = g'(x_1) + h(x_1). \end{aligned} \tag{3}$$

Introducing the vector $v = [f, \hat{f}]$, the vector $v_0 = [g, g' + h]$, the real Hermitian matrix

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4}$$

and the matrix

$$B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

the system (3) may be written as

$$\begin{aligned} &\frac{\partial}{\partial x_1} v(x_1, x_2) = A \frac{\partial}{\partial x_2} v(x_1, x_2) + Bv(x_1, x_2), \\ &v(x, 0) = v_1(x). \end{aligned} \tag{5}$$

Thus our initial problem (1) can be written in the form (5) to which the Friedrichs theorem applies. A number of other important partial differential equations and systems of partial differential equations arising in physics can either be written in the form (5), or have this form to begin with. Notable among these is the *Maxwell system* of electrodynamics (written in natural units):

$$\frac{\partial V}{\partial x_0} = A_1\frac{\partial V}{\partial x_1} + A_2\frac{\partial V}{\partial x_2} + A_3\frac{\partial V}{\partial x_3}, \tag{6}$$

$$V(0, x_1, x_2, x_3) = V_0(x_1, x_2, x_3),$$

where $V = [V_1, V_2, V_3]$ is a complex three-dimensional vector equal to the sum of the "electric" vector and the imaginary unit i times the magnetic vector, and where the matrices A_1, A_2, and A_3 are given by the formulae

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad A_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \quad A_3 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Both the above examples are special cases of the Cauchy problem for the general type of first order system described in the following theorem.

1 Theorem. (Friedrichs). *Let $n \geq 1, m \geqq 1$. Let $A_i(x; s)$, $i = 1, \ldots, n$, and $B(x; s)$ be a set of $m \times m$ dimensional matrices defined in $(n+1)$-dimensional Euclidean space (here we adopt the notational convention $x \in E^n$, $s \in E^1$ described in Section 2) and infinitely often differentiable there. Suppose that $A_j(x; s)$ is uniformly bounded and Hermitian for $1 \leq j \leq n$ and $[x, s]$ in E^{n+1}. Let $V_0(x)$ be an m-vector valued function defined in E^n and infinitely often differentiable there. Then there exists a unique m-vector valued function $V(x; s)$, defined and infinitely often differentiable in E^{n+1}, such that*

(a) $$\frac{\partial}{\partial s}V(x; s) = \sum_{j=1}^{n} A_j(x; s)\partial_{x_j}V(x; s) + B(x; s)V(x; s), \qquad [x, s] \in E^{n+1},$$

(b) $$V(x; 0) = V_0(x), \qquad x \in E^n.$$

Remark 1. It follows from Theorem 1 by the argument of the first part of Section 1 that we have local dependence of the solution

on the initial data in the sense explained in that section. We shall see, however, that this fact is needed in the course of the proof of Theorem 1, and shall prove it by a direct method where it is needed.

Remark 2. The theorem is false if no boundedness restriction is imposed on the coefficient matrices A_j, as the elementary system

$$\partial_y f(x, y) = -e^{-x}\partial_x f(x, y) + g(x, y)$$
$$\partial_y g(x, y) = \partial_x g(x, y)$$

shows. Here any pair of functions $f(x, y)$, $g(x, y)$ of the form $f(x, y) = h(y - e^x)$, $g(x, y) = 0$, is a solution of the system. Suppose that we let $h_N(s) = \varphi(Ns)$, where φ is a function in $C^\infty(-\infty, +\infty)$ which vanishes for $|s| \geq 1$, but not for $s = 0$. Then, if $f_N(x, y) = h_N(y - e^x)$ and $g_N(x, y) = 0$, we have $f_N(0, 1) = 1$ for all N, while $h_N(x, 0) = \varphi(Ne^x) = 0$ for $x > -\log N$. Thus the supposed validity of the existence and uniqueness theorem in this case would violate the result on local dependence proved in the introduction to the present chapter (Section 1)

PROOF (*of Theorem* 1). The proof will be given in a series of steps, some of which will be proofs of auxiliary assertions, others of which will be demonstrations that the proof of certain of these auxiliary assertions can be reduced to the proof of certain other auxiliary assertions.

Before beginning the proof, however, let us agree upon a number of notational and terminological conventions. In what follows, we shall use the word "function" either for a function with real or complex values, or for a function whose values are m-vectors, that is, for elements of complex m-dimensional unitary space U^m. The use of the term "function" intended in those instances to follow in which both possibilities are not admissible will either be clear from context, or will be specified in each particular case. In case it is desired to emphasize the fact that a space of functions under consideration consists of vector-valued functions, a caret will be added over the symbol for the corresponding space of scalar-valued functions. Thus, if C_1 denotes the cube

$$C_1 = \{y \in E^{n+1} \mid |y_j| \leq \pi,\ j = 1, \ldots, n+1\}$$

in E^{n+1}, then $\hat{C}^\infty(C_1)$ will denote the space of all infinitely often

differentiable m-vector valued functions defined in C_1. Similarly, $\hat{C}_\pi^\infty(C_1)$ and $\hat{C}_0^\infty(C_1)$ will denote the subspaces of $\hat{C}^\infty(C_1)$ consisting of all functions which are multiply periodic of period 2π and of all functions which vanish outside a compact subset of the interior of C_1, respectively. If $\mathring{C}_1$ is the interior of C, then $\hat{D}(\mathring{C}_1)$ and $\hat{D}_\pi(C_1)$ denote the sets of all linear functionals on $\hat{C}_0^\infty(C_1)$ and $\hat{C}_\pi^\infty(C_1)$, respectively. These functionals are assumed to be continuous in the sense that $F(f_n) \to F(f)$ whenever $f_n, f \in \hat{C}_0^\infty(C_1)$ and $f_n \rightrightarrows f$ in the topology of $\hat{C}^\infty(C_1)$ (and, in the case of $\hat{D}(\mathring{C}_1)$, all the functions f_n and f vanish outside a common compact subset of C_1). We take the topology of $\hat{C}^\infty(C_1)$ mentioned in the previous sentence to be the topology determined by the metric derived from the norm

$$|f| = \sum_{k=0}^{\infty} \sum_{\substack{j \geq 0 \\ |J|+j \leq k}} \frac{1}{2^k k!} \frac{\mu_{J,j}(f)}{1+\mu_{J,j}(f)},$$

where

$$\mu_{J,j}(f) = \sup_{y \in C_1} |\partial_s^j \partial^J f(y)| \qquad \text{and} \qquad y = [x_1, \ldots, x_n; s].$$

Let $e_1, \ldots, e_m$ be an orthonormal basis for U^m chosen once for the rest of the present proof. Then, if f is in $C_\pi^\infty(C_1)$, $e_j f$ is in $\hat{C}_\pi^\infty(C_1)$. Thus, if F is in $\hat{D}_\pi(C_1)$, the expression $F_j(f) = F(e_j f)$ evidently defines an element of $D_\pi(C_1)$. It is easily seen that the mapping $F \to [F_1, \ldots, F_m]$ is a one-to-one linear bicontinuous mapping of $\hat{D}_\pi(C_1)$ onto the direct sum of m replicas of $D_\pi(C_1)$. If we expand the vector function f into its components f_j, $j = 1, \ldots, n$, so that $f = [f_1, \ldots, f_m]$, then it is easily seen that $F(f) = \sum_{i=1}^m F^{(j)}(f_j)$. The elements $F^{(j)}$ in $D_\pi(C_1)$ will be called, correspondingly, the components of the element F in $\hat{D}_\pi(C_1)$. If all the components of F belong to $H_\pi^{(p)}(C_1)$, we shall write $F \in \hat{H}_\pi^{(p)}(C_1)$, and for two such F, G in $\hat{H}_\pi^{(p)}(C_1)$ we shall write $(F, G)_{(k)} = \sum_{j=1}^m (F^{(j)}, G^{(j)})_{(k)}$ and $|F|_{(k)} = ((F, F)_k)^{1/2}$. Thus $\hat{H}_\pi^{(p)}(C_1)$ is evidently a complete Hilbert space. Using this construction of components for an element of $\hat{D}_\pi(C_1)$, the whole theory of distributions developed in Sections 3 and 4 carries over to our present "vector-valued" setting; we shall use the "vector-valued" analogues of the results and definitions of Sections 3 and 4 without giving special proofs and without repeating the details of definitions, but merely by citing the relevant "scalar-valued"

theorems, lemmas, and definitions. Definitions will sometimes be cited implicitly by the device, explained above, of adding a caret over the symbol for a space of "scalar-valued" functions, distributions, etc., to denote the corresponding space of "vector-valued" functions, distributions, etc. The routine task of establishing the appropriate "vector-valued" modifications of the distribution-theoretic theorems, lemmas, and definitions as they are needed is left to the reader as an exercise.

Finally, let us agree to write τ for the formal differential operator defined for each f in $\hat{D}(E^n)$ by the equation

$$\tau f = \partial_s f - \sum_{i=1}^{n} A_i(\cdot)\partial_{x_i} f - B(\cdot)f(\cdot).$$

After this brief digression on notation and terminology we begin the series of steps which will constitute the proof of Theorem 1.

(A) The first step is to establish the following statement.

(i) For any r there exists a finite constant $K_0(r) > r$ so large that the equations

$$(\tau f)(x; s) = 0 \qquad |[x, s]| < K_0(r)$$

and

$$f(x; 0) = 0, \qquad |x| < K_0(r),$$

supposed to be valid for a function f in $\hat{C}^1(E^{n+1})$, imply that $f(y) = 0$ for $|y| < r$.

To prove (i), we argue as follows. Let f satisfy the hypotheses of (i), and let $F(x; s) = f(x; \varphi(x)s)$, where φ is a non-negative function in $C^\infty(E^n)$ which will be chosen below; we suppose however that $0 \leq \varphi(x) \leq 1$. Then $\partial_{x_i} F = \partial_{x_i} f + (\partial_{x_i}\varphi)\partial_s f$ and $\partial_s F = \varphi\partial_s f$. Thus, F satisfies the (system of) equation(s)

$$(1) \quad \Big(I + \sum_{i=1}^{n} A_i(x; \varphi(x)s)\partial_{x_i}\varphi(x)\Big)\partial_s F(x; s)$$
$$= \sum_{i=1}^{n} \big(\varphi(x)A_i(x; \varphi(x)s)\big)\partial_{x_i} F(x; s) + \varphi(x)B\big(x; \varphi(x)s\big)F(x; s)$$

in the set of points $[x, s]$ satisfying $|[x, \varphi(x)s]| < K_0(r)$. Let

$$\mu = \sup_{[x, s] \in E^n} |A_i(x; s)|.$$

Let ψ be a non-negative function in $C^\infty(E^n)$ such that $\psi(x) \leq 1$

for all x, $\psi(x) = 1$ for $|x| \leq 1$, and $\psi(x) = 0$ for $|x| > \frac{3}{2}$. Then, if ε is sufficiently small, $|\mu\varepsilon(\partial_{x_i}\psi)(\varepsilon x)| < 1/2n$ for all x. Choose some $\varepsilon < r^{-1}$ for which this is the case, and put $\varphi(x) = \psi(\varepsilon x)$ and $K_0(r) = 10\, n\varepsilon^{-1}$, so that $|r| < K_0(r)/10n$, and so that $\varphi(x) = 1$ for $|x| \leq r$. Put

$$H(x; s) = \sum_{i=1}^{n} A_i(x, \varphi(x)s)\,\partial_{x_i}\varphi(x)$$

$$\tilde{A}_i(x; s) = \varphi(x)A_i(x; \varphi(x)s); \qquad \tilde{B}(x; s) = \varphi(x)B(x; \varphi(x)s).$$

Then clearly $H(x; s)$ is Hermitian and $|H(x; s)| < \frac{1}{2}$, while F satisfies the (system of) equation(s)

$$\text{(2)} \quad (I - H(y))\,\partial_s F(y) = \sum_{i=1}^{n} \tilde{A}_i(y)\partial_{x_i}F(y) + \tilde{B}(y)F(y), \qquad |y| < K_0(r).$$

Moreover, it is clear that $F(x; s) - f(x; s) = 0$ if $s = 0$ and $|x| < K_0(r)$, and also that $F(x; s) = f(x; 0) = 0$ if $\frac{3}{2}\varepsilon^{-1} < |x| < K_0(r)$, i.e., if $3K_0(r)/20n < |x| < K_0(r)$. Since $H(x; s)$ is Hermitian and $|H(x; s)| < \frac{1}{2}$, it follows from Lemmas VII.3.4 and VII.3.11, and from Corollary X.2.8 and Theorem X.4.2, that $I - H(x; s)$ has a positive square root $H_1(x; s)$ which is also Hermitian and has a Hermitian inverse. We wish to show that this square root $H_1(x; s)$ is dependent on the parameters x, s in an infinitely often differentiable fashion. To do this, first note that by Definition VII.3.9,

$$(*) \qquad H_1(x; s) = \int_{\Gamma} (\lambda I - I + H(x; s))^{-1}\lambda^{1/2}\,d\lambda,$$

where the contour Γ of integration can be taken to be any closed contour lying in the right half plane and enclosing the spectrum of $I - H(x; s)$ exactly once in the customary positive sense of complex function theory. Since by Lemmas VII.3.4 and VII.3.8 the spectrum of $I - H(x; s)$ is contained in the disc

$$\{z \mid |z-1| \leq \tfrac{1}{2}\},$$

we can take Γ to be the circle

$$\{\lambda \mid |\lambda - 1| = \tfrac{3}{4}\}.$$

Then, from Lemma VII.3.4,

$$(\lambda I - I + H(x; s))^{-1} = \sum_{k=0}^{\infty} \frac{(H(x; s))^k}{(\lambda - 1)^{k+1}}.$$

Since $|H(x; s)| \leq \frac{1}{2}$ and $|\lambda - 1| = \frac{3}{4}$ for λ in Γ, this series, together with the series obtained from it by arbitrarily many term-by-term differentiations with respect to x, s, or λ, converges uniformly for all x and s and all λ in Γ, and hence defines a function which, in its dependence on the three parameters x, s, and λ, is infinitely often differentiable. It therefore follows immediately from formula (*) that $H_1(x; s)$ is dependent on the parameters x, s in an infinitely often differentiable fashion.

Let $G(y) = H_1(y)F(y)$. Then from (2)

$$
\begin{aligned}
(3)\quad \partial_s G(y) = \partial_s H_1(y)F(y) + H_1(y)\partial_s F(y) &= \big(\partial_s H_1(y)\big)F(y) \\
&\quad + H_1(y)^{-1}\sum_{i=1}^{n} \tilde{A}_i(y)\partial_{x_i}\big(H_1(y)^{-1}G(y)\big) \\
&\quad + H_1(y)^{-1}\tilde{B}(y)H_1(y)^{-1}G(y) \\
= \sum_{i=1}^{n} H_1(y)^{-1}\tilde{A}_i(y)H_1(y)^{-1}\partial_{x_i}G(y) &+ \big\{\big(\partial_s H_1(y)\big)H_1(y)^{-1} \\
&\quad + H_1(y)^{-1}\sum_{i=1}^{n} \tilde{A}_i(y)\big(\partial_{x_i} H_1(y)^{-1}\big) \\
&\quad + H_1(y)^{-1}B(y)H_1(y)^{-1}\big\}G(y),
\end{aligned}
$$

so that, putting

$$
\begin{aligned}
\mathring{A}_i(y) = H_1(y)^{-1}\tilde{A}_i(y)H_1(y)^{-1}; \qquad \mathring{B}(y) &= \big\{\big(\partial_s H_1(y)\big)H_1(y)^{-1} \\
+ H_1(y)^{-1}\sum_{i=1}^{n} \tilde{A}_i(y)\big(\partial_{x_i} H_1(y)^{-1}\big) &+ H_1(y)^{-1}B(y)H_1(y)^{-1}\big\}
\end{aligned}
$$

and putting σ equal to the formal partial differential operator

$$
\sigma = \sum_{i=1}^{n} \mathring{A}_i(y)\partial_{x_i} + \mathring{B}(y),
$$

we have

$$
(4)\qquad \partial_s G(x; s) = \sigma G(x; s), \qquad |x| < \tfrac{1}{2}K_0(r), \qquad |s| < \tfrac{1}{2}K_0(r).
$$

Since the matrices $\tilde{A}_i(y)$ and $H_1(y)$ are Hermitian, the matrices $H_1(y)^{-1}$ and $\mathring{A}_i(y) = H_1(y)^{-1}\mathring{A}_i(y)H_1(y)^{-1}$ are Hermitian. (Cf. XII.1.6(a) and (c)).

Now, if λ is an arbitrary real constant, the function $G_\lambda(x; s)$ defined by the equation $G_\lambda(x; s) = e^{-\lambda s}G(x; s)$ satisfies the condition

$$\partial_s G_\lambda(x; s) = (\sigma - \lambda) G(x; s).$$

Moreover, since $F(x; s) = 0$ if $s = 0$ and $|x| < K_0(r)$, and also if $K_0(r) > |x| > (3/20n)K_0(r)$, it follows immediately that $G_\lambda(x; s) = 0$ if $s = 0$ and $|x| < K_0(r)$, and also if $K_0(r) > |x| > (3/20n)K_0(r)$. Let D_r denote the cube

$$D_r = \left\{ x \in E^n \,\middle|\, |x_j| \leq \frac{1}{2n} K_0(r), \quad j = 1, \ldots, n \right\}$$

in Euclidean n-space. Then $G_\lambda(x; s)$ vanishes for x in a neighborhood of the boundary of D_r. Hence, integrating by parts, we find from (4) that

$$\begin{aligned}
(5) \quad \frac{d}{ds}\int_{D_r} |G_\lambda(x; s)|^2 dx &= \int_{D_r} \sum_{j=1}^{n} \left\{ (\mathring{A}_j(x; s)\partial_{x_j} G_\lambda(x; s), G_\lambda(x; s)\right\} \\
&\quad + \left\{ \left(\mathring{A}_j(x; s) G_\lambda(x; s),\ \partial_{x_j} G_\lambda(x; s)\right)\right\} dx \\
&\quad + \int_{D_r} \left\{ \left(\mathring{B}(y) G_\lambda(y), G_\lambda(y)\right) + \left(G_\lambda(y), \mathring{B}(y) G_\lambda(y)\right)\right\} dx \\
&\quad - 2\lambda \int_{D_r} |G_\lambda(x; s)|^2 dx \\
= \int_{D_r} \Bigg(\Bigg\{ \mathring{B}(x; s) + \mathring{B}^*(x; s) &+ \sum_{j=1}^{n} \partial_{x_j} \mathring{A}_j(x; s) \Bigg\} G_\lambda(x; s), G_\lambda(x; s) \Bigg) dx \\
&\quad - 2\lambda \int_{D_r} |G_\lambda(x; s)|^2 dx \leq (\mathring{\alpha} - 2\lambda) \int_{D_r} |G_\lambda(x; s)|^2 dx, \ |s| < \tfrac{1}{2} K_0(r).
\end{aligned}$$

where

$$(6) \qquad \alpha = \sup_{\substack{x \in D_r \\ |s| \leq 2r}} \left| \mathring{B}(x; s) + \mathring{B}^*(x; s) - \sum_{j=1}^{n} \partial_{x_j} \mathring{A}_j(x; s) \right|.$$

Thus, if $2\lambda > \mathring{\alpha}$, the function $\int_{D_r} |G_\lambda(x; s)|^2 dx$ is a decreasing function of s; since it vanishes for $s = 0$, it must vanish identically for $|s| < K_0(r)/2$. Hence $G_\lambda(y) = G_\lambda(x; s) = 0$ for x in D_r and $|s| < K_0(r)/2$, so that $G_\lambda(y) = F(y) = 0$ for x in D_r and $|s| < K_0(r)/2$. Thus $F(y) = 0$ for x in D_r and $|s| < K_0(r)/2$. If $|y| = |[x, s]| < r$, then, since $|r| < K_0(r)/10n$, $|x| < r$ and $|s| < K_0(r)/2$, so that x is in D_r and $F(y) = 0$. Moreover, as remarked above, $\varphi(x) = 1$ for $|x| < r$. Thus

$F(x; s) = f(x; \varphi(x)s) - f(x; s)$ for $\|[x, s]\| < r$. Hence we find that if $|y| < r$, $f(y) = 0$, and statement (i) is fully proved.

(B) The uniqueness of the function V of the theorem is an evident consequence of statement (i). Moreover, statement (i) enables us to reduce the proof of the existence of the function V to the proof of the following statement.

(ii) For each $r > 0$ and $p \geqq 1$, there exists a function V_r^p in $\hat{C}^p(E^n)$, such that $\tau V_r^p(x; s) = 0$ for $\|[x, s]\| < r$ and $V_r^p(x; 0) = V_0(x)$ for $|x| < r$.

Indeed, assuming (ii), and letting $K_0(r)$ be the function of r introduced in statement (i), define the function $V^p(x; s)$ by the equation

$$(*) \qquad V^p(x; s) = V^p_{K_0(r)}(x; s), \qquad \|[x, s]\| < r, \qquad p \geqq 1.$$

Since it follows from (i) that

$$(**) \quad V^p_{K_0(r)}(x; s) = V^{p'}_{K_0(r)}(x; s) \quad \text{for } p, p' \geqq 1 \text{ and } \|[x, s]\| < \min(r, r_1),$$

the function $V^p(x; s)$ is defined in a unique way by equation $(*)$. It is evident from $(*)$ that V^p is in $\hat{C}^p(E^{n+1})$, that $\tau V^p \equiv 0$, and that $V^p(x; 0) \equiv V_0(x)$. On the other hand, it follows from $(**)$ that $V^p = V^{p+1}$, so that, putting $V = V^1$, we have V in $\hat{C}^\infty(E^{n+1})$, and the present theorem is shown to be a consequence of (ii).

The remainder of the present proof is devoted to showing that (ii) is valid for each fixed r. Making a change of the scale of our variables, we may evidently, and shall henceforth, take $r = 1$. Let C_1 be the parallelepiped in E^{n+1} described in the second paragraph of the present proof, and let $C = (x \in E^n \mid |x_j| \leqq \pi,\ j = 1, \ldots, n)$. Let a_j, b, and v_0 be functions in $\hat{C}^\infty_\pi(C_1)$, $\hat{C}^\infty_\pi(C_1)$, $\hat{C}^\infty_\pi(C)$ respectively, chosen so that $a_j(y) = A_j(y)$ and $b(y) = B(y)$ for $|y| \leqq 1$, and so that $v_0(x) = V_0(x)$ for $|x| \leqq 1$. We may suppose in addition that $a_j(y)$ is a Hermitian matrix for each y in C_1 and that the functions a_j and b are periodic not only of period 2π in the variables y, but even periodic of period π in these variables. Indeed, to construct the functions a_j and b we put

$$C_2 = \left\{ y \in E^{n+1} \,\middle|\, |y_j| \leqq \frac{\pi}{2}, 1 \leqq j \leqq n+1 \right\}.$$

Then we use Lemma 2.1 to find a function $w(y)$ in $C_0^\infty(C_2)$ such that $w(y) \equiv 1$ for y in a neighborhood of the set $\{y \in E^{n+1} \mid |y| \leqq 1\}$, and put $a_j(y) = w(y)A_j(y)$ for y in C_1, $b(y) = w(y)B(y)$ for y in C_2. Since the functions a_j and b vanish identically in a neighborhood of the boundary of the cube C_2, it is clear that a_j is in $\hat{C}_\pi^\infty(C_2)$ and b is in $\hat{C}_\pi^\infty(C_2)$. Thus we may evidently extend a_j and b_j to functions multiply periodic of period π defined in all of E^{n+1}. We may continue to denote these extended functions by the same symbols a_j and b. This completes the construction of functions a_j and b with the desired properties. The function v_0 may be constructed in the same way.

Having constructed the functions a_j, b, v_0, we may now remark that (ii) is evidently an immediate consequence of the following statement:

(iii) For each $p \geq 1$, there exists a function v in $\hat{C}_{\pi_x}^p(C_1)$ such that

$$\partial_s v(x; s) = \sum_{j=1}^n a_j(x; s)\partial_{x_j} v(x; s) + b(x; s)v(x; s), \quad [x; s] \in C_1,$$

$$v(x; 0) = v_0(x), \qquad x \in C.$$

(C) To prove (iii), we proceed as follows. Let ρ denote the formal differential operator

$$\rho = \partial_s - \sum_{j=1}^n a_j(x; s)\partial_{x_j} - b(x; s).$$

For each $k \geq 0$, define the unbounded linear mapping W_k of $\hat{H}_\pi^{(k)}(C)$ into $\hat{H}_{\pi_x}^{(k)}(C_1)$ (cf. the two paragraphs in Section 6 following the proof of Lemma 6.16 for the definition of the space $\hat{H}_{\pi_x}^{(k)}(C_1)$) by putting f in $\mathfrak{D}(W_k)$ if f is in $\hat{C}_\pi^{k+1}(C)$ and there exists a g in $\hat{C}_\pi^{k+1}(C_1)$ such that $\rho g = 0$ and such that $f(x) = g(x; 0)$ for x in C, and putting $W_k f = g$ if such a g exists. (It follows from (i) that g is unique if it exists at all.) We shall show that

(iv) W_k is a single-valued and bounded mapping of a linear subspace of $\hat{H}_\pi^{(k)}(C)$ into $\hat{H}_{\pi_x}^{(k)}(C_1)$ for each $k \geqq 0$.

(v) $\mathfrak{D}(W_k)$ is dense in $\hat{H}_\pi^{(k)}(C)$ for each $k \geqq 0$.

Once (iv) and (v) are established, we may argue as follows. Let f be in $\hat{C}_\pi^\infty(C)$, and, using (v), let f_n be a sequence of elements in $\mathfrak{D}(W_{p+\nu})$, where $\nu = [(n+1)/2]$, such that $|f_n - f|_{(p+\nu)} \to 0$ as $n \to \infty$.

Then, by (iv), $W_{p+\nu}f_n$ converges in the norm of $\hat{H}^{(p+\nu)}_{\pi_x}(C_1)$ to some element g in $\hat{H}^{(p+\nu)}_{\pi_x}(C_1)$ (cf. 3.19). It follows from Sobolev's theorem (4.5) (cf. the remarks in Section 3 following the proof of Lemma 3.28 for the details of the way in which Sobolev's theorem is to be applied to $\hat{H}^{(k)}_{\pi_x}(C_1)$) that g is in $\hat{C}^p_{\pi_x}(C_1)$. It is clear from Lemma 3.22 that $\rho g = 0$. It follows from Corollary 4.6 that $\partial^J f_n \to \partial^J f$ in the topology of $\hat{C}^{(0)}(C)$ for each J with $|J| \leq p$, so that $f_n \to f$ in the topology of $\hat{C}^p_\pi(C)$; similarly, $W_{p+\nu}f_n \to g$ in the topology of $\hat{C}^p_{\pi_\alpha}(C_1)$, so that $g(x; 0) = f(x)$ for x in C. Thus, once (iv) and (v) are proved, (iii), and with it the present theorem, will follow immediately.

(D) Statement (iv) is readily proved directly, as follows: Let f be in $\mathfrak{D}(W_k)$, and let $g = W_k f$. Let

$$A_k(s) = A_k(g; s) = \sum_{|J| \leq k} \int_C |\partial^J g(x; s)|^2 dx, \qquad -\pi \leqq s \leqq \pi.$$

We differentiate this last equation with respect to s, and use the equation $\rho g = 0$, to find that

$$\begin{aligned}\frac{d}{ds} A_k(s) = & \sum_{|J| \leq k} \int_C \left(\sum_{j=1}^n a_j(x; s)\partial_{x_j}\partial^J g(x; s) + b(x; s)\partial^J g(x; s),\ \partial^J g(x; s)\right) dx \\ & + \sum_{|J| \leq k} \int_C \left(\partial^J g(x; s), \sum_{j=1}^n a_j(x; s)\partial_{x_j}\partial^J g(x; s) + b(x; s)\partial^J g(x; s)\right) dx.\end{aligned}$$

Next, using the fact that $a_j(y)$ is Hermitian, and using the multiple-periodicity of g, a_j, and b with respect to the variables x to integrate by parts, we find that

$$\begin{aligned}\frac{d}{ds} A_k(s) = & \sum_{|J| \leq k} \int_C \left(\partial^J g(x; s),\ -\sum_{j=1}^n \partial_{x_j}\{a_j(x; s)\partial^J g(x; s)\} + b^*(x; s)\partial^J g(x; s)\right) dx \\ & + \sum_{|J| \leq k} \int_C \left(\partial^J g(x; s), \sum_{j=1}^n a_j(x; s)\partial_{x_j}\partial^J g(x; s) + b(x; s)\partial^J g(x; s)\right) dx \\ = & \sum_{|J| \leq k} \int_C \left(\partial^J g(x; s), \left\{\left(-\sum_{j=1}^n \partial_{x_j} a_j(x; s)\right) + b^*(x; s) + b(x; s)\right\}\partial^J g(x; s)\right) dx. \end{aligned} \tag{7}$$

It follows from formula (7) and from Schwarz' inequality that for each $k \geqq 0$ there exists a positive constant $N_k < \infty$ such that

$$\left|\frac{d}{ds} A_k(g; s)\right| \leqq N_k A_k(g; 0), \qquad -\pi \leqq s \leqq \pi. \tag{8}$$

By (8) $A_k(g; s)e^{-N_k s}$ is non-increasing for $0 \leqq s \leqq \pi$; thus $A_k(g; s) \leqq e^{N_k \pi} A_k(g; 0)$ for $0 \leqq s \leqq \pi$. Using a similar argument in the interval $-\pi \leqq s \leqq 0$, we find that

$$A_k(g; s) \leqq e^{N_k \pi} A_k(g; 0), \qquad -\pi \leqq s \leqq \pi. \tag{9}$$

Since g satisfies the partial differential equation

$$\partial_s g(x; s) = \sum_{j=1}^{n} a_j(x; s)\partial_{x_j} g(x; s) + b(x; s)g(x; s), \qquad [x, s] \in C_1,$$

it is evident on repeated partial differentiation of this equation that any derivative $\partial^{J_1}g$ of g of order at most k can be expressed as a linear combination of the "pure x-derivatives" $\partial^J g$ of g of order at most k; that is, it is evident that there exists a family of coefficient matrices $C_{J_1, J}$ such that

$$\partial^{J_1} g(x; s) = \sum_{|J| \leqq |J_1|} C_{J_1, J}(x; s)\partial^J g(x; s), \qquad [x, s] \in C_0. \tag{10}$$

The matrices $C_{J_1, J}$, which are infinitely often differentiable as functions of $[x, s]$ for $[x, s]$ in C_1, could readily be calculated explicitly in terms of a_j and b by the use of Leibniz' rule. Since we shall have no use for these explicit expressions in what follows, we shall not give them. It follows, however, from the existence of the formulae (10), from (9), and from the definition of $A_k(g; s)$ that for each $k < p$ there exist finite positive constants $\tilde{N}_k$ such that $W_k f = g$ implies that

$$\left\{\int_{C_1} \partial^{J_1} g(x; s)dx\,ds\right\}^{\frac{1}{2}} < \tilde{N}_k |f|_k, \qquad |J_1| \leqq k. \tag{11}$$

Thus (iv) follows.

In the same way, it follows from the inequality $A_k(g; s) \leqq e^{N_k \pi} A_k(g; 0)$ established above that the following statement is valid.

(vi) If g is in $\hat{C}^{k+1}_{\pi_x}(C_1)$, $\rho g(x; s) = 0$, $[x, s] \in C_1$, and $g(x; 0) = f(x)$ for x in C, then for each $k > 0$ there exists a finite positive constant M_k such that $|g(\cdot, \pi)|_{(k)} \leqq M_k |f|_{(k)}$.

(E) Since (iv) has just been proved, the proof of Theorem 1 is now reduced to the proof of (v). To prove (v), we proceed as

follows. For each $k \geq 0$, define the linear mapping S_k of $\hat{H}_\pi^{(k)}(C)$ into $\hat{H}_\pi^{(k)}(C)$ by putting $\mathfrak{D}(S_k) = \mathfrak{D}(W_k)$, $(S_kf)(x) = (W_kf)(x;\pi)$ for x in C. Then, by (vi), S_k is bounded and of norm at most M_k. We shall show that

(vii) for each $k \geqq 0$, and for each sufficiently small positive $\alpha \leqq \alpha(k)$, the mapping $I - \alpha S_k$ has a range dense in $\hat{H}_\pi^{(k)}(C)$.

Suppose that (v) is false, but that (vii) has been established. Since (v) is false, and since $\mathfrak{D}(S_k) = \mathfrak{D}(W_k)$, there exists a $k \geqq 0$ with $k \leqq p$ and a non-zero element f_0 in $\hat{H}_\pi^{(k)}(C)$ which is such that $(f_0, f)_{(k)} = 0$ for each f in $\mathfrak{D}(S_k)$. Let a constant $\alpha < 1/2M_k^{-1}$ be chosen and, using (vii), let $\{h_n\}$ be a sequence of elements of $\mathfrak{D}(S_k)$ such that $h_n - \alpha S_k h_n \to f_0$ in the norm of $\hat{H}_\pi^{(k)}(C)$ as $n \to \infty$. Since $|\alpha S_k h_n|_{(k)} < \frac{1}{2}|h_n|_{(k)}$ by (6), we have

$$|h_n - \alpha S_k h_n|_{(k)} \geqq |h_n|_{(k)} - \tfrac{1}{2}|h_n|_{(k)} \geq \tfrac{1}{2}|h_n|_{(k)},$$

so that, since $h_n - \alpha S_k h_n \to f_0$ as $n \to \infty$, $|h_n|_{(k)}$ remains bounded as $n \to \infty$. Thus, there exists a constant Λ such that $|h_n|_{(k)} \leq \Lambda$ for $n \geq 1$. Hence we have

$$|(h_n - \alpha S_k h_n, h_n)_{(k)} - (f_0, h_n)_{(k)}| \leqq \Lambda |h_n - \alpha S_k h_n - f_0|_{(k)},$$

so that

$$(h_n - \alpha S_k h_n, h_n)_{(k)} = |h_n|^2_{(k)} - \alpha(S_k h_n, h_n)_{(k)} \to 0$$

as $n \to \infty$. Therefore, since $|\alpha S_k h_n|_{(k)} < \frac{1}{2}|h_n|_{(k)}$, $|h_n|_{(k)} \to 0$ as $n \to \infty$. But, since $h_n - \alpha S_k h_n \to f_0$, then $f_0 = 0$; and this contradiction shows that (vii) implies (v).

To prove (vii), we shall first prove the following statement:

(viii) Let $\lambda \geq \lambda(k')$ be real and sufficiently large, and let $k' \geq [(n+1)/2]+1 = \nu+1$. Then for each h in $C^{k'}_{\pi_s}(C_1)$ all of whose derivatives of order up to k' are periodic of period π in s, there exists a function g in $C^{k'-\nu}_{\pi_s}(C_1)$ all of whose derivatives of order up to $k' - \nu$ are periodic of period π in s, such that $((\rho+\lambda)g)(y) = h(y)$ for y in C_1.

Statement (vii) can be deduced from statement (viii) as follows. Let k' in (viii) be set equal to $k+\nu+1$, where $\nu = [(n+1)/2]$ and k is as in (vii). Let λ and α be related by $\alpha = e^{-\pi\lambda}$, so that "α sufficiently small" is equivalent to "λ sufficiently large." Let f be in $C_\pi^\infty(C)$. Let ρ_0 denote the formal partial differential operator

$$\rho_0 = \sum_{j=1}^{n} a_j(x;s)\partial_{x_j} + b(x;s), \tag{12}$$

so that $\rho = \partial_s - \rho_0$. Let η be a function in $C^\infty(-\infty, +\infty)$ which is identically equal to zero in $(-\infty, \frac{1}{4})$ and identically equal to minus one in $(\frac{1}{2}, \infty)$. Let $m = k'+1$ and let

$$g(x;s) = \eta(s)\sum_{j=0}^{m} \frac{(s-\pi)^j\big((\rho_0 - \lambda)^j f\big)(x)}{j!}, \quad x \in C, \quad 0 \leq s \leq \pi. \tag{13}$$

Then let $g(x;s)$ be defined for $x \in C$ and $-\pi \leq s < 0$ by the requirement that it be periodic of period π in s. It follows that $\big((\rho+\lambda)g\big)(x;s) = 0$ for x in C and $0 \leq s \leq \frac{1}{4}$, while

$$\begin{aligned}\big((\rho+\lambda)g\big)(x;s) &= \big((\partial_s - (\rho_0 - \lambda))g\big)(x;s)\\ &= -(m!)^{-1}(s-\pi)^m\big((\rho_0 - \lambda)^{m+1}f\big)(x), \quad \tfrac{1}{2} \leq s < \pi.\end{aligned}$$

These last expressions make it plain that the partial derivatives of the function $(\rho+\lambda)g$ of order up to $k' - m - 1$ are not only multiply periodic of period 2π in the variables x, but also periodic of period π in the variable s. Hence, by (viii), there exists a function h in $\hat{C}^{k'-\nu}_{\pi_x}(C_1)$, all of whose derivatives of order up to $k' - \nu$ are periodic of period π in s, such that $\big((\rho+\lambda)g\big)(y) = \big((\rho+\lambda)h\big)(y)$ for y in C_1. Let $g_1 = g - h$. Then clearly $\big((\rho+\lambda)g_1\big)(y) = 0$ for y in C_1, while it is evident from (13) that $g_1(x;0) - g_1(x;\pi) = g(x;0) - g_1(x;\pi) = f(x)$ for x in C. Since $(\rho+\lambda)g_1(y) = (\partial_s - \rho_0 + \lambda)g_1(y) = 0$, it is clear, by placing $g_2(x;s) = e^{+\lambda s}g_1(x;s)$, that we have $(\rho g_2)(y) = 0$. Moreover, $g_2(x;0) - \alpha g_2(x;\pi) = g_1(x;0) - g_1(x;\pi) = f(x)$. Thus f is in the range of $I - \alpha S_{k'-\nu-1} = I - \alpha S_k$. Hence, it follows that the range of $I - \alpha S_k$ includes every function in $\hat{C}^\infty_\pi(C)$. Since this last space is dense in $\hat{H}^{(k)}_\pi(C)$ for every k by Lemma 3.4, (vii) follows. Thus our theorem will be proved once (viii) is proved.

Note now that, in accordance with the statement made in the paragraph immediately preceding statement (iii), the coefficient matrices a_j and b of the formal partial differential operator ρ are multiply periodic of period π in the variables y. Thus, making the change of variable $x \to x$, $s \to 2s - \pi$, and using Sobolev's theorem (4.5) (cf. the remarks in Section 3 following the proof of Lemma 3.23 for more details of the way in which Sobolev's theorem is to be

applied to $\hat{H}^{(k)}_{\pi_s}(C_1)$), (viii) is an obvious consequence of the following statement.

(ix) Let $\hat{A}_j(y)$, $j = 1, \ldots, n$, and $\hat{B}(y)$ be $m \times m$ matrices dependent on the parameter y in E^{n+1} in an infinitely often differentiable way, and suppose that $\hat{A}_j(y)$ is Hermitian for $j = 1, \ldots, n$, and all y in E^{n+1}. Suppose also that $\hat{A}_j(y)$ and $\hat{B}(y)$ are multiply periodic of period 2π in all the variables y. Let $\hat{\tau}$ denote the formal partial differential operator

$$\hat{\tau} = \partial_s - \sum_{j=1}^{n} \hat{A}_j(x; s)\partial_{x_j} - \hat{B}(x; s).$$

Let $k \geq [(n+1)/2]+1$; then, for each sufficiently large real λ, and each h in $\hat{H}^{(k)}_{\pi}(C_1)$, there exists a g in $\hat{H}^{(k)}_{\pi}(C_1)$ such that $(\hat{\tau} - \lambda)g = h$.

Having reduced our whole problem to that of proving (ix), we now proceed to prove (ix). Consider the two unbounded operators in $\hat{H}^{(k)}_{\pi}(C_1)$ defined by the equations

$$\mathfrak{D}(T_k^{(0)}) = \hat{C}^{\infty}_{\pi}(C_1), \qquad T_k^{(0)}F = \hat{\tau}F, \qquad F \in \mathfrak{D}(T_k^{(0)}), \tag{14}$$

$$\mathfrak{D}(T_k^{(1)}) = \{f \in \hat{H}^{(k)}_{\pi}(C_1) | \hat{\tau}f \in \hat{H}^{(k)}_{\pi}(C_1)\}; \tag{15}$$
$$T_k^{(1)}F = \hat{\tau}F, \qquad F \in \mathfrak{D}(T_k^{(1)}).$$

By Definitions 3.2, 3.5, and 3.15, we have

$$\begin{aligned}(T_k^{(0)}f, g)_{(k)} &= \sum_{|J_1|, |\bar{J}_1| \leq k} \int_{C_1} \Big(\partial^{J_1}((\partial_s - \sum_{j=1}^{n} \hat{A}_j(x; s)\partial_{x_j} \\ &\qquad - \hat{B}(x; s))f)(x; s), \overline{\partial^{\bar{J}_1} g(x; s)}\Big) dx ds \\ &= \sum_{|J_1|, |\bar{J}_1| \leq k} \int_{C_1} \Big\{(\partial^{J_1} f)(x; s), \overline{\Big[\partial^{\bar{J}_1}\Big(\partial_s - \sum_{j=1}^{n} \hat{A}_j(x; s)\partial_{x_j} - \hat{B}(x; s)\Big)g\Big](x; s)}\Big\} dx ds \\ &\quad + \sum_{|J_1|, |\bar{J}_1| \leq k} \int_{C_1} [B_{J_1, \bar{J}_1}(x; s)(\partial^{J_1} f)(x; s), \overline{(\partial^{\bar{J}_1} g)(x; s)}] dx ds \\ &= (f, T_k^{(1)} g)_{(k)} + \sum_{|J_1|, |\bar{J}_1| \leq k} \int_{C_1} (B_{J_1; \bar{J}_1}(x; s)(\partial^{J_1} f)(x; s)\overline{(\partial^{\bar{J}_1} g)(x; s)}) dx\end{aligned} \tag{16}$$

for f in $\mathfrak{D}(T_k^{(0)})$ and g in $\mathfrak{D}(T_k^{(1)})$; here the matrices $B_{J_1, \bar{J}_1}(x; s)$, which are infinitely often differentiable, could readily be calculated explicitly in terms of $\hat{A}_j(x; s)$ and $\hat{B}(x; s)$ by the use of Leibniz' rule. Since we shall not have to use it in the following arguments, we shall not give the explicit expression here.

Suppose that f is in $\hat{C}_\pi^\infty(C_1)$, that g is in $\hat{H}_\pi^{(k)}(C_1)$, and that we let Γ denote the elliptic partial differential operator

$$\Gamma = \sum_{|J_1| \leq k} (-1)^{J_1} \partial^{J_1} \partial^{J_1}.$$

Then we have

$$(f, \Gamma g) = (f, g)_{(k)}$$

by Definitions 3.15, 3.2, and 3.5.

We remark for later use that if we regard the function g in (16) as a distribution, and make use of distribution theoretic notation to write the result of (16) as

$$(T_k^{(0)} f, g)_{(k)} = (\overline{\Gamma g})(f) + \sum_{|J_1|, |\bar{J}_1| \leq k} \int_{C_1} B_{J_1, \bar{J}_1}(x; s)$$
$$\times \partial^{J_1} f(x; s) \overline{\partial^{\bar{J}_1} g(x; s)} dx ds,$$

then we obtain a formula valid not only for f in $\mathfrak{D}(T_k^{(0)}) = \hat{C}_0^\infty(C_1)$ and g in $\mathfrak{D}(T_k^{(1)})$, but also for f in $\hat{C}_0^\infty(C_1)$ and g in $\hat{H}_\pi^{(k)}(C_1)$. This assertion follows, in the same way as (16) itself, from Definitions 3.2, 3.5, and 3.15.

The bilinear form

$$\Phi_k(g, h) = \sum_{|J_1|, |\bar{J}_1| \leq k} \int_{C_1} B_{J_1, \bar{J}_1}(x; s)(\partial^{J_1} g)(x; s)\overline{(\partial^{\bar{J}_1} h)(x; s)} dx ds$$

is evidently (by Schwarz' inequality) bounded if regarded as a bilinear form in $\hat{H}_\pi^{(k)}(C_1)$. Thus, by Theorem IV.4.5, $\Phi_k(g, h)$ may be written in the form

$$\Phi_k(g, h) = (B_k g, h)_{(k)}, \qquad g, h \in \hat{H}_\pi^{(k)}(C_1),$$

and it follows readily, again from Theorem IV.4.5, that the operator B_k is linear and bounded. Therefore, formula (16) may be written as

$$(T_k^{(0)} f, g)_{(k)} = (f, T_k^{(1)} g)_{(k)} + (B_k f, g)_{(k)}, \tag{17}$$
$$f \in \mathfrak{D}(T_k^{(0)}), \qquad g \in \mathfrak{D}(T_k^{(1)}).$$

Suppose next that g is in $\mathfrak{D}((T_k^{(0)})^*)$, so that $(T_k^{(0)})^*g \quad g^*$ is an element of $\hat{H}_\pi^{(k)}(C_1)$ satisfying the equation $(T_k^{(0)}f, g)_{(k)} = (f, g^*)_{(k)}$ for all f in $\mathfrak{D}(T_k^{(0)}) \quad \hat{C}_\pi^\infty(C_1)$. As was remarked in the third paragraph following formula (16), we have

$$(T_k^{(0)}f, g)_{(k)} \qquad (\overline{\Gamma\hat{\tau}g})(f)+(B_kf, g)_{(k)}$$

for f in $\mathfrak{D}(T_k^{(0)})$ and g in $\hat{H}_\pi^{(k)}(C_1)$. Thus

$$(f, g^*)_{(k)} \quad (f, B_kg)_{(k)} \quad \overline{(\Gamma\hat{\tau}g)}(f) = 0, \qquad f \in \hat{C}_\pi^\infty(C_1),$$

so that by the remarks in this same paragraph

$$\Gamma(g^* \quad B_kg \quad \hat{\tau}g) = 0.$$

By Corollary 6.4, $g^* \quad B_kg \quad \hat{\tau}g$ belongs to $\hat{C}_\pi^\infty(C_1)$ (cf. the remarks in Section 3 immediately following the proof of Lemma 3.28 for the detailed considerations justifying the application of 6.4 to the space $\hat{D}_\pi(C_1)$). Since g^*-B_kg is in $\hat{H}_\pi^{(k)}(C_1)$, it follows that $\hat{\tau}g$ is in $\hat{H}_\pi^{(k)}(C_1)$. Thus by (15) g is in $\mathfrak{D}(T_k^{(1)})$. Equation (17) now shows that $(f, (T_k^{(0)})^*g)_{(k)} = \quad (f, T_k^{(1)}g)_{(k)}+(f, B_k^*g)$ for each f in the subspace $\hat{C}_\pi^\infty(C_1)$ of $\hat{H}_\pi^{(k)}(C_1)$, which, by Lemma 3.41, is dense in $\hat{H}_\pi^{(k)}(C_1)$. Therefore this last equation holds by continuity for all f in $H_\pi^{(k)}(C_1)$. Thus we have shown that $g \in \mathfrak{D}((T_k^{(0)})^*)$ implies that $g \in \mathfrak{D}(T_k^{(1)})$, and that in this case $T_k^{(0)*}g = -T_k^{(1)}g+B_k^*g$. Since it follows immediately from (17) that this same equation is valid for g in $\mathfrak{D}(T_k^{(1)})$, we conclude that

$$(T_k^{(0)})^* = -T_k^{(1)}+B_k^*. \tag{18}$$

Hence, from Lemma XII.1.6 we see that $(T_k^{(0)}+B_k+\lambda I)^* \quad T_k^{(1)}+\lambda I$ for every real λ.

By Sobolev's theorem (4.5), and since we have assumed that $k \geq \nu+1 \quad [(n+1)/2]+1$, every function in $\hat{H}_\pi^{(k)}(C)$ belongs to $\hat{C}_\pi^1(C_1)$, and even to $\hat{C}_\pi^{k-\nu}(C_1)$. (Cf. at this point the remarks in Section 3 following the proof of Lemma 3.28, for the details of the way in which Sobolev's theorem is to be applied to the space $\hat{H}_\pi^{(k)}(C_1)$.) Suppose that $(T_k^{(0)}+B_k+\lambda I)^*f \quad (T_k^{(1)}+\lambda I)f = 0$. Then f is in $\hat{C}_\pi^1(C_1)$, so that, calculating exactly as in equations (5) and (6) of Part (A) of this proof, we find that for

$$\lambda > \hat{\alpha} \quad \sup_{y \in C_1} |\hat{B}(y)+\hat{B}(y)^* \quad \sum_{i=1}^n \partial_{x_i}\hat{A}_i(y)|,$$

the integral

$$\int_C |f(x; s)|^2 dx$$

is non-increasing for $-\pi \leq s \leq \pi$, and actually decreases as s goes from $-\pi$ to $+\pi$ unless $f(y) = 0$ for y in C_1. Since $f(y)$ is multiply periodic of period 2π, the displayed integral is periodic in s, and hence cannot decrease as s goes from $-\pi$ to π. Thus, $(T_k^{(0)} + B_k + \lambda I)^* f = 0$ and $\lambda > \hat{\alpha}$ together imply that $f = 0$. It follows from Lemma XII.1.6(a) that if $\lambda > \hat{\alpha}$, the mapping $T_k^{(0)} + B_k + \lambda I$ has a dense range.

Let g, f be in $\mathfrak{D}(T_k^{(0)}) = \hat{C}_\pi^\infty(C_1)$. Then, by (18), and since $T_k^{(0)} \subseteq T_k^{(1)}$ by definition, we have

$$\begin{aligned}(B_k^* f, g) = ((T_k^{(1)} + (T_k^{(0)})^*)f, g)_{(k)} &= ((T_k^{(0)} + (T_k^{(0)})^*)f, g)_{(k)} \\ &= (f, ((T_k^{(0)})^* + T_k^{(0)})g)_{(k)} \\ &= (f, B_k^* g)_{(k)}.\end{aligned}$$

This shows that the bounded operator B_k^* is self adjoint, from which it follows immediately that B_k is self adjoint. Consequently, if $\lambda > 0$, and f is in $\mathfrak{D}(T_k^{(0)}) = \hat{C}_\pi^\infty(C_1)$, we have, again by (18), and since $T_k^{(0)} \subseteq T_k^{(1)}$,

$$\begin{aligned}|(T_k^{(0)} - \tfrac{1}{2}B_k + \lambda I)f|_{(k)}^2 &= ((T_k^{(0)} - \tfrac{1}{2}B_k + \lambda I)f, (T_k^{(0)} - \tfrac{1}{2}B_k + \lambda I)f)_{(k)} \\ &= |(T_k^{(0)} - \tfrac{1}{2}B_k)f|_{(k)}^2 + \lambda\{((T_k^{(0)} - \tfrac{1}{2}B_k)f, f)_{(k)} \\ &\qquad + (f, (T_k^{(0)} - \tfrac{1}{2}B_k)f)_{(k)}\} + \lambda^2(f, f)_{(k)} \\ &\geq \lambda^2|f|_{(k)}^2 + \lambda((T_k^{(0)} + (T_k^{(0)})^* - B_k)f, f)_{(k)} \\ &= \lambda^2|f|_{(k)}^2.\end{aligned}$$

Thus,

$$(19) \quad |(T_k^{(0)} - \tfrac{1}{2}B_k + \lambda I)f|_{(k)} \geq \lambda|f|_{(k)}, \quad \lambda > 0, \quad f \in \hat{C}_\pi^\infty(C_1) = \mathfrak{D}(T_k^{(0)}).$$

Now, the operator $T_k^{(1)}$ is closed. Indeed, if $\{f_n\}$ is a sequence of elements of $\mathfrak{D}(T_k^{(1)})$ such that $f_n \to f$ and $T_k^{(1)} \to g$ in the topology of $\hat{H}_\pi^{(k)}(C_1)$, then by Lemma 3.22 and Definition 3.15 we have $\tau f = g$, so that f is in $\mathfrak{D}(T_k^{(1)})$ and $T_k^{(1)} f = g$ by (15). Thus, the graph $\Gamma(T_k^{(1)})$ of $T_k^{(1)}$ is a closed subspace of the Hilbert space $\hat{H}_\pi^{(k)}(C_1) \oplus \hat{H}_\pi^{(k)}(C_1)$. Let $T_k \subseteq T_k^{(1)}$ be the operator whose graph is the closure in $\Gamma(T_k^{(1)})$ of the graph $\Gamma(T^{(0)})$ of $T_k^{(0)}$. Then T_k is clearly closed, and since $T_k \supseteq T_k^{(0)}$,

it follows that for $\lambda > \hat{\alpha}$, $T_k + B_k + \lambda I$ has a dense range, while it follows by an evident "density argument" from (19) that

$$(20) \qquad |(T_k - \tfrac{1}{2}B_k + \lambda I)f|_{(k)} \geq \lambda |f|_{(k)}, \qquad f \in \mathfrak{D}(T_k), \qquad \lambda > 0.$$

Let $\lambda > \hat{\alpha}$, and let g be in $\hat{H}_\pi^{(k)}(C_1)$. Then we may find a sequence $\{f_n\}$ of elements in $\mathfrak{D}(T_k)$ such that $(T_k - 2^{-1}B_k + \lambda I)f_n \to g$ as $n \to \infty$. By (20), $\{f_n\}$ is a Cauchy sequence, and hence a convergent sequence with some limit f in $H_\pi^{(k)}(C_1)$. Thus

$$\{T_k f_n\} = \{(T_k - 2^{-1}B_k + \lambda I)f_n - (-2^{-1}B_k + \lambda I)f_n\}$$

is a convergent sequence, and, since T_k is closed, $(T_k - 2^{-1}B_k + \lambda I)f = g$. Thus, $T_k - 2^{-1}B_k + \lambda I$ maps its domain *onto* $\hat{H}_\pi^{(k)}(C_1)$. By (20), $T_k - 2^{-1}B_k + \lambda I$ is one-to-one and has a bounded inverse. Thus, if $\lambda > \hat{\alpha}$, we have $\lambda \notin \sigma(T + B_k)$; in this case it follows from (20) that $|R(-\lambda; T_k + B_k)| \leq \lambda^{-1}$.

Now let $\beta_k = |B_k|$, and $\lambda > 2\beta_k + \hat{\alpha}$. Then it follows from what has just been shown that $|B_k R(-\lambda; T_k - 2^{-1}B_k)| < \frac{1}{2}$, so that, from Lemma VII.3.4, the operator $(I - 2^{-1}B_k)R(-\lambda; T_k - 2^{-1}B_k)^{-1}$ exists, is everywhere defined, and is bounded. We have

$$\begin{aligned}&(T_k + \lambda I)R(-\lambda; T_k - 2^{-1}B_k)\big(I - \tfrac{1}{2}B_k R(-\lambda; T_k - 2^{-1}B_k)\big)^{-1}f \\ &= \big(-I + \tfrac{1}{2}B_k R(-\lambda; T_k - 2^{-1}B_k)\big)\big(I - \tfrac{1}{2}B_k R(-\lambda; T_k - 2^{-1}B_k)\big)^{-1}f = -f, \\ &\hspace{22em} f \in \hat{H}_\pi^{(k)}(C_1),\end{aligned}$$

and similarly

$$\begin{aligned}&R(-\lambda; T_k - 2^{-1}B_k)\big(I - \tfrac{1}{2}B_k R(-\lambda; T_k - 2^{-1}B_k)\big)^{-1}(T_k + \lambda I)f = -f, \\ &\hspace{22em} f \in \mathfrak{D}(T_k).\end{aligned}$$

Thus $(T_k + \lambda I)^{-1}$ exists, is bounded, and is everywhere defined for $\lambda > 2\beta_k + \alpha$. Since $T_k \subseteq T_k^{(1)}$ and since we have seen above that $T_k^{(1)} + \lambda I$ is one-to-one for $\lambda > 2\beta_k + \alpha$, this means that for each g in $\hat{H}_\pi^{(k)}(C_1)$, the equation $(\tau + \lambda)f = g$ has a unique solution f in $\hat{H}_\pi^{(k)}(C_1)$. Thus (ix) is proved and hence Theorem 1 follows. Q.E.D.

8. Parabolic Equations and Semi-Groups

In the present short section we shall develop the theory of the boundary value problem associated with the parabolic equation

$$u_t = -\tau u, \qquad t \geq 0,$$

where τ is a formal elliptic partial differential operator satisfying the hypotheses of Gårding's inequality, Lemma 6.10.

Every major fact that we shall establish is stated in the following theorem.

1 THEOREM. *Let τ be an elliptic formal partial differential operator of even order $2p$, defined in a domain I_0 in E^n. Suppose that*

$$\tau = \sum_{|J| \leq 2p} a_J(x)\partial^J,$$

that $a_J(x)$ is real if $|J| = 2p$, and that

$$(-1)^p \sum_{|J|=2p} a_J(x)\xi^J > 0, \qquad x \in I_0, \qquad \xi \in E^n, \qquad \xi \neq 0.$$

Let I be a bounded open set whose closure is contained in I_0. Let T be the extension of $T_0(\tau)$ defined by the equations

$$\mathfrak{D}(T) = \mathfrak{D}(T_1(\tau)) \cap H_0^{(p)}(I),$$
$$Tf = T_1(\tau)f, \qquad f \in \mathfrak{D}(T).$$

Then

(i) *T generates a strongly continuous semi-group $S(t)$, $t \geq 0$, in $L_2(I)$. For $t > 0$, $k \geq 0$, and f in $L_2(I)$, $S(t)f$ is in $\mathfrak{D}(T^k)$, $T^kS(t)f$ is analytic in t, and $(S(t)f)(x)$ is analytic in t and is infinitely often differentiable in x for $t > 0$ and x in I.*

(ii) *Let f be in $L_2(I)$. There exists one and only one function $u(t, x)$ defined and continuously differentiable in t and x for $t > 0$ and x in I such that*

$$u(t, \cdot) \in L_2(I), \qquad t \geq 0,$$

$$\lim_{t \to s} |u(t, \cdot) - u(s, \cdot)| = 0, \qquad s \geq 0,$$

$$u(0, \cdot) = f(\cdot),$$

$$u(t, \cdot) \in \mathfrak{D}(T), \qquad t > 0,$$

$$\lim_{t \to s} |Tu(t, \cdot) - Tu(s, \cdot)| = 0, \qquad s > 0,$$

$$\frac{\partial}{\partial t} u(t, x) = -\tau u(t, x), \qquad t > 0, \qquad x \in I.$$

This function is determined by the equation $u(t, x) = (S(t)f)(x)$.

PROOF. Let $\varepsilon > 0$, and let $\varepsilon < 10^{-1}\pi$. Let A be the reflex angle $(\lambda | \, |\arg \lambda| > \varepsilon)$ in the complex λ-plane. By Corollary 6.27 there exists a finite constant $K > 0$ such that $\lambda \notin \sigma(T)$ and $|R(\lambda; T)| \leq K|\lambda|^{-1}$ if λ is in A and $|\lambda| > K$. It follows immediately from the Hille-Yosida theorem (VIII.1.18) that $-T$ generates a semi-group. We shall derive a representation for this semi-group which will make all the parts of (i) evident.

Put

$$S(t) = \frac{1}{2\pi i}\int_C e^{-tz} R(z; T)dz, \tag{1}$$

where C is a contour consisting of the ray $\{z | \arg z = 2\varepsilon, |z| \geqq 2K)$ traversed inward from ∞ to $2Ke^{2i\varepsilon}$, the circular arc from $2Ke^{+2i\varepsilon}$ to $2Ke^{-2i\varepsilon}$ traversed in the positive sense, and the ray

$$\{z| \arg z = -2\varepsilon, \qquad |z| \geq 2K)$$

traversed outward from $2Ke^{-2i\varepsilon}$ to ∞. From the bound on $|R(\lambda; T)|$ stated above, the integral in (1) evidently converges absolutely and uniformly in any bounded closed subset of the region $\rho = (t| \, |\arg t| < \pi/2 - 2\varepsilon, \ t \neq 0\}$ and is consequently analytic in ρ. Since

$$TR(z; T) = zR(z; T) - I,$$

the integral

$$\frac{1}{2\pi i}\int_C e^{-tz} TR(z; T)dz$$

also converges absolutely and uniformly and is equal to

$$\begin{aligned} \frac{1}{2\pi i}\int_C ze^{-tz}R(z; T)dz &- \frac{1}{2\pi i} I \int_C e^{-tz}dz \\ &= \frac{1}{2\pi i}\int_C ze^{-tz}R(z; T)dz \\ &= -\frac{d}{dt}S(t). \end{aligned}$$

Thus, by Theorem III.6.20, $S(t)f$ is in $\mathfrak{D}(T)$ for each f in $L_2(I)$, and

$$\frac{d}{dt}S(t) = -TS(t), \qquad t \in \rho. \tag{2}$$

Let t be in ρ and f be in $L_2(I)$; then by (2) the quantity $f_{\Delta t}$ defined by the equation $f_{\Delta t} = (\Delta t)^{-1}(S(t+\Delta t)-S(t))f$ belongs to $\mathfrak{D}(T)$, and $Tf_{\Delta t}$ equals $(\Delta t)^{-1}(d/dt)(S(t+\Delta t)f - S(t)f)$. Letting $\Delta t \to 0$, and noting that by Corollary 6.11 and Lemma XII.1.6, T is closed, it follows that $S(t)f$ is in $\mathfrak{D}(T^2)$, and that

$$\left(\frac{d}{dt}\right)^2 S(t) = (-T)^2 S(t), \qquad t \in \rho.$$

It follows by induction that $S(t)f$ is in $\mathfrak{D}(T^k)$ for each $k \geqq 1$ and each f in $L_2(I)$, and that

$$\left(\frac{d}{dt}\right)^k S(t) = (-T)^k S(t), \qquad t \in \rho.$$

Thus $T^k S(t)f$ is analytic for t in ρ and f in $L_2(I)$. It follows immediately from Theorem 6.23 and Corollary 6.24 that $T^k S(t)f(x)$ is analytic in t and infinitely often differentiable in x for t in ρ and x in I.

Next suppose that f is in $\mathfrak{D}(T)$. Then

$$R(z; T)Tf + f = zR(z; T)f,$$

so that

$$S(t)f = \frac{1}{2\pi i}\int_C z^{-1}e^{-zt}R(z; T)Tf\,dz + \frac{1}{2\pi i}\int_C \frac{f}{z}\,dz$$
$$= \frac{1}{2\pi i}\int_C z^{-1}e^{-zt}R(z; T)Tf\,dz + f.$$

Using the fact that $|R(z; T)| = O(|z|^{-1})$ as $|z| \to \infty$, z remaining in A, the Lebesgue dominated convergence theorem (III.6.16), and Cauchy's integral theorem, we find that

$$\lim_{t\to 0} |S(t)f - f| = 0.$$

Thus, if f is in $\mathfrak{D}(T)$, the function $u(t, x) = (S(t)f)(x)$ satisfies all the hypotheses of (ii). We shall show that if $S_1(t)$ is the semi-group generated by $-T$, then any function $u(t, x)$ satisfying the hypotheses of (ii) is given by $S_1(t)f$. Thus it will follow that $S(t) = S_1(t)$ for $t > 0$, so that both (i) and (ii) will be proved.

Let u be as in (ii), f in $L_2(I)$ being arbitrary. Let $s > 0$, and consider the function

$$\varphi(s, t, \cdot) = S_I(s-t)u(\cdot, t).$$

As $t \to 0$, $\varphi(s, t, \cdot) \to (S_I(s)f)(\cdot)$ in the norm of $L_2(I)$ by the strong continuity of S_I and by hypothesis. Similarly, $\varphi(s, t, \cdot) \to u(\cdot, s)$ as $t \to s$. We shall prove that $(\partial/\partial t)\varphi(s, t, \cdot) = 0$, from which (ii) evidently will follow.

It is clear that

$$\begin{aligned}\frac{\partial}{\partial t}\int_I u(t, x)f(x)dx &= \int_I \frac{\partial}{\partial t} u(t, x)f(x)dx \\ &= -\int_I (Tu)(t, x)f(x)dx, \qquad t > 0,\end{aligned}$$

for each f in $L_2(I)$ which is continuous and vanishes outside a compact subset of I. Thus

$$\int_I u(t, x)f(x)dx - \int_I u(s, x)f(x)dx = -\int_s^t\int_I (Tu)(t, x)f(x)dxdt$$

for $0 < s < t < \infty$ and each f in $L_2(I)$ which vanishes outside a compact subset of I. By Fubini's theorem (III.11.9), by Theorem III.11.17, and by Theorem III.2.20, it follows from the density of $C_0^\infty(I)$ in $L_2(I)$ that

$$u(t, \cdot)-u(s, \cdot) = -\int_s^t (Tu)(t, \cdot)dt, \qquad 0 < s < t < \infty.$$

Hence, since $(Tu)(t, \cdot)$ is continuous in t, it follows immediately that

$$\frac{d}{dt} u(t, \cdot) = -(Tu)(t, \cdot), \qquad t > 0.$$

Thus

$$\frac{\partial}{\partial t} S_I(s-t)u(t, \cdot) = S_I(s-t)Tu(t, \cdot)-S_I(s-t)Tu(t, \cdot) = 0,$$

by Lemma VIII.1.7(b), and our proof is complete. Q.E.D.

2 Corollary. *Suppose that the hypotheses of Theorem* 1 *are satisfied, and that in addition I is bounded by a smooth surface Σ, no point of which is interior to the closure $\bar{I}$ of I.*

(i) *Then the function $(S(t)f)(x)$ is analytic in t and infinitely often differentiable in x for $t > 0$ and x in I, and satisfies the equation*

$$S(t)f(x) - \big(\partial_\nu(\Sigma)S(t)f\big)(x) = \ldots - \big(\partial_\nu^{p-1}(\Sigma)S(t)f\big)(x) = 0,$$
$$t > 0, \qquad x \in \Sigma.$$

(ii) *Letting f be in $L_2(I)$, there exists one and only one function $u(t, x)$ defined for $t > 0$ and x in $\bar{I}$, continuously differentiable in t and $2p$ times continuously differentiable in x for $t > 0$ and x in $\bar{I}$, such that*

$$\frac{\partial}{\partial t} u(t, x) = \tau u(t, x), \qquad t > 0, \qquad x \in I,$$

such that

$$u(t, x) - \partial_\nu(\Sigma)u(t, x) - \ldots - \partial_\nu^{p-1}(\Sigma)u(t, x) = 0, \qquad t > 0, \qquad x \in \Sigma,$$

and such that

$$\lim_{t \to 0} |u(t, \cdot) - f(\cdot)| = 0.$$

PROOF. Statement (i) follows from the preceding theorem and Theorem 6.23. Statement (ii) follows from statement (ii) of the preceding theorem, since a function satisfying the hypotheses of the present statement (ii) evidently (cf. Theorem 6.23) satisfies the hypotheses of statement (ii) of the preceding theorem. Q.E.D.

APPENDIX

Hilbert space is a linear vector space $\mathfrak{H}$ over the field Φ of complex numbers, together with a complex function $(\cdot, \cdot)$ defined on $\mathfrak{H}\times\mathfrak{H}$ with the following properties:

(i) $(x, x) = 0$ if and only if $x = 0$;

(ii) $(x, x) \geq 0, \quad x \in \mathfrak{H}$;

(iii) $(x+y, z) = (x, z)+(y, z), \quad x, y, z \in \mathfrak{H}$;

(iv) $(\alpha x, y) = \alpha(x, y), \quad \alpha \in \Phi, \quad x, y \in \mathfrak{H}$;

(v) $(x, y) = \overline{(y, x)}$;

(vi) if $x_n \in \mathfrak{H}$, $n = 1, 2, \ldots$, and if $\lim_{n,m\to\infty}(x_n-x_m, x_n-x_m) = 0$, then there is an x in $\mathfrak{H}$ with $\lim_n (x_n-x, x_n-x) = 0$.

The function $(\cdot, \cdot)$ is called the *scalar* or *inner* product in $\mathfrak{H}$ and (x, y) is called the *scalar* or *inner product of x and y*. The *norm* in $\mathfrak{H}$ is $|x| = (x, x)^{1/2}$.

Remark. Hilbert space has been defined by a set of abstract axioms. It is noteworthy that some of the concrete spaces defined above satisfy these axioms, and hence are special cases of abstract Hilbert space. Thus, for instance, the n-dimensional unitary space E^n is a Hilbert space, if the inner product (x, y) of two elements $x = [\alpha_1, \ldots, \alpha_n]$ and $y = [\beta_1, \ldots, \beta_n]$ in E^n is defined by the formula

$$(x, y) = \sum_{i=1}^{n} \alpha_i \bar{\beta}_i.$$

In the same way, complex l_2 is a Hilbert space if the scalar product (x, y) of the vectors $x = \{\alpha_n\}$, $y = \{\beta_n\}$ is defined by the formula

$$(x, y) = \sum_{n=1}^{\infty} \alpha_n \bar{\beta}_n.$$

Also the complex space $L_2(S, \Sigma, \mu)$ is a Hilbert space with the scalar product

$$(f, g) = \int_S f(s)\overline{g(s)}\mu(ds) \qquad f, g \in L_2(S, \Sigma, \mu).$$

Of the infinite dimensional B-spaces, Hilbert space is the most closely related, especially in its elementary geometrical aspects, to the Euclidean or finite dimensional unitary spaces. It is not immediate from the definition that a Hilbert space is a B-space, but this fact is established in the following theorem. Throughout this discussion of Hilbert space the conditions (i), . . ., (vi) in the definition will be used without explicit reference and the symbol $\mathfrak{H}$ will always be used for a Hilbert space.

1 Theorem. *A Hilbert space $\mathfrak{H}$ is a complex B-space and*

$$|(x, y)| \leq |x||y|, \qquad x, y \in \mathfrak{H}.$$

Proof. The above inequality, known as the *Schwarz inequality*, will be proved first. It follows from the postulates for $\mathfrak{H}$ that the Schwarz inequality is valid if either x or y is zero. Hence suppose that $x \neq 0 \neq y$. For an arbitrary complex number α

$$\begin{aligned} 0 &\leq (x+\alpha y, x+\alpha y) \\ &= |x|^2 + |\alpha|^2|y|^2 + \alpha(y, x) + \bar{\alpha}(x, y) \\ &= |x|^2 + |\alpha|^2|y|^2 + 2\mathscr{R}(\alpha(y, x)), \end{aligned}$$

where the symbol $\mathscr{R}(\lambda)$ is used for the real part of λ. If $\alpha = re^{i\theta}$ and if θ is chosen properly, it follows from the above inequality that

$$|x|^2 + r^2|y|^2 \geq 2r|(x, y)|$$

for every positive r. Upon placing $r = |x|/|y|$ the Schwarz inequality follows.

To complete the proof of the theorem it will suffice to show that $|x+y| \leq |x| + |y|$. First note that

$$(x, y) + (y, x) = 2\mathscr{R}(x, y) \leq 2|x||y|$$

and hence that

$$\begin{aligned} |x+y|^2 &= |x|^2 + |y|^2 + (x, y) + (y, x) \\ &\leq |x|^2 + |y|^2 + 2|x||y| \\ &= (|x| + |y|)^2. \qquad \text{Q.E.D.} \end{aligned}$$

Remark. It should be noted that the above proofs of the Schwarz

inequality and the triangle inequality $|x+y| \leq |x|+|y|$ do not require that $\mathfrak{H}$ be complete or that (x, x) vanish only when $x = 0$.

2 Lemma. *Let x be an element of $\mathfrak{H}$ and let K be a subset of $\mathfrak{H}$ with the property that $\frac{1}{2}(K+K) \subseteq K$. Let $\{k_i\}$ be a sequence in K with the property that*

$$\lim_i |x-k_i| = \inf_{k \in K} |x-k|.$$

Then $\{k_i\}$ is a convergent sequence.

Proof. The identity

$$|x+y|^2+|x-y|^2 = 2|x|^2+2|y|^2, \qquad x, y \in \mathfrak{H},$$

called the *parallelogram identity*, follows immediately from the axioms. If $\delta = \inf_{k \in K} |x-k|$, the preceding identity shows that

$$\begin{aligned} |k_i-k_j|^2 &= 2|x-k_i|^2+2|x-k_j|^2-4|x-(k_i+k_j)/2|^2 \\ &\leq 2|x-k_i|^2+2|x-k_j|^2-4\delta \to 0. \end{aligned}$$

Q.E.D.

3 Definition. Two vectors x, y in $\mathfrak{H}$ are said to be *orthogonal* if $(x, y) = 0$. Two manifolds $\mathfrak{M}, \mathfrak{N}$ in $\mathfrak{H}$ are *orthogonal manifolds* if $(\mathfrak{M}, \mathfrak{N}) = 0$. We write $x \perp y$ to indicate that x and y are orthogonal, and $\mathfrak{M} \perp \mathfrak{N}$ to indicate that $\mathfrak{M}$ and $\mathfrak{N}$ are orthogonal. The *orthocomplement* of a set $A \subseteq \mathfrak{H}$ is the set $\{x|(x, A) = 0\}$. It is sometimes denoted by $\mathfrak{H} \ominus A$, or, if $\mathfrak{H}$ is understood, by $A^{\perp}$.

→ 4 Lemma. *The orthocomplement $\mathfrak{N}$ of a closed linear manifold $\mathfrak{M}$ in $\mathfrak{H}$ is a closed linear manifold complementary to $\mathfrak{M}$*, i.e., $\mathfrak{H} = \mathfrak{M} \oplus \mathfrak{N}$.

Proof. It follows from the linearity and the continuity of the scalar product (Theorem 1) that the orthocomplement of any set $\mathfrak{M}$ is a closed linear manifold. If $\mathfrak{M}$ is a closed linear manifold and if x is an arbitrary point in $\mathfrak{H}$ there is, by Lemma 2, an $m \in \mathfrak{M}$ such that $|x-m| = \delta = \inf_{m_1 \in \mathfrak{M}} |x-m_1|$. It will now be shown that the element $n = x-m$ is in $\mathfrak{N}$. For any complex number α and any m_1 in $\mathfrak{M}$ the vector $m+\alpha m_1$ is in $\mathfrak{M}$ and hence $|x-(m+\alpha m_1)| \geqq \delta$. Thus

$$\begin{aligned} 0 \leq |x-(m+\alpha m_1)|^2-|n|^2 &= |n-\alpha m_1|^2-|n|^2 \\ &= -\alpha(m_1, n)-\bar{\alpha}(n, m_1)+|\alpha|^2|m_1|^2. \end{aligned}$$

Let $\alpha = \lambda(n, m_1)$ where λ is an arbitrary real number. Then

$$0 \leq -2\lambda|(n, m_1)|^2 + \lambda^2|(n, m_1)|^2|m_1|^2$$

which is possible only if $(n, m_1) = 0$. Thus $n \in \mathfrak{N}$. To complete the proof, note that $x \in \mathfrak{M} \cap \mathfrak{N}$ implies $|x| = (x, x) = 0$. Thus $\mathfrak{M} \cap \mathfrak{N} = \{0\}$ and $\mathfrak{H} = \mathfrak{M} \oplus \mathfrak{N}$. Q.E.D.

→ 5 THEOREM. *Every y^* in $\mathfrak{H}^*$ uniquely determines a y in $\mathfrak{H}$ such that*

$$y^*x = (x, y), \qquad x \in \mathfrak{H}.$$

This map $\sigma : y^* \to y$ is a one-to-one isometric map of $\mathfrak{H}^*$ onto all of $\mathfrak{H}$ and $\sigma(y^* + z^*) = \sigma(y^*) + \sigma(z^*)$, $\sigma(\alpha y^*) = \bar{\alpha}\sigma(y^*)$.

PROOF. If $y^* = 0$ let $y = 0$. If $y^* \neq 0$ the set $\mathfrak{M} = \{x | y^*x = 0\}$ is a proper closed linear manifold in $\mathfrak{H}$ and its orthocomplement $\mathfrak{N}$ contains, by Lemma 4, a vector $y_1 \neq 0$. Let $y = \alpha y_1$ where $\bar{\alpha} = y^*(y_1)/|y_1|^2$. For an arbitrary vector x in $\mathfrak{H}$ the vector $x - (y^*x)/(y^*y_1)y_1$ is in $\mathfrak{M}$ so that $(x, y) = y^*x(y_1, y)/y^*y_1 = y^*x$, which proves the existence of the desired y. To see that y is unique, let y' be an element of $\mathfrak{H}$ such that $y^*x = (x, y')$ for every $x \in \mathfrak{H}$. Then $(x, y - y') = 0$ for every $x \in \mathfrak{H}$ and, in particular, $(y-y', y-y') = 0$, so $y = y'$. Thus σ is well defined. Since $|(x, y)| \leq |x||y|$ it follows that $|y^*| \leq |\sigma(y^*)|$, and since $(y, y) = |y|^2$ it follows that $|y^*| \geq |\sigma(y^*)|$. Therefore σ is an isometry. The remaining properties to be proved for σ follow immediately from the postulated properties of the scalar product. Q.E.D.

6 COROLLARY. *The space $\mathfrak{H}^*$ is also a Hilbert space and $\mathfrak{H}$ is reflexive.*

PROOF. If the scalar product in $\mathfrak{H}^*$ is defined by

$$(x^*, y^*)_1 = (\sigma(y^*), \sigma(x^*)),$$

then $\mathfrak{H}^*$ is clearly a Hilbert space. For $y^{**} \in \mathfrak{H}^{**}$ there is, according to the theorem, an element $y^* \in \mathfrak{H}^*$ such that

$$y^{**}x^* = (x^*, y^*)_1 = (\sigma(y^*), \sigma(x^*)) = x^*y, \qquad x^* \in \mathfrak{H}^*,$$

where $y = \sigma(y^*)$. Q.E.D.

7 COROLLARY. *Hilbert space is weakly complete and a subset is weakly sequentially compact if and only if it is bounded.*

PROOF. This follows from 6, II.3.28, and II.3.29. Q.E.D.

8 **Definition.** A set $A \subseteq \mathfrak{H}$ is called an ***orthonormal*** set if each vector in A has norm one and if every pair of distinct vectors in A is orthogonal. An orthonormal set is said to be ***complete*** if no non-zero vector is orthogonal to every vector in the set, i.e., A is complete if $(0) = \mathfrak{H} \ominus A$. We recall that a projection is a linear operator E with $E^2 = E$. A projection E in $\mathfrak{H}$ is called an ***orthogonal projection*** if the manifolds $E\mathfrak{H}$ and $(I-E)\mathfrak{H}$ are orthogonal.

It has been shown in Lemma 4 that

$$\mathfrak{H} = \mathfrak{M} \oplus (\mathfrak{H} \ominus \mathfrak{M})$$

where $\mathfrak{M}$ is an arbitrary closed linear manifold in $\mathfrak{H}$. If $x = y + z$ where $y \in \mathfrak{M}$ and $z \in \mathfrak{H} \ominus \mathfrak{M}$ let us define the transformation E in $\mathfrak{H}$ by placing $Ex = y$. It is clear that E is a projection, i.e., $E^2 = E$, and that E is an orthogonal projection. It is the uniquely determined orthogonal projection with $E\mathfrak{H} = \mathfrak{M}$. For if D is an orthogonal projection with $D\mathfrak{H} = \mathfrak{M}$ then $ED = D$ and, since $(I-D)\mathfrak{H} \subseteq \mathfrak{H} \ominus \mathfrak{M}$, we see that $E(I-D) = 0$. Thus

$$D = ED + E(I-D) = E.$$

This uniquely determined orthogonal projection E with $E\mathfrak{H} = \mathfrak{M}$ is called *the orthogonal projection on* $\mathfrak{M}$ or sometimes simply *the projection on* $\mathfrak{M}$.

9 **Lemma.** *If $\{y_i\}$ is an orthonormal sequence and $\{\alpha_i\}$ is a sequence of scalars, then the series $\Sigma\alpha_i y_i$ converges if and only if $\Sigma|\alpha_i|^2 < \infty$, and in this case*

$$|\Sigma\alpha_i y_i| = (\Sigma|\alpha_i|^2)^{1/2}.$$

When it converges, the series $\Sigma\alpha_i y_i$ is independent of the order in which its terms are arranged.

Proof. For $m > n$

$$|\sum_{i=n}^{m} \alpha_i y_i|^2 = (\sum_{i=n}^{m} \alpha_i y_i, \sum_{j=n}^{m} \alpha_j y_j) = \sum_{i=n}^{m} \sum_{j=n}^{m} \alpha_i \bar{\alpha}_j (y_i, y_j) = \sum_{i=n}^{m} |\alpha_i|^2,$$

and hence one series converges if the other does. If, in the above equality, one puts $n = 1$ and allows m to increase indefinitely, the second conclusion of the lemma follows. Finally let $z = \sum_{n=1}^{\infty} \alpha_{i_n} y_{i_n}$

be a series obtained from $x = \sum \alpha_i y_i$ by a rearrangement of its terms. Then

$$|x-z|^2 = (x, x) - (x, z) - (z, x) + (z, z),$$

and a direct computation, similar to that above, shows that each of these scalar products is $\sum |\alpha_i|^2$. Thus $z = x$. Q.E.D.

→ 10 Theorem. *Let A be an orthonormal set in $\mathfrak{H}$ and let x be an arbitrary vector in $\mathfrak{H}$. Then $(x, y) = 0$ for all but a countable number of y in A. The series*

$$Ex = \sum_{y \in A} (x, y) y, \qquad x \in \mathfrak{H}$$

converges and is independent of the order in which its non-zero terms are arranged. The operator E is the orthogonal projection on the closed linear manifold determined by A.

Proof. Let $y_1, \ldots, y_n$ be distinct elements of A and let $y = \sum_{i=1}^n (x, y_i) y_i$ so that (by Lemma 9), $|y|^2 = \sum_{i=1}^n |(x, y_i)|^2$ and

$$0 \leq |x-y|^2 = |x|^2 - (x, y) - (y, x) + |y|^2,$$

$$(x, y) = \sum_{i=1}^n \overline{(x, y_i)}(x, y_i) = |y|^2,$$

$$(y, x) = \sum_{i=1}^n (x, y_i)\overline{(x, y_i)} = |y|^2.$$

Thus $|y|^2 \leq |x|^2$, i.e.,

$$\sum_{i=1}^n |(x, y_i)|^2 \leq |x|^2.$$

This shows that at most a finite number of vectors $y_1, \ldots, y_n$ in A can have $|(x, y_i)|$ greater than a preassigned positive number and proves that at most a countable number of the scalar products (x, y) with y in A fail to vanish. Since

$$\sum_{y \in A} |(x, y)|^2 \leq |x|^2,$$

the preceding lemma shows that the series defining Ex converges and is independent of the order of its terms.

Now it is clear that E is a linear operator with $Ex = x$ for x in A. Thus $Ex = x$ for x in the closed linear manifold $\mathfrak{A}_1$ determined by A.

Also $Ex = 0$ if x is orthogonal to A. Thus E is the orthogonal projection on $\mathfrak{A}_1$. Q.E.D.

11 DEFINITION. A set A is called an *orthonormal basis* for the linear manifold $\mathfrak{N}$ in $\mathfrak{H}$ if A is an orthonormal set contained in $\mathfrak{N}$ and if

$$x = \sum_{y \in A} (x, y)y, \qquad x \in \mathfrak{N}.$$

12 THEOREM. *Every closed linear manifold in $\mathfrak{H}$ contains an orthonormal basis for itself.*

PROOF. If the orthonormal sets in the closed linear manifold $\mathfrak{M}$ are ordered by inclusion, it is seen from Zorn's lemma (I.2.7) that there is a maximal one A which determines the closed linear manifold $\mathfrak{A}_1 \subseteq \mathfrak{M}$. Since A is maximal then $\mathfrak{M} \ominus \mathfrak{A}_1 = 0$. But by Lemma 4, $\mathfrak{M} = \mathfrak{A}_1 \oplus (\mathfrak{M} \ominus \mathfrak{A}_1)$, and so $\mathfrak{M} = \mathfrak{A}_1$. The desired conclusion now follows from Theorem 10. Q.E.D.

13 THEOREM. *For an orthonormal set $A \subset \mathfrak{H}$ the following statements are equivalent:*

(i) *the set A is complete;*

(ii) *the set A is an orthonormal basis for $\mathfrak{H}$;*

(iii) $|x|^2 = \Sigma_{y \in A} |(x, y)|^2, \qquad x \in \mathfrak{H}.$

PROOF. The equivalence of statements (i) and (ii) is clear from Theorem 10. That either one of these implies (iii) follows from Theorem 10 and Lemma 9. Now assume (iii) and let x be an arbitrary vector in $\mathfrak{H}$. By Lemma 4, $x = u + v$ where $u \in \overline{\text{sp}}(A)$ and $v \in \mathfrak{H} \ominus \overline{\text{sp}}(A)$. Therefore we have $|x|^2 = |u|^2 + |v|^2$. But, by Theorem 10 and Lemma 9, $|u|^2 = \sum_{y \in A} |(u, y)|^2$. Hence $|x|^2 = |u|^2$ and $v = 0$. This means that $\overline{\text{sp}}(A) = \mathfrak{H}$ from which (i) follows. Q.E.D.

The next result enables us to define the *dimension* of a Hilbert space.

14 THEOREM. *All orthonormal bases of a given Hilbert space $\mathfrak{H}$ have the same cardinality.*

PROOF. If $\mathfrak{H}$ is finite dimensional, the result is a well-known result in algebra. Suppose then that $\mathfrak{H}$ is infinite dimensional, and let $\{u_\alpha\}$ and $\{v_\beta\}$ be two orthonormal bases for $\mathfrak{H}$. We shall say that the vectors u_α and $u_{\alpha'}$ in the basis $\{u_\alpha\}$ are equivalent if there exists a finite chain

$$[*] \qquad u_\alpha, v_{\beta_1}, u_{\alpha_1}, \ldots, u_{\alpha_k}, v_{\beta_{k+1}}, u_{\alpha'},$$

of vectors in which the scalar product of any two successive vectors is non-zero and in which the terms alternate between vectors in $\{u_\alpha\}$ and $\{v_\beta\}$. The equivalence of two vectors v_β and $v_{\beta'}$ in $\{v_\beta\}$ is defined similarly. It follows immediately from Theorem 10 that any equivalence class of vectors is either finite or countable. An equivalence class U of vectors u_α will be said to correspond to an equivalence class V of vectors v_β if there is a pair of vectors, one from U and one from V, with a non-zero inner product. Suppose that U and V are corresponding equivalence classes and that $u_\alpha \in U$. Consider an arbitrary element v_β in the basis $\{v_\beta\}$ for which $(u_\alpha, v_\beta) \neq 0$. It will be shown that $v_\beta \in V$. Since U and V are corresponding classes there are elements $u_{\alpha'}, v_{\beta'}$ in U, V respectively with $(u_{\alpha'}, v_{\beta'}) \neq 0$. Now since $u_{\alpha'} \in U$ there is a finite chain of the form given in [*] above in which successive vectors have non-zero scalar products. Thus by forming the chain $v_\beta, u_\alpha, \ldots, u_{\alpha'}, v_{\beta'}$ it is seen that v_β is equivalent to $v_{\beta'}$ and thus that v_β is in V. Since $\{v_\beta\}$ is a basis, the vector u_α has an expansion of the form $u_\alpha = \sum_\beta (u_\alpha, v_\beta) v_\beta$, so that u_α is in the closed linear manifold determined by those v_β with $(u_\alpha, v_\beta) \neq 0$. Since such v_β are in V we have $u_\alpha \in \overline{\mathrm{sp}}[V]$ and thus $\overline{\mathrm{sp}}[U] \subseteq \overline{\mathrm{sp}}[V]$. Similarly $\overline{\mathrm{sp}}[V] \subseteq \overline{\mathrm{sp}}[U]$. It is thus seen that corresponding equivalence classes U and V determine the same closed linear manifold $\mathfrak{M}$. Hence, if one of U and V is finite, $\mathfrak{M}$ is finite dimensional, and therefore the other of U and V is finite and has the same cardinality. If neither of U and V is finite, both are countable. Thus $\{u_\alpha\}$ and $\{v_\beta\}$ break up into a disjoint union of corresponding pairs U, V of equivalence classes, each U having the same cardinality as the corresponding V. Consequently $\{u_\alpha\}$ and $\{v_\beta\}$ have the same cardinality. Q.E.D.

15 Definition. The cardinality of an arbitrary orthonormal basis of a Hilbert space $\mathfrak{H}$ is its *dimension*.

16 Theorem. *Two Hilbert spaces are isometrically isomorphic if and only if they have the same dimension.*

Proof. Let U be an isometric isomorphism of $\mathfrak{H}_1$ onto $\mathfrak{H}_2$. Then if x and y are orthogonal elements of $\mathfrak{H}_1$,

$$\begin{aligned}|U(x+\lambda y)|^2 &= |x+\lambda y|^2 = |x|^2+|\lambda|^2|y|^2 = |Ux+\lambda Uy|^2\\ &= |Ux|^2+|\lambda|^2|Uy|^2+(Ux,\lambda Uy)+\overline{(Ux,\lambda Uy)}\\ &= |x|^2+|\lambda|^2|y|^2+(Ux,\lambda Uy)+\overline{(Ux,\lambda Uy)}.\end{aligned}$$

This shows that for arbitrary λ

$$0 = (Ux,\lambda Uy)+\overline{(Ux,\lambda Uy)},$$

and if we let $\lambda = (Ux, Uy)$ in this equation it is seen that $(Ux, Uy) = 0$. Thus U maps an orthonormal basis for $\mathfrak{H}_1$ onto an orthonormal basis for $\mathfrak{H}_2$, and consequently $\mathfrak{H}_1$ and $\mathfrak{H}_2$ have the same dimension.

Conversely, let $\mathfrak{H}_1$ and $\mathfrak{H}_2$ have the same dimension, and let $\{u_\alpha, \alpha\in A\}$ and $\{v_\alpha, \alpha\in A\}$ be orthonormal bases for $\mathfrak{H}_1$ and $\mathfrak{H}_2$ respectively. For each scalar function C on A with $C(\alpha) = 0$ for all but a countable set of indices α and such that $\sum|C(\alpha)|^2<\infty$, let

$$U\left(\sum C(\alpha)u_\alpha\right) = \sum C(\alpha)v_\alpha.$$

It follows from Theorem 13 that U is an isometric isomorphism of $\mathfrak{H}_1$ onto $\mathfrak{H}_2$. Q.E.D.

Direct Sums of Hilbert Spaces

We recall (cf. I.11) that the direct sum

$$\mathfrak{X} = \mathfrak{X}_1\oplus\ldots\oplus\mathfrak{X}_n$$

of the vector spaces $\mathfrak{X}_1,\ldots,\mathfrak{X}_n$ is the set $\mathfrak{X}_1\times\mathfrak{X}_2\times\ldots\times\mathfrak{X}_n$ in which addition and scalar multiplication are defined by the formulas

$$\begin{aligned}[x_1,\ldots,x_n]+[y_1,\ldots,y_n] &= [x_1+y_1,\ldots,x_n+y_n],\\ \alpha[x_1,\ldots,x_n] &= [\alpha x_1,\ldots,\alpha x_n].\end{aligned}$$

The space $\mathfrak{X}_i$ is algebraically equivalent to the subspace $\mathfrak{M}_i$ of $\mathfrak{X}$ consisting of all vectors $[x_1,\ldots,x_n]$ in $\mathfrak{X}$ with $x_j = 0$ for $j\neq i$. It is sometimes convenient to refer to the space $\mathfrak{X}_i$ itself as a subspace of $\mathfrak{X}$ and, when such reference is made, it is the equivalent space $\mathfrak{M}_i$ that is to be understood. The map

$$[x_1,\ldots,x_n]\to[0,\ldots,x_i,\ldots,0]$$

of $\mathfrak{X}$ onto $\mathfrak{M}_i$ is a projection and is sometimes called *the projection of* $\mathfrak{X}$

onto $\mathfrak{M}_i$. Equivalently, the map $[x_1, \ldots, x_n] \to x_i$ is called *the projection of* $\mathfrak{X}$ *onto* $\mathfrak{X}_i$. If each of the spaces $\mathfrak{X}_1, \ldots, \mathfrak{X}_n$ is a linear topological space, then the direct sum $\mathfrak{X}$, with the product topology (cf.I.8), is also a linear topological space in which the subspace $\mathfrak{M}_i$ is topologically as well as algebraically equivalent to $\mathfrak{X}_i$. If a topology in each of the summands $\mathfrak{X}_i$, $i = 1, \ldots, n$, is given by a norm $|\cdot|_i$, i.e., if each of the spaces $\mathfrak{X}_i$ is a normed linear space, then the space $\mathfrak{X}$ is a normed linear space. The norm in $\mathfrak{X}$ may be introduced in a variety of ways; for example, any one of the following norms defines the product topology in $\mathfrak{X}$.

(i) $|[x_1, \ldots, x_n]| = |x_1|_1 + |x_2|_2 + \ldots + |x_n|_n$,

(ii) $|[x_1, \ldots, x_n]| = \sup\limits_{1 \leq i \leq n} |x_i|_i$,

(iii) $|[x_1, \ldots, x_n]| = (|x_1|_1^2 + \ldots + |x_n|_n^2)^{1/2}$.

Whenever the direct sum of normed linear spaces is used as a normed space, the norm will be explicitly mentioned. If, however, each of the spaces $\mathfrak{X}_1, \ldots, \mathfrak{X}_n$ are Hilbert spaces then it will always be understood, sometimes without explicit mention, that $\mathfrak{X}$ is the uniquely determined Hilbert space with scalar product

$$\text{(iv)}\quad ([x_1, \ldots, x_n], [y_1, \ldots, y_n]) = \sum_{i=1}^{n} (x_i, y_i)_i,$$

where $(\cdot, \cdot)_i$ is the scalar product in $\mathfrak{X}_i$. Thus the norm in a direct sum of Hilbert spaces is always given by (iii). To summarize, we state the following definition.

17 Definition. For each $i = 1, \ldots, n$, let $\mathfrak{H}_i$ be a Hilbert space with scalar products $(\cdot, \cdot)_i$. The *direct sum of the Hilbert spaces* $\mathfrak{H}_1, \ldots, \mathfrak{H}_n$ is the linear space $\mathfrak{H} = \mathfrak{H}_1 \oplus \ldots \oplus \mathfrak{H}_n$ in which a scalar product is defined by (iv).

Let $\mathfrak{H} = \mathfrak{H}_1 \oplus \ldots \oplus \mathfrak{H}_n$ be the direct sum of the Hilbert spaces $\mathfrak{H}_1, \ldots, \mathfrak{H}_n$. Then for $i \neq j$ the manifolds $\mathfrak{H}_i$ and $\mathfrak{H}_j$ are orthogonal in $\mathfrak{H}$ and the projection of $\mathfrak{H}$ onto $\mathfrak{H}_i$ is the same as the orthogonal projection of $\mathfrak{H}$ onto $\mathfrak{H}_i$. Thus, for example, the subspace $\mathfrak{H}_2 \oplus \ldots \oplus \mathfrak{H}_n$ of $\mathfrak{H}$ is the orthocomplement of $\mathfrak{H}_1$.

The following definition generalizes Definition 17 to cover the case of an infinite family of direct summands.

18 Definition. For each ν in an index set A let $\mathfrak{H}_\nu$ be a Hilbert space. The *direct sum* $\sum \mathfrak{H}_\nu$ of the Hilbert spaces $\mathfrak{H}_\nu$ is defined to be the family of all functions $\{x_\nu\}$ on A such that for each ν, $x_\nu \in \mathfrak{H}_\nu$ and such that $\sum_{\nu \in A} |x_\nu|^2 < \infty$.

It is clear that $\sum \mathfrak{H}_\nu$ is a vector space if addition and scalar multiplication are defined by the formulas

$$\alpha\{x_\nu\} = \{\alpha x_\nu\}, \qquad \{x_\nu\}+\{y_\nu\} = \{x_\nu + y_\nu\}.$$

Moreover, one may define an inner product in $\sum \mathfrak{H}_\nu$ by the formula

$$(\{x_\nu\}, \{y_\nu\}) = \sum_\nu (x_\nu, y_\nu),$$

the series converging unconditionally since

$$\begin{aligned}\sum_\nu |(x_\nu, y_\nu)| &\leqq \sum_\nu |x_\nu||y_\nu| \\ &\leqq (\sum_\nu |x_\nu|^2)^{1/2}(\sum_\nu |y_\nu|^2)^{1/2}.\end{aligned}$$

Properties (i)–(v) of Definition 2.26 may readily be verified.

19 Lemma. *If $\{\mathfrak{H}_\nu\}$, $\nu \in A$, is a family of Hilbert spaces, their direct sum $\sum \mathfrak{H}_\nu$ is a Hilbert space.*

Proof. As remarked above, it only remains to prove the completeness of $\sum \mathfrak{H}_\nu$. If $\{x_\nu^n\}$, $n = 1, 2, \ldots$, is a Cauchy sequence in $\sum \mathfrak{H}_\nu$, it is clear that for fixed ν, $\{x_\nu^n\}$ is a Cauchy sequence in $\mathfrak{H}_\nu$ converging to some element x_ν^0. For any finite subset $\pi \subset A$ and any integer n,

$$\begin{aligned}\sum_{\nu\in\pi} |x_\nu^n - x_\nu^0|^2 &= \lim_{m\to\infty} \sum_{\nu\in\pi} |x_\nu^n - x_\nu^m|^2 \\ &\leq \limsup_{m\to\infty} |\{x_\nu^n\} - \{x_\nu^m\}|^2.\end{aligned}$$

It follows that

$$\limsup_{n\to\infty} \sum_\nu |x_\nu^n - x_\nu^0|^2 \leq \limsup_{m,n\to\infty} |\{x_\nu^n\} - \{x_\nu^m\}| = 0$$

showing that $\{x_\nu^0\}$ is in $\sum \mathfrak{H}_\nu$ and that $\{x_\nu^n\}$ converges to $\{x_\nu^0\}$. Q.E.D.

We conclude this section by listing, in the following lemma, a few useful properties of the orthocomplement.

→ 20 Lemma. *Let B be a set in $\mathfrak{H}$ and $\mathfrak{M}$ a closed linear manifold in $\mathfrak{H}$. Then*

(i) $\mathfrak{H} = \mathfrak{M} \oplus (\mathfrak{H} \ominus \mathfrak{M})$;

(ii) $\mathfrak{M} = \mathfrak{H} \ominus (\mathfrak{H} \ominus \mathfrak{M})$;

(iii) $\overline{\mathrm{sp}}(B) = \mathfrak{H} \ominus (\mathfrak{H} \ominus B)$.

PROOF. Equation (i) is merely a restatement of Lemma 4. Equation (ii) may be proved by replacing $\mathfrak{M}$ by $\mathfrak{H} \ominus \mathfrak{M}$ in equation (i). This shows that $\mathfrak{H} \ominus (\mathfrak{H} \ominus \mathfrak{M})$, as well as its closed subspace $\mathfrak{M}$, is a complementary manifold to $\mathfrak{H} \ominus \mathfrak{M}$. Thus $\mathfrak{M} = \mathfrak{H} \ominus (\mathfrak{H} \ominus \mathfrak{M})$. To prove (iii) note that for an arbitrary set $B \subseteq \mathfrak{H}$ the condition $(B, x) = 0$ on an element x in $\mathfrak{H}$ is equivalent to the condition $(\overline{\mathrm{sp}}(B), x) = 0$. Thus $\mathfrak{H} \ominus B = \mathfrak{H} \ominus \overline{\mathrm{sp}}(B)$ and (iii) follows by placing $\mathfrak{M} = \overline{\mathrm{sp}}(B)$ in (ii). Q.E.D.

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NOTATION INDEX

SUBJECT INDEX

Section numbers are followed by page numbers in parentheses

C

E

F

I

M

Errata, Part I

Italic line numbers indicate the number of lines counted from the bottom of the page

page viii, line *3*: **Change netrospect to retrospect.**

page viii, line *12*: **Change Comments to Remarks.**

page 4, line 12: **Insert after respectively:** If f is a real function defined on an open interval containing zero then the equations

$$\limsup_{x\to 0} f(x) = \inf_{a>0} \sup f((-a, a)),$$

$$\liminf_{x\to 0} f(x) = \sup_{a>0} \inf f((-a, a)),$$

$$\limsup_{x\to 0^+} f(x) = \inf_{a>0} \sup f((0, a)),$$

$$\liminf_{x\to 0^+} f(x) = \sup_{a>0} \inf f((0, a)),$$

define the symbols on their left sides. Similar definitions hold for the symbols $\limsup_{x\to 0^-} f(x)$, $\liminf_{x\to 0^-} f(x)$.

page 21, line 1: **Change The to the.**

page 33, line *6*: **Change bounded to periodic.**

page 41, line 12: **Change last $\cup$ to $\cap$.**

page 42, line *6*: **Change $(h_0)0$ to $h_0(0)$.**

page 43, line 10: **Change the period to a comma.**

page 44, line *8*: **Insert after dimensional: linear.**

page 46, line *4*: **Change to to of.**

page 55, line 7: **Change $|Tx| < \varepsilon$ to $|Tx| < \varepsilon$.**

page 66, line 11: **Change $\hat{\mathfrak{X}}^{**}$ to $\mathfrak{X}^{**}$.**

page 69, line *3*: **Change x_n to x_m.**

page 70, line *4*: **Change $z^*f(y)$ to z^*f.**

page 71, lines *12, 13*: **Change closure of the set consisting of to closed linear manifold determined by.**

page 72, line 7: **Insert after 14: Let $\mathfrak{X}$ be an F-space or a normed space.**

page 72, line 13: Change If $\mathfrak{X}$ is a B-space to The function.
page 72, line 15: Insert after 15: (Banach).
page 72, lines 16, 17, 18: Delete sentence beginning Show that and substitute: Show that there is a number $N > 0$ such that for every sequence $y_n \quad \to y_0$ there is a sequence $x_n \to x_0$ with $|x_n| < N|y_n|$ and $Tx_n \quad y_n$, $n \quad 0, 1, \ldots$.
page 72, line *10*: Change into to onto all of.
page 72, line *5*: Change $\mathfrak{Z}^{\perp\perp} - \mathfrak{Z}$ to $\mathfrak{Z}^{\perp\perp} \quad \kappa\mathfrak{Z}$.
page 75, line *7*: Change p_i to P_i.
page 75, line *5*: Change p_i^{-1} to P_i^{-1}.
page 76, line 7: Change $\mathscr{R}(\alpha) > 0$ to $\mathscr{R}(\alpha) > 0$.
page 76, line *11*: Change $(1 \quad z)\sum_{n=0}^{\infty} s_n$ to $(1 \quad z)\sum_{n=0}^{\infty} s_n z^n$.
page 76, line *2*: Change lower limit on second and third summation signs from $n \quad 0$ and $m \quad 0$ to $m \quad 0$ and $n \quad 0$.
page 78, lines *14*, *12*: Delete uniformly.
page 91, line *13*: Change $|F(x)| \quad |x|$ to $|F(x) \quad F(y)| \quad |x \quad y|$.
page 96, line 2: Change if $\mu(\phi) = 0$ and to if τ contains the void set ϕ, if $\mu(\phi) \quad 0$, and if.
page 96, line *1*: Insert after non-negative: and additive.
page 97, line 1: Insert after if μ is bounded: and additive.
page 98, line 3: Insert after set function: μ.
page 98, line *2*: Change in to *in*.
page 101, line *15*: Change $\int f(s)\mu \ (ds)$ to $\int f(s)\mu(ds)$.
page 103, line 11: Change F to f.
page 103, line *12*: Change *everwhere* to *everywhere*.
page 105, line 4: Change $F(S)$ to $F(S) \times F(S)$.
page 105, line 13: Change $\varepsilon/|\alpha|$, to $\varepsilon/|\alpha|$)..
page 106, line *14*: Delete $TM(S)$ *into itself*, and substitute: *the space of totally measurable scalar functions into itself*.
page 106, line *12*: Add after the second $\beta(s)$:).
page 107, line 14: Add after positive number: less than one.
page 107, line 15: Change M to $M + 1$.
page 110, line 11: Insert before subsets: disjoint.
page 111, line *6*: Change $f_n^2 d\mu$ to $\int_E f_n^2 d\mu$.
page 112, line 6: Change $\dfrac{\varepsilon}{v(A)}$ to $\dfrac{\varepsilon}{v(A)+1}$.

page 113, line *8*: Insert after and: , for a fixed m, .
Change $\{|f_n(\cdot) - f(\cdot)|\}$ to $\{|f_n(\cdot) - f_m(\cdot)|\}$.
After determines: delete zero and substitute $|f(\cdot) - f_m(\cdot)|$.

page 114, line 5: Change g to f.

page 114, line *1*: Change Theorem to Lemma.

page 115, line *8*: Change Theorem to Lemma.

page 116, line *9*: Change $>$ to $\geq$.

page 117, line 17: Change Theorem to Lemma.

page 118, line 7: Delete sentence beginning Then, since g is μ-integrable, and substitute: Since $2|g| > |f|$ we have $|g(s)| \geq |x|/2$ on E. The argument of the preceding lemma then shows that $v(\mu, E) < \infty$.

page 120, line 2: Add after PROOF: In case either $|f|_p$ or $|g|_q$ is zero the lemma follows from Lemmas 2.12, 2.21, and Theorem 2.20(d) and so we shall assume that neither of these norms is zero.

page 120, line 7: Change $f(s)$ and $g(s)$ to $|f(s)|$ and $|g(s)|$.

page 120, line *1*: Should read:

$$\leq \int_S \{|f_1(s)| + |f_2(s)|\}|f_1(s) + f_2(s)|^{p-1}v(\mu, ds)$$

page 121, line 1: Should read: $\int_S |f_1(s)|\, |f_1(s) + f_2(s)|^{p-1}v(\mu, ds)$

page 121, line 2: Should read: $+ \int_S |f_2(s)|\, |f_1(s) + f_2(s)|^{p-1}v(\mu, ds)$

page 122, line *6*: Change ds to E.

page 124, line 4: Change Theorem to Lemma.

page 125, line 1: Change $g(s)|$ to $|g(s)|$.

page 125, line 12: Change the second f to f_α.

page 128, line 1: Change measure to set function.

page 128, line *2*: Change non-decreasing to non-increasing.

page 128, line *1*: Change the period to a comma.

page 129, line 1: Change $\geq$ to $\leq$.

page 129, line 9: Change 7 to 8.

page 130, line 5: Insert before then: and $B_0 = \cap_{n=1}^{\infty} B_n$. .

page 130, line *11*: Delete unique.

page 133, line 3: Insert after is called: a.

page 133, line *6*: Delete comma after E_1.

page 137, line *7*: Insert after from: the remark following.
Change extended to bounded and.

page 188, line 8: Change subscript i to subscript n.
page 138, line 14: Change $>$ to $\geqq$.
page 139, line 3: Add after Theorem 4: , Theorem 8,
page 139, line 11: Change all three $\subset$ to $\subseteqq$.
page 139, line *15*: Change Σ to Σ_1.
page 140, line *4*: Change $(c+\varepsilon]$ to $(c, c+\varepsilon]$.
page 141, line 8: Change ε_{n_2} to ε_2.
page 141, line *3*: Change the first if in the line to is.
page 142, line 6: Change $[a, b]$ to $[a, b_1]$.
page 143, line 9: Insert before vector: countably additive.
page 145, line *6*: Change f_n to f_{n_i}.
page 146, line 2: Change 3 to 2.
page 146, line 16: Change f_{n_1} to f_{n_i}.
page 146, line *5*: Change $|f_n(s)^p$ to $|f_n(s)|^p$.
page 146, line *3*: Change ∞ to 0.
Change the second $\rightarrow$ in the line to $=$.
page 147, line 7: Change $v(\mu, E)$ to $v(\mu, F)$.
page 147, line 15: Change $\cap$ to $\cup$.
page 148, line 4: Change G_n to $f_k^{-1}(G_n)$.
page 148, line 5: Add after E: $\in \Sigma$.
page 148, line 12: Change $\cup_{i=1}^{\infty}$ to $\cup_{n=1}^{\infty}$.
page 149, line *12*; Insert after $\mathfrak{X}$: . If $G = \phi$, $f^{-1}(G) = \phi \in \Sigma^*$. If $G \neq \phi$, let $\{y_n\}$ be that subset of $\{x_n\}$ that is in G.
page 149, line *11*: Change the first x_n in the line to y_n.
page 149, line *10*: Change subscript m to subscript n.
page 149, line *2*: Change $f_n(s)$ to $f_m(s)$.
page 152, line 8: Change for $n \geqq N$, to $n = 1, 2, \ldots$.
page 152, line 6: Change $n \geqq N$ to in n.
Change of course to since ε is arbitrary.
page 154, line *11*: Change δ_1 to δ,.
page 155, line 3: Change the second $\leqq$ in the line to $<$.
page 155, line 4: Change the second $\leqq$ in the line to $<$.
page 159, line *1*: Change $<$ to $\leqq$.
page 160, line 13; Insert after Now let: $\lambda_n = 0$ if $\mu_n = 0$ and otherwise let.
page 163, line *11*: Change $\lim\limits_n$ to $\lim_n$.
page 163, line *10*: Change $\lim$ to $\lim_n$.

page 169, line 7: Change the interval $(-\infty, +\infty)$ to the finite interval $(a, b]$.

page 170, lines *15, 14*: Delete sentence after 20 and substitute: (Langlands) A regular complex valued additive set function defined on a field of sets in a compact space is countably additive.

page 170, line *13*: Insert after topological spaces: with S_1 a Hausdorff space.

page 170, line *11*: Change of sets in to containing the open subsets of.

page 170, line *8*: Change that to with.

page 172, line 4: Insert after generating Σ: and suppose that μ is σ-finite on Σ_1.

page 172, lines 14–17: Delete these lines and substitute: Find a sequence $\{f_n\}$ of positive functions in $L_1(S, \Sigma, \mu)$ for which the preceding inequalities are not all valid.

page 174, line 6: Change with ε to as ε decreases.

page 180, line 10: Change measurable to integrable.

page 183, line 2: Change $\phi(E_1 \cup \phi(E_2)$ to $\phi(E_1) \cup \phi(E_2)$.

page 183, line 11: Change S to S_2.

page 183, line *4*: Change E to the to E to be the.

page 185, line 13: Change f to F.

page 185, line *13*: Change $(E_1 \times E_2 \times E_2')$ to $(E_1 \times E_2 \times E_3')$.

page 192, line 4: Change $g(r)$ to $f(r)$.

page 193, line 2: Change Σ to Σ_S.

page 197, line *9*: Change $F_n \to 0$ to $F_n \to F$.

page 198, line *9*: Change I to T.

page 215, lines 10–15: Delete these lines and substitute: We will show now that $d\lambda/d\mu$ is μ-measurable. Let $C(p, \alpha)$ denote the closed cube with center p and side length α. Let

$$\mu_m(p, \alpha) = 2m \int_{\alpha+1/2m}^{\alpha+1/m} \mu\big(C(p, \beta)\big)d\beta,$$

$$\lambda_m(p, \alpha) = 2m \int_{\alpha+1/2m}^{\alpha+1/m} \lambda\big(C(p, \beta)\big)d\beta.$$

Then $\lambda_m(p, \alpha)/\mu_m(p, \alpha)$ is a continuous function of p for each $\alpha > 0$, and hence $\big(C(p, \alpha)\big)/\mu\big(C(p, \alpha)\big) = \lim_{m\to\infty} \lambda_m(p, \alpha)/\mu_m(p, \alpha)$ is μ-measurable. Consequently, .

page 232, line *17*: Change [1] to [2].

page 241, line 18: Insert after bounded: μ-measurable.
page 242 is numbered wrong.
page 249, line *13*: Change subscript m to subscript m_1.
page 249, line *6*: Change N to $\mathfrak{N}$.
page 250, line *7*: Change are to is.
page 256, line 3: Change are Hilbert spaces to is a Hilbert space.
page 267, line 10: Insert before compact: conditionally.
page 274, line *16*: Change s to S.
page 276, line *8*: Add after if: points are closed and
page 290, line *9*: Insert after S_α: , and such that $\mu(K) = 0$ for every set K in Σ for which $\mu(KS_\alpha) = 0$ for all α,
page 311, line 6: Change the to then.
page 346, line *10*: Delete uniformly.
page 361, line 5: Change continuous to equivalent to a continuous function.
page 370, line *3*: Change both p's to exponents.
Change subscript $d\theta$'s to text size $d\theta$'s set on the line.
page 371, line 1: Change both p's to exponents.
Change subscript $d\theta$'s to text size $d\theta$'s set on the line.
page 402, line *12*: Change $\int_{E^n}$ to $\int_e$.
page 476, line 8: Change x^* to x.
page 504, line 17: Change 7 to 6.
page 557, line *5*: Change superscript αi to superscript α_i.
page 561, line 1: Insert after $\lim\limits_{n\to\infty}$: $\dfrac{1}{n}$.
page 561, line 6: Change p_{kj} to p_{jk}.
page 562, line *6*: Insert after $j = \nu(\lambda)$: or $j = 0$.
page 565, line 14: Change exercise to Exercise.
page 633, line *3*: The convolution symbol $*$ should be lowered here and elsewhere throughout Section VIII.1.
page 643, lines *5*, *1*: Star should be centered, it is not a superscript.
page 644, lines 9, 12: Star should be centered.
page 645, line *3*: Star should be centered.
page 678, line *4*: Insert after $2p$: $\int_S$.

page 722, line 11: Change f in denominator to g.
page 739, line *5*: Change N. to H.
page 827, column 2, line 10: Change f^*g to $f*g$.
page 828, line *5*: After this line insert: $S(x, \varepsilon)$ (19)
$S(A, \varepsilon)$ (19)
page 829, column 1, line 11: Change Alexiewiez to Alexiewicz.
page 831, column 2, line 3: Move Inaba, M., 474 down two spaces.
page 832, column 2, line 1: Change McShano to McShane.
page 840, column 1, line *23*: Change 539 to 598.
page 841, column 2, line 4: Change 267 to 26–27.
page 847, column 2, line 15: Change *Jenssen* to *Jessen*.
page 347, column 2, line *3*: Change Lalpace to Laplace.
page 848, column 1, line 23: Change 3 to 4.
page 854, column 2, line 7: Insert after Set(s): Borel, III.5.10 (137).